ON ANALYSIS OF FACTORIAL EXPERIMENTS, COMPLETE AND FRACTIONAL

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CONTENTS

I. INTRODUCTION

The general symmetric factorial experiment consists of s^n combinations coming from n factors each at s levels. Both for construction and analysis of designs of such experiments the totality of s^{n-1} D.F. are made into $\frac{s^{n-1}}{s-1}$ components each of (s-1) D.F.. Some of these components belong to the main effects while others belong to the interactions. The interaction involving two specific factors will have (s-1) components each of (s-1) D.F.. Though these components can be interpreted geometrically, they do not always have physical understandable interpretation. As such the significance of any one or more of such components can not usually be connected with any real physical implications. We shall call this method of analysis as analysis through geometrical partition or simply geometrical analysis. Sometimes the total D.F. due to main effects or interactions are sub-divided into linear, quadratic etc. components and their interactions. For example, if we have two factors A and B the four D.F. due to the interaction AB is split into the components, A_LB_L, A_LB_Q, A_QB_L and A_QB_Q where A_LB_Q denotes interaction between linear contrast of A and quadratic contrast of B, and so on for other components. Such components are more meaningful and possess more realistic interpretation. Each such component has only one D.F.. We shall call this method of analysis as component analysis.

The method of analysis of factorial experiments given by Yates(1937) through sum and difference of treatment totals gives directly the results of component analysis. When the design is constructed by adopting geometrical techniques in Galois fields, it becomes difficult to have its component analysis particularly in respect of those interactions which are confounded. Similar difficulties are encountered when the design is fractionally replicated. It appears that this type of difficulty can be overcome if a link between the contrasts in the two types of analysis can be established. One of the main purposes of the present investigation is to establish such a link.

It appears that the method of analysis of fractionally replicated designs through Yates' technique is not yet perfected when factors are at more than two levels. This is because, though the aliases of geometrical components are available, such aliases are not available for linear, quadratic etc. components and their interactions. We have attempted to obtain a solution of this problem also through the present investigation.

Further an investigation has been made to give specific direction for the suppression of factors in case of fractionally replicated designs involving factors each at two levels.

When a large number of factors is involved in a factorial experiment, the analysis through Yates' method becomes complicated as a very large number of combinations have to be taken and operated upon. We have given a modified method whereby suitable groups of the treatment combinations are formed and then each group is analysed separately. The separate analysis of these groups are then combined to get the final results. We have discussed these results in the subsequent chapters beginning with analysis of complete factorial experiments through Yates' method.

2.(a) ANALYSIS OF FACTORIAL EXPERIMENTS OF THE SERIES 2^n THROUGH THE SUM AND DIFFERENCE METHOD OF YATES'

When the number of factors in the experiment of the series 2^n is large, it becomes inconvenient first to write all the treatment combinations and then carry out the operations on them as required by the method of sum and difference of Yates' for obtaining contrasts corresponding to various main effects and interactions. We have discussed below a convenient method for their analysis through which the totality of treatment combinations can first be made into groups of equal size and then each group is analysed separately through the method of sum and difference. Subsequently those results are analysed again through the application of the method of sum and difference for obtaining the final results. We shall discuss this method with reference to an example. General results follow from them.

Let there be eight factors (each at 2 levels) denoted by A, B, C, D, E, F, G and H. There are 2^8 treatment combinations . Instead of writing the whole of the 2^8 treatment combinations along with their observation totals, we can first make them into groups of suitable size and analyse each group separately. Let us decide to make groups of size 32 so that there will be 8 groups. We can conveniently make the groups (without any regard to blocking adopted at the time of construction of the design), say, by confounding the main effects and interactions of the last three factors, viz., F, G and H. The first group then can be written as required in Yates' technique by using the first five factors A, B, C, D, and E. Another group is then obtained by multiplying each of the treatment combinations in the first group by f and writing them in the corresponding order. In this way by multiplying the first group in order by g, fg, h, fh, gh and fgh we shall be getting six other groups. Each of these groups is taken separately and the observation totals are written against the corresponding treatment combinations. The sum and difference operations are carried out in each group separately, there being 5 cycles of operations as required by 2^5 treatment combinations. Next let us consider the 8 contrasts available against the treatment combination ac and those corresponding to ac in other groups, namely, acf, acg, acfg, ach, acfh, acgh and acfgh. These 8 contrasts are then written in a column and linked with the combinations I, f, g, fg, h, fh, gh, fgh of the factors F, G and H in the order as required by Yates' method; each of them, however, has to be multiplied by ac. With these 8 contrasts 3 cycles of operations are carried out as in Yates' method. From this table we shall get the 8 contrasts appropriate for 2^8 design corresponding to interactions AC, ACF, ACG, ACFG, ACH, ACFH, ACGH and ACFGH. In this way corresponding to each interaction of the first 5 factors, we shall get 8 interactions of the final design. The divisors of these contrasts are the same as in the 2^8 design.

It will be seen that the total number of 256 treatment combinations were first made into 8 groups of size 32, each of which was then analysed. Subsequently, 32 groups each of size 8 were formed and analysed so as to give the final results. The total number of operations in the original and the present methods are same. Though we made groups of size 32, this is not necessary. The group size could have been made 16 also so as to get 16 groups. The method has been illustrated by analysing a 2^5 experiment breaking into groups of size 8. The data used are fictitious.

