Statistics and Applications
Volume 12, Nos. 1&2, 2014 (New Series), pp. 71-79

Structurally Incomplete Row-Column Designs with Multiple Units per Cell*

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Abstract

Row-column designs with multiple units per cell are used when the number of treatments is substantially large with limited number of replicates and controlling heterogeneity in two directions. There may also be situations when some units may not be available for experimentation or some of the treatments may be scarce. Structurally incomplete row-column designs with multiple units per cell are designs in which corresponding to the intersection of row and column, there is at least one cell which does not contain any treatment. Here, some methods of constructing row-column designs with multiple units per cell with equal/ unequal cell sizes and equal/ unequal replications have been developed that are structurally incomplete.

Key words: Empty cells; structurally incomplete; variance balanced; partially variance balanced; elementary contrast; nested group divisible

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1 Introduction

Row-column designs with multiple units per cell are used when the number of treatments is substantially large with limited number of replicates and controlling heterogeneity in two directions. In row-column designs with multiple units per cell, sometimes there may be situations when some units may not be available for experimentation or some of the treatments may be scarce. In such situations, it may not be possible for the experimenter to have the treatments in one or more cells of the row-column intersections i.e. there may be empty cells.

A row-column design with multiple units per cell is said to be structurally incomplete if corresponding to the intersection of any row and column, there is at least one cell which does not contain any treatment. Suppose the units corresponding to second row and first column is not available, then the design becomes structurally incomplete as shown below.

A B	C	D
-	A D	B C
C D	B	A

Row-column designs with multiple experimental units per cell have been studied in the literature as semi-Latin square/ Trojan square designs. An $(n \times n)/k$ semi-Latin square is an arrangement of nk symbols (treatments) in an $(n \times n)$ square array such that each row-column intersection contains k symbols and each symbol occurs once in each row and each column. Trojan squares were first discussed by Harshbarger and Davis (1952) but then it was named as Latinized Near Balanced Rectangular Lattices having $k = n-1$. Later, Darby and Gilbert (1958) discussed the general case for $k < n$ and introduced the name Trojan square designs where $k > 2$. However, all designs of the Latinized Rectangular Lattice type are now commonly described as Trojan squares for any $1 < k < n$.

Preece and Freeman (1983) discussed the combinatorial properties of semi-Latin squares and related designs. Bailey (1988 and 1992) gave methods of constructing a range of semi-Latin and Trojan square designs, studied their efficiencies and showed that the Trojan squares are the optimal choice of semi-Latin squares for pair-wise comparisons of treatment means. These are particularly suitable for crop research experiments either in field or in the glasshouse. Trojan squares are normally the best choice of semi-Latin squares for crop research (Edmondson, 1998).

Bedford and Whitaker (2001) have given several methods of construction of semi-Latin squares. Complete Trojan squares of size $(n \times n)/k$ have n^2 blocks of size k and require n replicates of nk treatments. Sometimes, design or cost constraints make complete Trojan squares impossible [Cheng and Bailey (1991) and Bailey (1992)] because of the unavailability of the homogeneous experimental units corresponding to the sources of variability considered and then incomplete Trojan squares of size $[(n-1) \times n]/ k$ or of size $[n \times (n-1)]/ k$ can be useful. Such incomplete Trojan squares can be constructed by omitting any complete row or column from any Trojan design of size (n x n)/k. Dharmalingam (2002) gave an application of Trojan square designs and used it to obtain partial triallel crosses. Edmondson (2002) constructed generalized incomplete Trojan square designs, denoted by $(m \times n)/k$ where m denotes the number of replicates of nk treatments, based on a set of k cyclic generators. Parsad (2006) discussed a method of constructing Semi-Latin square for $v = 2n$ treatments in n rows, n columns and $k = 2$ by developing an initial column.

Bailey and Chigbu (1997) gave three optimal $(4 \times 4)/4$ semi-Latin squares for sixteen treatments in blocks of size four. Since these squares do not have the same concurrences, there was a need for distinguishing one square from the others and determining the most preferred square in a given context. Chigbu (2003) obtained the best of the three optimal $(4 \times 4)/4$ semi-Latin squares by finding and comparing the variances of elementary contrasts of treatments for the squares.

Jaggi *et al.* (2010) defined generalized incomplete Trojan-type designs and developed method of constructing these designs. SahaRay (2001) studied designs with unequal row and column sizes having more than one observations per cell. Varghese and Jaggi (2001) obtained generalized row-column designs and showed their application in obtaining mating plans.

In this paper, some methods of constructing series of structurally incomplete row-column designs with equal or unequal cell sizes are obtained. Statistical properties of the designs constructed are examined.

2 Methods of Construction

Three methods of constructing structurally incomplete row-column designs with multiple units per cell have been developed. Two series are for equal cell sizes, one of which has treatments with different replications and one series is for structurally incomplete designs with unequal cell sizes.