Analysis of 2^5 experiment by breaking into groups of size 8

Group I		Group II		Group III		Group IV	
Treat. comb.	Obs. total	Treat. comb.	Obs. total	Treat. comb.	Obs. total	Treat comb.	Obs total
1	1	d	3	e	3	de	4
a	2	ad	2	ae	4	ade	6
b	4	bd	1	be	1	bde	7
ab	6	abd	2	abe	1	abde	5
c	7	cd	7	ce	2	cde	5
ac	9	acd	8	ace	3	acde	4
bc	8	bcd	6	bce	4	bcde	3
abc	4	abcd	5	abce	5	abcde	2

Analysis of Group I

Treat. comb.	Obs. total	1	2	3
1	1	3	13	41
a	2	10	28	1
b	4	16	3	3
ab	6	12	-2	-5
c	7	1	7	15
ac	9	2	-4	-5
bc	8	2	1	-11
abc	4	-4	-6	-7

Analysis of Group II

Treat. comb.	Obs. total	1	2	3
d	3	5	8	34
ad	2	3	26	0
bd	1	15	0	-6
abd	2	11	0	0
cd	7	-1	-2	18
acd	8	1	-4	0
bcd	6	1	2	-2
abcd	5	-1	-2	-4

Analysis of Group III

Treat. comb.	Obs. total	1	2	3
e	3	7	9	23
ae	4	2	14	3
be	1	5	1	-1
abe	1	9	2	-1
ce	2	1	-5	5
ace	3	0	4	1
bce	4	1	-1	9
abce	5	1	0	1

Analysis of Group IV

Treat. comb.	Obs. total.	1	2	3
de	4	10	22	36
ade	6	12	14	-2
bde	7	9	0	-2
abde	5	5	-2	-4
cde	5	2	2	-8
acde	4	-2	-4	-2
bcde	3	-1	-4	-6
abcde	2	-1	0	4

Now we form 8 groups of size 4 and analyse them to get the final results.

Treat. comb.	Contrast values	1	final contrasts 2	Treat. comb.	Contrast values	1	final contrasts 2
1	41	75	134	a	1	1	2
d	34	59	6	ad	0	1	-6
e	23	-7	-16	ae	3	-1	0
de	36	13	20	ade	-2	-5	-4
b	3	-3	-6	ab	-5	-5	-10
bd	-6	-3	-10	abd	0	-5	2
be	-1	-9	0	abe	-1	5	0
bde	-2	-1	8	abde	-4	-3	-8
c	15	33	30	ac	-5	-5	-6
cd	18	-3	-10	acd	0	-1	2
ce	5	3	-36	ace	1	5	4
cde	-8	-13	-16	acde	-2	-3	-8
bc	-11	-13	-10	abc	-7	-11	-6
bcd	-2	3	-6	abcd	-4	5	6
bce	9	9	16	abce	1	3	16
bcde	-6	-15	-24	abcde	4	3	0

The S S. due to any contrast, say, AB can now be obtained by squaring the contrast against ab, namely, -10(*as read from the second series of tables*) and dividing by 32r. Thus, S.S. due to AB = $(-10)^2/32r$, *where r is the number of replications.*

2.(b) ANALYSIS OF FACTORIAL EXPERIMENTS OF THE SERIES 3^n THROUGH THE EXTENDED YATES' METHOD.

As in 2^n series, the totality of the treatment combinations in 3^n series can also be broken into groups of equal size when the treatment combinations are large and then each group is analysed through the extended Yates' method (Davies, the Design and analysis of Industrial experiments). Subsequently, the results from each group can be analysed likewise for obtaining the final results. We shall discuss this method with reference to an example. General results follow from them. Let there be 5 factors A, B, C, D and E each at 3 equispaced levels. There are 3^5 treatment combinations. Let us decide to make groups of size 27 so that there will be 3^2 groups. We can conveniently make the groups (without any regard to blocking adopted at the time of construction of the design), say, by confounding the main effects and interactions of the last two factors, namely, D and E. The first group can be written as required in Yates' technique by using the first 3 factors A, B and C. The next group then to be obtained by multiplying each of the treatment combinations in the first group by d and writing them in the corresponding order. The third group is obtained by multiplying each of the treatment combinations in the first group by d^2 and writing them in the corresponding order. In this way by multiplying in order by e, de, d^2e, e^2, de^2 and d^2e^2 we shall get 6 other groups. Each of these groups is taken separately and the observation totals are written against the corresponding treatment com-binations. The extended Yates' method is applied to analyse each group separately. Next let us consider the 9 contrasts available against the treatment combination, say, ab and those corresponding to ab in other groups, namely, abd, abd^2, abe, abde, abd^2e, abe^2, $abde^2$ and abd^2e^2. These 9 treatment combinations are then written in a column. This is better done first writing the combinations of the factors D and E in that order as required by Yates' method and then multiplying each of them by ab. The 9 contrasts obtained earlier are then written against the corresponding treatment combinations and 2 cycles of operations are carried out on them. From this table we shall get 9 contrasts appropriate for 3^5 design corresponding to interactions A_LC_L, $A_LC_LD_L$, $A_LC_LD_Q$, $A_LC_LE_L$, $A_LC_LD_LE_L$, $A_LC_LD_QE_L$, $A_LC_LE_Q$, $A_LC_LD_LE_Q$ and $A_LC_LD_QE_Q$.

It is to be noted that corresponding to a squared letter, we get a quadratic component of the factor corresponding to that letter. In this way corresponding to each interaction of the first three factors, we shall get 9 interactions of the final design. The divisors of these contrasts are the same as in the 3^3 design. The method is illustrated by analysing 3^3 experiment breaking into groups of size 9. The data used are factitious.

Analysis of 3^3 experiment by breaking into groups of size 9

Analysis of group I				Analysis of group II				Analysis of group III			
Treat. comb.	Obs. total	1	2	Treat. comb.	Obs. total	1	2	Treat. comb.	Obs. total	1	2
1	2	14	41	c	3	15	39	c^2	1	11	41
a	5	13	-3	ac	5	9	-7	ac^2	4	20	-5
a^2	7	14	-19	a^2c	7	15	-9	a^2c^2	6	10	-1
b	3	-5	0	bc	4	-4	0	bc^2	7	-5	1
ab	9	2	-5	abc	3	2	1	abc^2	8	2	-3
a^2b	1	0	3	a^2bc	2	-5	9	a^2bc^2	5	-2	-5
b^2	4	-1	2	b^2c	1	0	12	b^2c^2	3	-1	-19
ab^2	6	-14	-9	ab^2c	8	0	-13	ab^2c^2	2	-4	-11
a^2b^2	4	-4	23	a^2b^2c	6	-9	-9	$a^2b^2c^2$	5	4	11

Now we form 9 groups of size 3 and analyse them to get the final results.

Treat. Comb.	Contrast Value	Final Contrast	Treat. Comb.	Contrast Value	Final Contrast	Treat. Comb.	Contrast Value	Final Contrast
1	41	121	a	-3	-15	a^2	-19	-29
c	39	0	ac	-7	2	a^2c	-9	-18
c^2	41	4	ac^2	-5	6	a^2c^2	-1	-2
b								
b	0	1	ab	-5	-7	a^2b	3	7
bc	0	-1	abc	1	-2	a^2bc	9	8
bc^2	1	1	abc^2	-3	-10	a^2bc^2	-5	-20
b^2	2	-5	ab^2	-9	33	a^2b^2	23	25
b^2c	12	21	ab^2c	-13	2	a^2b^2c	-9	12
b^2c^2	-19	-41	ab^2c^2	-11	6	$a^2b^2c^2$	11	52

The S.S. due to any interaction, say, $A_Q C_L$ can now be obtained by squaring the contrast against a^2c, namely, -18(as read from the second series of tables) and dividing by 36r.

$$\text{S.S. due to } A_Q C_L = (-18)^2 / 36r$$

where r is the number of replications.

2.(c) ANALYSIS OF FACTORIAL EXPERIMENTS OF THE MIXED SERIES THROUGH YATES' METHOD

Combined Yates' methods - If there are p factors each at 2 levels and q factors each at 3 levels, we have $2^p \times 3^q$ treatment combinations. The combined Yates' method for analysing the data of this experiment is to carry out p cycles of operations as we do in 2^n series and then carry out q cycles of operations as we do in 3^n series to get the final results.

As in 2^n and 3^n series, the totality of treatment combinations in mixed series can be broken into groups of equal size when the treatment combinations are large and then each group is analysed through the combined Yates' method as described above. Subsequently, these results are also analysed for obtaining the final results. We shall discuss this method also with reference to an example. General results follow from them. Let there be 3 factors A, B and C at 2 levels and 3 factors D, E and F at 3 levels. There are $2^3 \times 3^3$ treatment combinations. Let us decide to make groups of size 2×3^2, so that there will be $2^2 \times 3$ groups. We can conveniently make the groups (without any regard to blocking adopted at the time of construction of the design), say, by confounding the main effects and interactions corresponding to the last 2 factors at 2 levels, namely, B and C and the last factor at 3 levels, namely, F. The first group can be written by using the factors, A, D and E in systematic order. The next group then to be obtained by multiplying each of the treatment combinations in the first group by b and writing them in the corresponding order. In this way by multiplying in order by c, bc, f, bf, cf, bcf, f^2, bf^2, cf^2 and bcf^2 we shall be getting 10 other groups. Each of these groups is taken separately and the observation totals are written against the corresponding treatment combinations. The combined Yates' method is carried out in groups separately, there being one cycle of operation as in 2^n series followed by 2 cycles of operations as in 3^n series. Next let us consider the 12 contrasts available against the treatment combination ad and those corresponding to ad in other groups, namely, abd, acd, abcd, adf, abdf, acdf, abcdf, adf^2, $abdf^2$, $acdf^2$ and $abcdf^2$. These 12 treatment combinations are then written in a column by introducing the factors B, C and F in that order as required by Yates' method. The 12 contrasts obtained earlier are written against the corresponding treatment combinations and 2 cycles of operations are carried out as in 2^n series and then one cycle of operation is carried out as in 3^n series. From this table we shall get 12 contrasts appropriate for $2^3 \times 3^3$ design corresponding to interactions AD_L, ABD_L, ACD_L, $ABCD_L$, AD_LF_L, ABD_LF_L, ACD_LF_L, $ABCD_LF_L$, AD_LF_Q, ABD_LF_Q, ACD_LF_Q and $ABCD_LF_Q$. In this way corresponding to each interaction in the first group, we shall get 12 interactions of the final design. The divisors of these contrasts are the same as in the $2^3 \times 3^3$ design. The method is illustrated by analysing a $2^2 \times 3^3$ experiment breaking into groups of size 6. The data used are factitious.

Analysis of $2^2 \times 3^2$ by breaking into groups of size 6

Group I				Group II				Group III			
Treat. comb.	Obs. total	1	2	Treat. comb.	Obs. total	1	2	Treat. Comb.	Obs. total	1	2
1	2	7	20	b	3	7	21	d	1	3	18
a	5	10	-2	ab	4	6	9	ad	2	12	4
c	7	3	4	bc	1	8	-1	cd	4	3	0
ac	3	3	4	abc	5	1	-3	acd	8	1	2
c^2	2	-4	-10	bc^2	2	4	3	c^2d	2	4	-18
ac^2	1	-1	10	abc^2	6	4	-3	ac^2d	1	-1	-8

Group IV				Group V				Group VI			
bd	6	9	18	d^2	1	10	21	bd^2	8	13	28
ab d	3	6	0	ad^2	9	5	11	abd^2	5	9	-2
bcd	2	3	6	cd^2	2	6	4	bcd^2	3	6	7
abcd	4	-3	-4	acd^2	3	8	6	$abcd^2$	6	-3	-1
bc^2d	1	2	0	c^2d^2	2	1	6	bc^2d^2	4	3	1
abc^2d	2	1	-6	ac^2d^2	4	2	8	abc^2d^2	2	-2	-11

Now we form 6 groups of size 6 and analyse them to get the final results.