Method 2.1: This method is for structurally incomplete row-column designs with multiple units per cell with differential replication.

Consider v (odd) treatments. Develop the contents of the h^{th} (h = 1, 2,...,v) column (mod v) with (v-1) cells of size two each as follows:

h	h + 1
h + 1	h + 3
h + 2	h + 5
h + 3	h + 7
.	.
.	.
.	.
h + (v - 2)	h + (v - 3)

for all h = 1,2,…,v. In the (v-1) × (v-1) square array starting from second column the positions or cell contents of the back diagonal are left blank.

The design so obtained is a structurally incomplete row-column design with two units per cell and differential replication of treatments. The parameter of the design are v (odd), m = (v-1) rows of size 2(v-1) each, n = v columns [one column of size 2(v-1) and remaining of size 2(v-2) each], k = 2, r_1 (replication of first v-1 treatments) = 2v-3 and r_2 (replication of the v^{th} treatment) = v-1.

The information matrix is given by

$$\mathbf{C}=\begin{pmatrix} \left(v-\frac{k-1}{2}\right)\mathbf{I}-\mathbf{1}\mathbf{1}' & -0.5\mathbf{1} \\ -0.5\mathbf{1}' & \frac{v-1}{2} \end{pmatrix}.$$

The elementary contrast pertaining to the (v-1) treatments having same replication are estimated with same variance.

Example 2.1.1: For v = 7, a structurally incomplete row- column design with parameters of the design are v = 7, m = 6 rows of size 12 each, n = 7 columns [one column of size 12 and remaining of size 10 each], k = 2, r_1 = 11 and r_2 = 6.

	Columns													
Rows	1	2	2	3	3	4	4	5	5	6	6	7	-	
	2	4	3	5	4	6	5	7	6	1	-		1	3
	3	6	4	7	5	1	6	2	-		1	4	2	5
	4	1	5	2	6	3	-		1	5	2	6	3	7
	5	3	6	4	-		1	6	2	7	3	1	4	2
	6	5	-		1	7	2	1	3	2	4	3	5	4

The C matrix for estimating treatment effects is given by

$$\mathbf{C}=\begin{pmatrix} 6.5\mathbf{I}-\mathbf{11}' & -0.5\mathbf{1} \\ -0.5\mathbf{1}' & 3 \end{pmatrix}$$

The elementary treatment contrasts are estimated with two different variances, viz., $0.3077\sigma^2$ and $0.4615\ \sigma^2$.

Remark 2.1.2.1: Replacing (v-2) replication of any treatment from (v-1) treatment with a new treatment (say, v + 1) we obtain a structurally incomplete row-column design with two units per cell and three types of replication.

Example 2.1.2: In Example 5.2.1.1, five replications of treatment number 6 are replaced by a new treatment 8 in the columns where treatment 6 appears twice. A structurally incomplete row-column design with v=8, m = 6, n = 7, k = 2, r_1= 11 (treatment 1 to 5), r_2 = 6 (treatment 6 and 7) and r_3 = 5 (treatment 8) is obtained as given below:

	Columns													
Rows	1	2	2	3	3	4	4	5	5	8	6	7	-	
	2	4	3	5	4	8	5	7	6	1	-		1	3
	3	8	4	7	5	1	6	2	-		1	4	2	5
	4	1	5	2	6	3	-		1	5	2	8	3	7
	5	3	6	4	-		1	8	2	7	3	1	4	2
	6	5	-		1	7	2	1	3	2	4	3	5	4

Here, variance of contrast for treatment 1 with 2 or 3 or 4 or 5 is $0.3077\ \sigma^2$, variance of contrast for treatment 1 with 6 or 7 is $0.4660\ \sigma^2$ and variance of contrast for treatment 1 with 8 is $0.5231\ \sigma^2$.

Method 2.2: Consider a resolvable BIB design with parameters v*, b*, r*, k*, λ*. In this class of designs, the blocks are partitioned into r* groups each containing b*/r* blocks such that each treatment occur in each group once. A structurally incomplete row-column design with r* rows of size $\frac{k^*b^*}{r^*}$ and $\frac{b^*}{r^*}$ +1 columns of size $\frac{k^*b^*}{r^*}$ is obtained by arranging $\frac{b^*}{r^*}$ blocks of a group in a $\frac{b^*}{r^*}$ cells in a row leaving one cell blank. Similarly, arrange blocks of second group in second row leaving one cell blank and so on. The blocks are arranged in the row-column set up such that there should not be more than one blank cell in each row and column.