Treat. Comb.	Contrast Value	Final Contrast 1	Final Contrast 2	Treat. Comb.	Contrast Value	Final Cont. 1	Final Cont. 2	Treat. Comb.	Cont. Value	Final Cont. 1	Final Cont. 2
1	20	41	126	a	-2	7	20	c	4	3	20
b	21	36	8	ab	9	4	-6	bc	-1	6	4
d	18	49	-8	ad	4	9	-2	cd	0	11	-8
bd	18	1	-6	abd	0	11	24	bcd	6	-5	-8
d^2	21	0	18	ad^2	11	-4	8	cd^2	4	6	2
bd^2	28	7	8	abd^2	-2	-13	6	bcd^2	7	3	-14

ac	4	1	4	c^2	-10	-7	-18	ac^2	1	-2	-19
abc	-3	-2	-20	bc^2	3	-18	26	abc^2	-3	-14	-21
acd	2	5	-4	c^2d	-18	7	-14	ac^2d	-8	-3	1
abcd	-4	-7	0	bc^2d	0	13	-18	abc^2d	-6	-4	15
acd^2	6	-6	10	c^2d^2	6	18	36	ac^2d^2	8	2	23
$abcd^2$	-1	-7	-2	bc^2d^2	1	-5	-28	abc^2d^2	-11	-19	-27

The S.S. due to any contrast, say, BC_Q can now be obtained by squaring the contrast against bc^2, namely, 26(as read from second series of tables) and dividing by 72*r*, where *r* is the number of replications.

SOME REMARKS :

1. In the method discussed above, we thought it desirable to form groups of small size. For larger experiments the number of groups may

thus be large. In such cases the contrasts against any treatment combination may be analysed by grouping the contrasts against it by following exactly the same method.

2. Chances of errors through the present method are likely to be smaller as we are handling small operations each time.

3. In some cases we have verified that this method gives correct results in less time then the usual method when the number of factors are large.

3. ANALYSIS OF SYMMETRIC FRACTIONAL FACTORIAL EXPERIMENTS

As in complete factorial experiments, the totality of treatment combinations in fractional factorial experiments can also be broken into groups of equal size when the treatment combinations are large. But the only problem is of writing the treatment combinations in a systematic order. Once the treatment combinations are written in a systematic order, they can be broken into groups of equal size on the basis of the existing factors and analysed as before. We are to suppress one or more letters to write the treatment combinations in a systematic order. In 2^n series it can be shown that any letter or letters which together do not have an even number of letters common with any interactions in Identity group (I-group) can be suppressed for this purpose. This ensures that all the treatment combinations present in the fraction when written after suppressing the letters will be distinct.

e.g. 1. Consider $\frac{1}{2^2}(2^6)$

Let I = ABCE = ABDF = CDEF

We can suppress any two letters except ab, ce and df for writing treatment combinations in a systematic order.

e.g. 2. Consider $\frac{1}{2^3}(2^8)$

The defining contrasts are BCDH, BDFG, CFGH, ABCEF, ABEGH, ACDEG, and ADEFH.

We can suppress any three letters except abd, ahc, afg, bcg, bde, bfh, cdf, ceh, dgh, and efg as each one of them is having an even numbers of letters common with each of the interaction in the identity group.

In fractional replication of 3^n series we can suppress letter or letters which are not contained in the fraction as treatment combination to write treatment combinations in a systematic order. But here we have two more problems, one is of finding the effect corresponding to letter or letters suppressed and interactions involving that effect as aliases of components of one D.F. are not well defined; the second is of finding the adjusted linear, quadratic etc. components corresponding to the interactions confounded. Both these problems are discussed in subsequent chapters.

4. ON LINKING DIFFERENT SETS OF CONTRASTS OF OBSERVATIONS.

So far we have discussed the analysis of factorial designs through the method of Yates' and its extension. As these designs are usually constructed through geometrical methods (Bose and Kishen, 1940) and geometrical components are confounded, it is necessary to find a relationship between geometrical [illegible]mponents and the linear, quadratic etc. components [illegible]ed in Yates' method ~~extended to~~ the case of 3^n series. [illegible]is is particularly necessary to adjust the linear, [illegible]dratic etc. components for block effects and also to [illegible]lyse fractionally replicated designs through Yates' [illegible]hnique when the factors are at 3 or more levels.

The following theorem gives us a very useful [illegible]is for establishing such relationship.

[illegible]EOREM: - Let there be N independent observations $y_1, y_2, \ldots, y_N$ drawn from a population with variance σ^2 a[illegible] let $Q_1, Q_2, \ldots, Q_{N-1}$ be (N-1) independent contrasts a[illegible]g the observations. Now if P_j be any other contrast an[illegible]g the observations, then P_j can be expressed as a lin[illegible] function of Q_i ($i = 1, 2, \ldots, N-1$) through the following rel[illegible]n

$$P_j = \sum_i C_{ji} \frac{Q_i}{d_i}$$

4. ON LINKING DIFFERENT SETS OF CONTRASTS OF OBSERVATIONS.

So far we have discussed the analysis of factorial designs through the method of Yates' and its extension. As these designs are usually constructed through geometrical methods (Bose and Kishen, 1940) and geometrical components are confounded, it is necessary to find a relationship between geometrical components and the linear, quadratic etc. components used in Yates' method ~~extended to~~ the case of 3^n series. This is particularly necessary to adjust the linear, quadratic etc. components for block effects and also to analyse fractionally replicated designs through Yates' technique when the factors are at 3 or more levels.

The following theorem gives us a very useful basis for establishing such relationship.

<u>THEOREM:</u> - Let there be N independent observations $y_1, y_2, \ldots, y_N$ drawn from a population with variance σ^2 and let $Q_1, Q_2, \ldots, Q_{N-1}$ be (N-1) independent contrasts among the observations. Now if P_j be any other contrast among the observations, then P_j can be expressed as a linear function of Q_i ($i = 1, 2, \ldots, N-1$) through the following relation

$$P_j = \sum_i C_{ji} \frac{Q_i}{d_i}$$

where $C_{ji}\sigma^2$ is the covariance between P_j and Q_i and d_i is the sum of squares of the coefficients of observations in Q_i.   ∴ variance of $P_j = \sigma^2 \Sigma \frac{c_i^2}{d_i}$

Proof. - Let $P_j = \sum_i C_{ji} \frac{Q_i}{d_i}$, where c_i's are assumed unknowns

$$\therefore \quad V(P_j) = \sum_i C_{ji}^2 \frac{V(Q_i)}{d_i^2}.$$

$$= \sum_i \frac{C_{ji}^2}{d_i^2} \cdot d_i \cdot \sigma^2$$

$$= \sum_i \frac{C_{ji}^2}{d_i} \sigma^2 \qquad \ldots (1)$$

$$\text{Also} \quad V(P_j) = \text{Cov}\left(P_j, \sum_i C_{ji} \frac{Q_i}{d_i}\right)$$

$$= \sum_i \frac{C_{ji}}{d_i} \text{Cov}(P_j, Q_i) \ldots\ldots \quad (2)$$

Now (1) and (2) can be equal only if $\text{Cov}(P_j, Q_i) = C_{ji}\sigma^2$

Therefore, $\text{Cov}(P_j, Q_i) = C_{ji}\sigma^2$

Q.E.D.

We shall apply this theorem for the analysis of fractionally replicated designs of the series 3^n. Subsequently, the analysis of confounded symmetrical and asymmetrical factorial designs will be discussed.

ANALYSIS OF FRACTIONALLY REPLICATED DESIGNS OF THE SERIES 3^n

In fractionally replicated experiments of 2^n series, application of Yates' technique is straightforward, particularly because each interaction component obtained through the method represents the whole of its alias group of interactions. But in experiments of 3^n series this advantage is not available because the alias relations connect interaction components each of two degrees of freedom, while through Yates' method we get interaction components each of one degree of freedom. Thus when the interaction components in such designs are obtained by suppressing one or more letters through extended Yates' method, the main effects and interaction components involving the suppressed letters can not be directly obtained as in the case of 2^n series. It thus becomes necessary to establish a relationship between main effect or interaction components involving at least one of the suppressed letters on the one hand and main effects and interaction components of the non-suppressed letters that could be obtained through the extended Yates' method.

It will be seen that all the main effects and interaction components of the non-suppressed letters (Q_i, $i = 1, 2, \ldots, N-1$) give us a complete set of contrast while a main effect or an interaction contrast involving at least one suppressed letter is any other contrast P_j. Now we can find a linear relationship of Q_i's which is equal to P_j with the help of the theorem just proved. We have established such a relationship in an actual case as described below. Let us take the fractional design $\frac{1}{3}(3^4)$ involving the factors A, B, C and D and obtained through the identity group ABCD = I. In table* given below the treatment combinations along with a factitious set of data have been presented. For applying Yates' techniqe for the analysis of the data the factor A has been suppressed and in column 2 of the table, treatment combinations have been written as required by the method, showing the level a or a^2 in the bracket wherever necessary. Through this analysis we could get all main effects and interaction components involving the factors B, C and D as shown in column 4.

Now suppose we are interested to obtain A_L, where A_L denotes linear component of the main effect of A (similarly A_Q will denote the quadratic component of the main effect of A). For this purpose we have to connect A_L with all the main effects and interaction components of the factors B, C and D. We have established through the theorem that the coefficient of B_L in this relation is

* See page 27

the coefficient of the error mean square in the covariance of A_L and B_L, divided by the divisor of B_L. The covariance as obtained through the usual method(i.e. sum of products of the coefficients of the common observations in the two contrasts) comes out to be zero. Thus B_L has no contribution to A_L. From the aliases of A we find that only the components of the three factor interaction BCD will be related with A_L. So now we take the component $B_LC_LD_L$, the covariance of A_L and $B_LC_LD_L$ comes out to be $-3\sigma^2$, the divisor for $B_LC_LD_L$ (as obtained by sum of squares of the coefficients of the observations in $B_LC_LD_L$) is evidently 8 as only 8 observations are involved in $B_LC_LD_L$, each with coefficient +1 or -1. Thus the coefficient of $B_LC_LD_L$ in A_L is $-\frac{3}{8}$. In this way by obtaining the coefficients of all other components involving all the factors B, C and D we get

$$\begin{aligned} A_L = &-\frac{3}{8} B_LC_LD_L + \frac{9}{24} B_LC_LD_Q + \frac{9}{24} B_LC_QD_L \\ &+ \frac{9}{24} B_QC_LD_L + \frac{9}{72} B_LC_QD_Q + \frac{9}{72} B_QC_LD_Q \\ &+ \frac{9}{72} B_QC_QD_L - \frac{27}{216} B_QC_QD_Q \quad \ldots (1) \end{aligned}$$

We have given below some more salient relations.

$$A_Q = \frac{9}{8} B_L C_L D_L + \frac{9}{24} B_L C_L D_Q + \frac{9}{24} B_L C_Q D_L + \frac{9}{24} B_Q C_L D_L - \frac{27}{72} B_L C_Q D_Q - \frac{27}{72} B_Q C_L D_Q$$

$$- \frac{27}{72} B_Q C_Q D_L - \frac{27}{216} B_Q C_Q D_Q$$

$$A_L B_L = - \frac{3}{12} C_L D_L + \frac{9}{36} C_L D_Q + \frac{9}{36} C_Q D_L + \frac{9}{108} C_Q D_Q + \frac{3}{8} B_L C_L D_L + \frac{3}{24} (B_L C_L D_Q + B_L C_Q D_L$$

$$- B_Q C_L D_L) + \frac{9}{72} (- B_L C_Q D_Q + B_Q C_L D_Q + B_Q C_Q D_L) + \frac{9}{216} B_Q C_Q D_Q$$

$$A_Q B_L = \frac{9}{12} (C_L D_L + C_L D_Q/3 + C_Q D_L/3) - \frac{27}{108} C_Q D_Q + \frac{3}{8} B_L C_L D_L - \frac{9}{24} (B_L C_L D_Q + B_L C_Q D_L - B_Q C_L D_L)$$

$$- \frac{9}{72} (B_L C_Q D_Q - B_Q C_L D_Q - B_Q C_Q D_L) - \frac{27}{216} B_Q C_Q D_Q$$

$$A_L B_Q = \frac{9}{12} (C_L D_L + C_L D_Q/3 + C_Q D_L/3) - \frac{27}{108} C_Q D_Q - \frac{3}{8} B_L C_L D_L + \frac{9}{24} (B_L C_L D_Q + B_L C_Q D_L - B_Q C_L D_L)$$

$$+ \frac{9}{72} (B_L C_Q D_Q - B_Q C_L D_Q - B_Q C_Q D_L) + \frac{27}{216} B_Q C_Q D_Q$$