The parameters of the design are v = v*, m = r* of size $\frac{k^*b^*}{r^*}$, n = $\frac{b^*}{r^*}$ +1 of size $\frac{k^*b^*}{r^*}$, k = k*. The design is variance balanced as all the elementary contrasts of treatment effects are estimated with same variance. The information matrix for estimating the contrasts pertaining to the treatment effects is obtained as $\mathbf{C} = \frac{v}{k}(\mathbf{I} - \frac{\mathbf{J}}{v})$.

Example 2.2.1: Consider a resolvable BIB design with parameters v* = 9, b* = 12, r* = 4, k* = 3, λ* = 1. The four groups each containing three blocks are as follows:

1 2 3
4 5 6
7 8 9

1 4 7
2 5 8
3 6 9

1 6 8
2 4 9
3 5 7

1 5 9
2 6 7
3 4 8

The resulting design which is a structurally incomplete row-column design with parameters v = 9, m = 4 of size 9, n = 4 of size 9, k = 3. 4 is given by

	Columns			
Rows	1 , 2 , 3	4 , 5 , 6	7 , 8 , 9	-
	-	1 , 4 , 7	2 , 5 , 8	3 , 6 , 9
	3 , 5 , 7	-	1 , 6 , 8	2 , 4 , 9
	2 , 6 , 7	3 , 4 , 8	-	1 , 5 , 9

The C matrix for estimating treatment effects is given by
$\mathbf{C} = 3\mathbf{I} - 0.333\mathbf{J}$

Example 2.2.2: For v = 4, a structurally incomplete row-column design in 3 rows of size 4 each and 3 columns of size 4 each with cells containing 2 units is obtained as

	Columns		
Rows	1 , 2	3 , 4	-
	-	1 , 3	2 , 4
	1 , 4	-	2 , 3

The C matrix for estimating treatment effects is given by $\mathbf{C} = 2\mathbf{I} - 0.5\mathbf{J}$.

Remark 2.2.1: The above method can be extended by taking any α–resolvable BIB design which will also result in a structurally incomplete balanced row-column design with multiple units per cell.

Example 2.2.3: Consider a 2- resolvable BIB design with parameters v* = 6, b* = 15, r* = 10, k* = 4, λ* = 6. Then the resulting design is incomplete balanced row-column design with v = 6, m = 5 rows of size 12, n = 4 columns of size 16, k = 4 and r = 10.

	Columns			
Rows	1 2 3 4	1 2 5 6	3 4 5 6	-
	-	1 2 3 5	1 4 5 6	2 3 4 6
	2 4 5 6	-	1 2 3 6	1 3 4 5
	1 3 4 6	2 3 5 6	-	1 2 4 5
	1 2 4 6	1 3 5 6	2 3 4 5	-

Method 2.3: For v = 4s (s > 1), the treatments are arranged in three rows and three columns as follows:

Divide the 4s treatments in four groups of size s each. The contents of 1st two groups are put in the first cell of the first row and the contents of the 3rd and 4th group are allotted to the second and third cells of the first row respectively. Now, for second row the contents of the 3rd and 4th groups are arranged in first cell, 2nd group in second cell and 1st group in third cell. In third row, the contents of the 1st and 4th groups are allotted to second cell, the contents of the 2nd and 3rd groups are allotted to third cell and first cell is left blank.

This arrangement will give rise to a structurally incomplete row-column designs with parameters v = 4s, m = n = 3, r = 3, $k_1 = \frac{v}{2}$, and $k_2 = \frac{v}{4}$.

The elementary treatment contrasts in the above design are estimated with three types of variances. The treatments follow three associate class nested group divisible association scheme and are arranged in two groups of two rows each as given below:

$$\begin{array}{c} 1 \quad 2 \quad \ldots \quad s \\ 2s+1 \quad 2s+2 \ldots \quad 3s \\ \hline s+1 \quad s+2 \ldots \quad 2s \\ 3s+1 \quad 3s+2 \ldots \quad 4s \end{array}$$

The treatments in the same row of the same group are first associate to each other, the treatments of the different row in same group are second associates and the treatments of the other group are third associates.

Example 2.3.1: Let s = 3 and so v = 12. The 12 treatments are arranged in 4 distinct groups of size 3 each.