$$A_QB_Q = \frac{9}{12} C_LD_L - \frac{27}{36} (C_LD_Q + C_QD_L) - \frac{27}{108} C_QD_Q + \frac{9}{8} B_LC_LD_L$$

$$+ \frac{9}{24} (B_LC_LD_Q + B_LC_QD_L - B_QC_LD_L) - \frac{27}{72} (B_LC_Q D_Q - B_QC_LD_Q$$

$$- B_QC_QD_L) + \frac{27}{216} B_QC_QD_Q$$

$$A_LB_LC_L = - \frac{3}{18} D_L + \frac{9}{54} D_Q + \frac{3}{12} B_LD_L + \frac{3}{36} (B_LD_Q - B_QD_L)$$

$$+ \frac{9}{108} B_QD_Q + \frac{3}{12} C_LD_L + \frac{3}{36} (C_LD_Q - C_QD_L) + \frac{9}{108} C_QD_Q$$

$$+ \frac{1}{8} B_LC_LD_L - \frac{3}{24} (B_LC_LD_Q - B_LC_QD_L - B_QC_LD_L)$$

$$+ \frac{3}{72} (B_LC_QD_Q + B_QC_LD_Q - B_QC_QD_L) + \frac{9}{216} B_QC_QD_Q$$

The S.S. due to the contrasts, A_L etc. can be obtained by squaring the contrast and dividing it by its usual divisor. The divisor can also be obtained from the variance of A_L and is equal to $\Sigma \frac{C_{ji}^2}{d_i^2} \cdot d_i$ where C_{ji} and d_i's are as explained in the theorem.

Table 1

Analysis of $\frac{1}{3}(3^4)$ experiment.   I = ABCD

S.No.	Treatment Combination	Observation Total	Final Contrast
1	1	1	101
2	$(a^2)b$	2	6
3	$(a)b^2$	4	2
4	$(a^2)c$	6	-1
5	$(a)bc$	3	-19
6	b^2c	5	-19
7	$(a)c^2$	8	11
8	bc^2	2	15
9	$(a^2)b^2c^2$	1	-25
10	$(a^2)d$	4	2
11	$(a)bd$	5	2
12	b^2d	7	20
13	$(a)cd$	2	-5
14	bcd	3	-7
15	$(a^2)b^2cd$	4	13
16	c^2d	6	-19
17	$(a^2)bc^2d$	5	-7
18	$(a)b^2c^2d$	3	13
19	$(a)d^2$	2	-16
20	bd^2	9	12
21	$(a^2)b^2d^2$	1	2
22	cd^2	2	-7
23	$(a^2)bcd^2$	1	-1
24	$(a)b^2cd^2$	4	-25
25	$(a^2)c^2d^2$	6	-25
26	$(a)bc^2d^2$	3	3
27	$b^2c^2d^2$	2	-25

Note : - The intermediate steps for getting the final contrasts are not shown.

$A_0 = 37$; $A_1 = 34$; $A_2 = 30$

Therefore, $A_L = 7$, $A_Q = -1$

Now substituting the value of the components of $B_LC_LD_L$, $B_LC_LD_Q$, $B_LC_QD_L$, $B_QC_LD_L$, $B_LC_QD_Q$, $B_QC_LD_Q$, $B_QC_QD_L$ and $B_QC_QD_Q$ as obtained in the table we get

$A_L = 7$, similarly we get $A_Q = -1$

Sum of squares due to $A_L = \frac{(7)^2}{188}$ and sum of squares due to $A_Q = \frac{(-1)^2}{54}$.

Adding these two sum of squares we get the sum of squares due to main effect A which is identical with the sum of squares obtained through the usual method. Similarly the sum of squares due to A_LB_L etc. can be obtained.

5.(a) CONFOUNDED DESIGNS OF THE SERIES 3^n

In symmetrical designs of the series 3^n components like ABC, ABC^2 etc. are confounded either completely or partially. It becomes sometimes necessary to obtain interaction components like $A_L B_L C_L$ in presence of such confounding. If at least one three factor interaction component involving the factors, say, A, B and C be confounded, evidently the components like $A_L B_L C_L$ have to be adjusted for block differences. Such an adjustment is possible by utilising the theorem. For this purpose, we define a total set of independent contrasts as below:

A_L, A_Q, B_L, B_Q etc. for the main effects, and $(AB)_L$, $(AB)_Q$, $(AB^2)_L$, $(AB^2)_Q$ etc. for the interactions.

$(AB)_L$ has been taken to be $(AB)_O - (AB)_2$ where $(AB)_O$ denotes, as usual, the sum of the treatment combinations involving the combinations OO, 12, 21 of the factors A and B. Similarly $(AB)_Q = (AB)_O - 2(AB)_1 + (AB)_2$

Now, let us take any other contrast $A_L B_L C_L$, it can be shown that the covariance of $A_L B_L C_L$ with any interaction component involving one or two letters like A_L, A_Q, $(AB)_L$, $(AB)_Q$ etc. is zero. As a matter of fact the covariances of $A_L B_L C_L$ with components involving all the three factors A, B and C like $(ABC)_L$, $(ABC)_Q$, $(AB^2C^2)_L$, $(AB^2C^2)_Q$ etc. are non-zero while its covariance with any other component is zero. Finding the covariance of $A_L B_L C_L$ with those components which are not confounded, we get the following relation

$$A_L B_L C_L = \frac{3}{18}(A^2B^2C)_L + \frac{9}{54}(A^2B^2C)_Q + \frac{3}{18}(A^2BC^2)_L$$

$$+ \frac{9}{54}(A^2BC^2)_Q + \frac{3}{18}(AB^2C^2)_L + \frac{9}{54}(AB^2C^2)_Q$$

assuming that in the design the interaction ABC is confounded and therefore the covariance of block adjusted $(ABC)_L$ and $A_L B_L C_L$ is zero. Also covariance of block adjusted $(ABC)_Q$ and $A_L B_L C_L$ is zero.

$$\text{Again, } A_L B_L C_L = \frac{3}{18}(ABC)_L - \frac{9}{54}(ABC)_Q$$

$$+ \frac{3}{18}(A^2B^2C)_L + \frac{9}{54}(A^2B^2C)_Q + \frac{3 \times 2}{18 \times 2}(A^2BC^2)_L$$

$$+ \frac{9 \times 2}{54 \times 2}(A^2BC^2)_Q + \frac{3 \times 2}{18 \times 2}(AB^2C^2)_L + \frac{9 \times 2}{54 \times 2}(AB^2C^2)_Q$$

when ABC and A^2B^2C are partially confounded in 2 replications.

The variance of $A_L B_L C_L$ as obtained from the first of the above two relations is equal to 6 and hence the divisor for the component obtained from the first design is 6. Again the variance of the component $A_L B_L C_L$ from the second relation above is 12 and therefore the divisor for finding its sum of squares from the second design is 12.

Though we have introduced the method through a particular case, the method is quite general and can be applied for adjusting the interaction components of any order in presence of either total or partial confounding. The results corresponding to a 3^3 design with complete confounding of, say, ABC are identical with those of the design $\frac{1}{3}(3^4)$ split into blocks confounding the interaction component, ABC. Thus, the relation of $A_L B_L C_L$ with the components$(ABC)_L$ etc. depends more on the total number of observations in the design then on the individual observations.

Suppose we have established a relationship between a component, say, $A_L B_L C_L$ with the components $(ABC)_L$ etc. through a design 3^n, then the corresponding relation for the design 3^{n+p} can be obtained by multiplying both numerator and denominator of the previous relation by 3^p. This shows that the relation remains unaffected by increasing the number of factors in the design but the divisor of the contrast changes and becomes multiplied by 3^p where p is the number of factors subsequently added. The above result is, however, subject to the fact that similar confounding of the interaction components involving the factors A, B and C exists in both basic design 3^n and augmented design 3^{n+p}. It may be further pointed out that instead of increasing the number of factors, the design is replicated r times, then also the coefficients remain unaffected but the divisors get multiplied by r.

It may be mentioned here that Kishen, K(1940, 1942 and 1950) expressed any single degree of freedom of treatment combinations in s^m non-confounded complete factorials in terms of its sets of main effects and interactions and worked out such relations in some particular cases. His results can be obtained in a straightforward way through the present technique.

We have illustrated these results by analysing the data of the $\frac{1}{3}(3^4)$ experiment split into 3 blocks of 9 plots each. For this purpose we have utilized 3 of the 9 blocks each of the size 9 of 3^4 confounded design adopted for an actual irrigation-cum-manuring experiment for wheat. The fraction taken corresponded to identity group I = ACD. The interaction component A^2B^2D is confounded with the 3 blocks. The data are presented below.

$\frac{1}{3}(3^4)$ experiment

I = ACD, confounded interaction is A^2B^2D

Block I		Block II		Block III	
Treat. comb.	Yield in kg/plot	Treat. comb.	Yield in kg/plot	Treat. comb.	Yield in kg/plot.
a^2bc	4	cd^2	4	a^2bd	4
bc^2d	3	ab^2d^2	4	a^2c	5
acd	2	a^2b^2c	3	ad^2	5
abd^2	4	b	4	abc^2	3
ab^2c^2	3	abcd	7	bcd^2	4
b^2cd^2	5	b^2c^2d	6	ab^2cd	5
1	3	a^2d	2	b^2	4
$a^2c^2d^2$	4	$a^2bc^2d^2$	4	$a^2b^2c^2d^2$	4
a^2b^2d	4	ac^2	4	c^2d	3

we get,

$(ABD)_L = 5$; $(A^2BD^2)_L = -5$; $(AB^2D^2)_L = -7$

$(ABD)_Q = -1$; $(A^2BD^2)_Q = -1$; $(AB^2D^2)_Q = 5$

We intended to obtain the component $A_LB_LD_L$ by linking it with the contrasts $(ABD)_L$, $(ABD)_Q$, $(A^2B^2D)_L$, $(A^2B^2D)_Q$, $(A^2BD^2)_L$, $(A^2BD^2)_Q$, $(AB^2D^2)_L$ and $(AB^2D^2)_Q$

The relation established earlier is

$$A_LB_LD_L = \frac{3}{18}(ABD)_L - \frac{9}{54}(ABD)_Q + \frac{3}{18}(A^2BD^2)_L$$

$$+ \frac{9}{54}(A^2BD^2)_Q + \frac{3}{18}(AB^2D^2)_L + \frac{9}{54}(AB^2D^2)_Q$$

obtaining the right hand components from the data and substituting their values in the relation we got an estimate of the contrast $A_L B_L D_L = -\frac{1}{3}$.

In addition to obtaining the contrast $A_L B_L D_L$ we can obtain the other 7 contrasts from the relations given below:

$$A_L B_L D_Q = \frac{9}{18}(ABD)_L + \frac{9}{54}(ABD)_Q + \frac{9}{18}(A^2BD^2)_L - \frac{9}{54}(A^2BD^2)_Q + \frac{9}{18}(AB^2D^2)_L - \frac{9}{54}(AB^2D^2)_Q$$

$$A_L B_Q D_L = \frac{9}{18}(ABD)_L + \frac{9}{54}(ABD)_Q - \frac{9}{18}(A^2BD^2)_L + \frac{9}{54}(A^2BD^2)_Q + \frac{9}{18}(AB^2D^2)_L - \frac{9}{54}(AB^2D^2)_Q$$