Groups			
I	II	III	IV
1 2 3	4 5 6	7 8 9	10 11 12

As per the method described, the following structurally incomplete row-column design in 3 rows and 3 columns with cells of size 6 and 3 is obtained such that each treatment is replicated r = 3 times in the design:

	Columns		
Rows	1 , 2 , 3 , 4 , 5 , 6	7 , 8 , 9	10 , 11 , 12
	7, 8 , 9 ,10 ,11 , 12	4 , 5 , 6	1 , 2 , 3
	-	1 , 2 , 3 , 10, 11 , 12	4 , 5 , 6 , 7 , 8 , 9

The information matrix for this design is obtained as follows:

$$\mathbf{C} = \begin{bmatrix} 2.33 & -0.67 & -0.67 & -0.17 & -0.17 & -0.17 & 0 & 0 & 0 & -0.17 & -0.17 & -0.17 \\ -0.67 & 2.33 & -0.67 & -0.17 & -0.17 & -0.17 & 0 & 0 & 0 & -0.17 & -0.17 & -0.17 \\ -0.67 & -0.67 & 2.33 & -0.17 & -0.17 & -0.17 & 0 & 0 & 0 & -0.17 & -0.17 & -0.17 \\ -0.17 & -0.17 & -0.17 & 2.33 & -0.67 & -0.67 & -0.17 & -0.17 & -0.17 & 0 & 0 & 0 \\ -0.17 & -0.17 & -0.17 & -0.67 & 2.33 & -0.67 & -0.17 & -0.17 & -0.17 & 0 & 0 & 0 \\ -0.17 & -0.17 & -0.17 & -0.67 & -0.67 & 2.33 & -0.17 & -0.17 & -0.17 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.17 & -0.17 & -0.17 & 2.33 & -0.67 & -0.67 & -0.17 & -0.17 & -0.17 \\ 0 & 0 & 0 & -0.17 & -0.17 & -0.17 & -0.67 & 2.33 & -0.67 & -0.17 & -0.17 & -0.17 \\ 0 & 0 & 0 & -0.17 & -0.17 & -0.17 & -0.67 & -0.67 & 2.33 & -0.17 & -0.17 & -0.17 \\ -0.17 & -0.17 & -0.17 & 0 & 0 & 0 & -0.17 & -0.17 & -0.17 & 2.33 & -0.67 & -0.67 \\ -0.17 & -0.17 & -0.17 & 0 & 0 & 0 & -0.17 & -0.17 & -0.17 & -0.67 & 2.33 & -0.67 \\ -0.17 & -0.17 & -0.17 & 0 & 0 & 0 & -0.17 & -0.17 & -0.17 & -0.67 & -0.67 & 2.33 \end{bmatrix}$$

From this it is clear that the treatments can be arranged in two groups of two rows each as given below:

$$\begin{array}{c} 1 \;\; 2 \;\; 3 \\ 7 \;\; 8 \;\; 9 \\ \hline 4 \;\; 5 \;\; 6 \\ 10 \; 11 \; 12 \end{array}$$

The various associates of the treatments are as follows:

Treatments	1st Associate		2nd Associate			3rd Associate					
1	2	3	7	8	9	4	5	6	10	11	12
2	1	3	7	8	9	4	5	6	10	11	12
3	1	2	7	8	9	4	5	6	10	11	12
4	5	6	10	11	12	1	2	3	7	8	9
5	4	6	10	11	12	1	2	3	7	8	9
6	4	5	10	11	12	1	2	3	7	8	9
7	8	9	1	2	3	4	5	6	10	11	12
8	7	9	1	2	3	4	5	6	10	11	12
9	7	8	1	2	3	4	5	6	10	11	12
10	11	12	4	5	6	1	2	3	7	8	9
11	10	12	4	5	6	1	2	3	7	8	9
12	10	11	4	5	6	1	2	3	7	8	9

3 List of Designs

The parameters of row-column designs with multiple units per cell obtained using methods described above have been listed in Table 3.1. The list contains number of treatments ($v \leq 20$), cell sizes (k_j), number of rows (m), number of columns (n) and replications (r_i). A SAS code has been written in PROC IML to calculate the information matrix (**C**-matrix) of treatment effects and to study the properties of the designs under the four-way model with treatments, rows, columns and cells as the four classifications.

Table 3.1: List of structurally incomplete row-column designs with multiple units per cell

S. No.	v	m	n	r_i	k_j	Method
1	4	3	3	3	2	2.2
2	8	3	3	3	4 , 2	2.3
3	9	4	4	4	3	2.2
4	12	3	3	3	6 , 3	2.3
5	16	3	3	3	4, 8	2.3
6	16	5	5	5	4	2.2
7	20	3	3	3	10, 5	2.3

4 Conclusions

Here, some methods of constructing structurally incomplete row-column designs with multiple units per cell have been developed which are very much useful when some units may not be available for experimentation or some of the treatments may be scarce. Two series of structurally incomplete designs have been constructed for equal cell sizes, one of which has treatments with different replications. One series is for structurally incomplete designs with unequal cell sizes. The design with equal cell sizes and equal replications is balanced and the elementary contrasts of treatment effects are estimated with same variance. The treatments in the designs with differential replications are estimated with two types of variances. In the third series for v = 4s, the elementary contrasts of treatment effects are estimated with different variance following nested group divisible association scheme.

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Received: 15 June, 2014 *Revised: 18 November, 2014* *Accepted: 7 December, 2014*