	$(ABD)_L$	$(ABD)_Q$	$(A^2BD^2)_L$	$(A^2BD^2)_Q$	$(AB^2D^2)_L$	$(AB^2D^2)_Q$
$A_Q B_L D_L =$	$\frac{9}{18}$	$\frac{9}{54}$	$\frac{9}{18}$	$-\frac{9}{54}$	$-\frac{9}{18}$	$\frac{9}{54}$
$A_L B_Q D_Q =$	$-\frac{9}{18}$	$\frac{27}{54}$	$\frac{9}{18}$	$\frac{27}{54}$	$-\frac{9}{18}$	$-\frac{27}{54}$
$A_Q B_L D_Q =$	$-\frac{9}{18}$	$\frac{27}{54}$	$-\frac{9}{18}$	$-\frac{27}{54}$	$\frac{9}{18}$	$\frac{27}{54}$
$A_Q B_Q D_L =$	$-\frac{9}{18}$	$\frac{27}{54}$	$\frac{9}{18}$	$\frac{27}{54}$	$\frac{9}{18}$	$\frac{27}{54}$
$A_Q B_Q D_Q =$	$-\frac{27}{18}$	$-\frac{27}{54}$	$\frac{27}{18}$	$-\frac{27}{54}$	$\frac{27}{18}$	$-\frac{27}{54}$

It can be easily seen that only 6 of these contrasts are independent and the remaining two can be obtained as linear functions of those six. This is otherwise evident also as only six of the 8 degrees of freedom belonging to the interaction components involving the factors A, B and D can be obtained as the other 2 d.f. are confounded.

5.(b) CONFOUNDED ASYMMETRICAL FACTORIAL DESIGNS

The same technique can be applied for the component analysis of confounded asymmetrical factorial designs. As in many designs two factor interaction components can not be saved, the application of present technique plays a more important role in such designs. For example, in the design $3^2 \times 2$ in 6 plots blocks involving the factors A, B and C in order balanced over two replications, the two factor interaction component AB is partly affected by block differences. The experimenter is often-times interested in the component $A_L B_L$ and the present technique is very helpful to obtain $A_L B_L$ from the four contrasts $(AB)_L$, $(AB)_Q$, $(AB^2)_L$ and $(AB^2)_Q$. The relation connecting $A_L B_L$ with the latter contrasts comes out to

$$A_L B_L = \frac{12}{24}(AB)_L - \frac{12}{72}(AB)_Q + \frac{12}{24}(AB^2)_L + \frac{12}{72}(AB^2)_Q$$

when none of the components AB and AB^2 are confounded. The coefficients of $(AB)_L$ etc. in this relation can also be obtained from the corresponding relation in 3^3 design with two replications, by multiplying the coefficients by $\frac{2}{3}$.

As in the present design AB is affected by the blocks it is necessary to adjust the coefficients $\frac{C_{ji}}{d_i}$

of the contrasts $(AB)_L$ and $(AB)_Q$. It is found that such adjustments are easily affected by multiplying both the numerator and denominator i.e. c_{ji} and d_i in the above relation by the relative information recovered in the design i.e. by $\frac{3}{4}$ in the present design. For getting the diviser for finding the S.S. of $A_L B_L$, we have to use the thus adjusted coefficients. The relation actually comes out as

$$A_L B_L = \frac{9}{18}(AB)_L - \frac{9}{54}(AB)_Q + \frac{12}{24}(AB^2)_L + \frac{12}{72}(AB^2)_Q ,$$

Similarly,

$$A_L B_Q = \frac{9}{18}(AB)_L + \frac{27}{54}(AB)_Q + \frac{12}{24}(AB^2)_L - \frac{36}{72}(AB^2)_Q ,$$

$$A_Q B_L = \frac{9}{18}(AB)_L + \frac{27}{54}(AB)_Q - \frac{12}{24}(AB^2)_L + \frac{36}{72}(AB^2)_Q , \text{ and}$$

$$A_Q B_Q = -\frac{27}{18}(AB)_L + \frac{27}{54}(AB)_Q + \frac{36}{24}(AB^2)_L + \frac{36}{72}(AB^2)_Q$$

The divisors for these contrasts come out as 14, 42, 42 and 126 respectively and their S.S. can be found as described earlier.

Similarly the components of three factor interaction in this case can be obtained from the following relations:

$$\begin{bmatrix} A_L B_L C \\ A_L B_Q C \\ A_Q B_L C \\ A_Q B_Q C \end{bmatrix} = \begin{bmatrix} \frac{3}{6} & -\frac{3}{18} & \frac{12}{24} & \frac{12}{72} \\ \frac{3}{6} & \frac{9}{18} & \frac{12}{24} & -\frac{36}{72} \\ -\frac{3}{6} & \frac{9}{18} & -\frac{12}{24} & \frac{36}{72} \\ -\frac{9}{6} & \frac{9}{18} & \frac{36}{24} & \frac{36}{72} \end{bmatrix} \begin{bmatrix} (AB)_L C \\ (AB)_Q C \\ (AB^2)_L C \\ (AB^2)_Q C \end{bmatrix}$$

where $(AB)_L C = \left[(AB)_0 (C_0 - C_1) - (AB)_2 (C_0 - C_1) \right]$

$$(AB)_Q C = \left[(AB)_0 (C_0 - C_1) - 2(AB)_1 (C_0 - C_1) + (AB)_2 (C_0 - C_1) \right]$$

The divisors for these contrasts come out as 10, 30, 30 and 90 respectively and their S.S. can be found. Though this method has been described with a particular case, it is quite general and can be applied to similarly analyse any balanced asymmetrical factorial design.

SUMMARY

When a large number of factors is involved in a factorial experiment, the analysis through Yates' method becomes complicated as a very large number of combinations have to be taken and operated upon. We have given a modified method whereby suitable groups of the treatment combinations are formed and then each group is analysed separately. The separate analysis of these groups are then combined to get the final results.

Further an investigation has been made to give specific direction for the suppression of factors in case of fractionally replicated designs involving factors each at two levels.

It appears that the method of analysis of fractionally replicated designs through Yates' technique is not yet perfected when factors are at more than two levels. This is because, though the aliases of geometrical components are available, such aliases are not available for linear, quadratic etc. components and their interactions. We have attempted to obtain a solution of this problem.

The extended Yates' method of analysis of factorial experiments gives directly the results of component analysis. When the design is constructed by adopting geometrical techniques in Galois fields, it becomes difficult to have its component analysis particularly in respect of those interactions which are confounded. Similar difficulties are encountered when the design is fractionally replicated. It appears that this type of difficulty can be overcome if a link between the contrasts in the two types of analysis can be established. One of the main purposes of the present investigation is to establish such a link.

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