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Resolvable Block Designs for Factorial Experiments with Full Main Effects Efficiency

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SUMMARY

The purpose of this article is to propose unified methods of construction of resolvable incomplete block designs for factorial experiments. These designs have orthogonal factorial structure, have balance, estimate all main effects with full efficiency and have control over the interaction efficiencies. These designs have applications in crop-sequence experiments. A catalogue of designs is prepared for number of levels of any factor at most 12.

Keywords: Orthogonal factorial structure, Balance, Efficiency, Structure *K*, Replacement technique.

1. INTRODUCTION

An experiment on 'rice-wheat' sequence was to be conducted in the Division of Agronomy, IARI, New Delhi. During the kharif crop (rice), 5 herbicidal treatments were to be applied. On the following rabi crop (wheat), 4 herbicidal treatments were given. The purpose of the experiment was (i) to compare direct effects of kharif and rabi treatments, (ii) residual effects of kharif treatments, and (iii) the interaction between residual effects of kharif and direct effects of rabi treatments.

Another experiment was to be planned in the Division of Agronomy, IARI, New Delhi for evaluation of sulphur sources at varying rates in aerobic rice-wheat cropping system for improved productivity and soil health. During kharif-rice, treatments comprising of combinations of two sources of sulphur, viz., Gypsum and Phosphogypsum and three levels of sulphur as 0, 30 and 60 kgs. / hectare are to be applied. During rabi-wheat three rates of sulphur to be applied through respective sources as in Kharif-rice are 0, 15 and 30 kgs./hectare. The study needs to be conducted for two years. The interest of the experimenter is to study (i) the direct effect of sulphur sources and levels of sulphur on kharif-rice; (ii) the residual effect of sulphur treatments applied to kharif-rice on succeeding rabi-wheat; (iii) the direct effect of sulphur applied to rabi-wheat; (iv) the interaction between residual effects of sulphur treatments applied during Kharif-rice and the direct effect of sulphur treatments applied during rabi-wheat; (v) the residual effect of sulphur applied to rabi-wheat on succeeding kharif-rice in the second year; (vi) the cumulative effect of sulphur applied to kharif-rice and rabi-wheat in first year on the kharif-rice and rabi-wheat of second year, and so on…

The experimenter would like to run such an experiment in a design in three replications. This ensures that even if one replication is lost during experimentation, the experimenter has two replications to run through the analysis. Moreover, the experimenter would be more interested in the main effects and two

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factor interactions may also be of interest to him. It is, therefore, desirable that the design permits estimation of main effects without any loss of information (or full efficiency) and the interaction efficiencies are also controlled and reasonably high.

It is indeed possible that there are more than two crops in a crop sequence. Further, the treatments in the individual cropping seasons could also have a factorial structure.

These experiments are generally run as a split plot design. But this is a typical example of an experiment to be run as a factorial experiment in an incomplete block design [see also, Parsad *et al*. 2007, Gupta and Parsad 2009]. To keep the discussion general, consider a factorial experiment involving m factors F_1, F_2, ... , F_m, the j^{th} factor F_j being experimented with q_j (≥ 2) levels, $j = 1, 2, ... m$. A particular selection of levels is an m-tuple, $s = (s_1, ..., s_j, ..., s_m)$, $0 \leq s_j \leq q_j - 1$ and will be termed the s^{th} treatment combination. The totality of all such m-tuples is $v = \prod_{j=1}^{m} q_j$. We shall also assume that the v treatments are run in an incomplete block design with b blocks and each block contains k ($<v$) distinct treatments. A factorial experiment run in an incomplete block design is said to have the *Orthogonal Factorial Structure* (OFS) if the adjusted treatment sum of squares can be split up orthogonally into various components corresponding to factorial effects like main effects and interactions. In a block design with OFS *inter-effect orthogonality* holds if the best linear unbiased estimates (BLUE) of the estimable treatment contrasts belonging to different factorial effects are mutually orthogonal or *uncorrelated*. A factorial effect is said to be *balanced* if all the normalized contrasts belonging to that effect are estimable and are estimated with the same variance, i.e., the loss of information on all the normalized contrasts of any factorial effect is the same. A factorial design is *balanced* if it is balanced for every factorial effect. For an equireplicated design, full information is retained on an effect if the effect is *balanced* and there is no loss of information on any contrast belonging to that effect (relative to the comparable complete block design). For details, readers may refer to Kurkjian and Zelen (1963) and Mukerjee (1982, 1986).

The problem of obtaining incomplete block designs having OFS has been studied extensively in the literature. *Extended Group Divisible* (EGD) designs have OFS with balance. Generalized cyclic designs also have OFS with balance. An alternative approach to generate designs having OFS with balance is to employ Kronecker type products of unstructured block designs, popularly known as varietal designs. For details the reader may see, David and Wolock (1965), Dean and John (1975), John (1966, 1973, 1987), Gupta (1983, 1985, 1987), and Gupta and Mukerjee (1980a, 1980b, 1981, 1984, 1989a, 1989b).

The designs available in the literature for such experimental situations have high main effects efficiencies but the interaction efficiencies also need to be controlled. However, there are some methods of construction available in the literature that generate designs in which it is possible to estimate main effects with full efficiencies. But despite a burst of activity in generating these designs, there is still a need to generate designs with full efficiency on main effects and controlled efficiency on interactions. The purpose of this paper is to describe a unified but very simple method of constructing such designs. However, the experimenter is always keen to have a design in which all the treatments appear at one place in a replication and, therefore, there is a tendency to adopt a complete block design, even at the cost of getting high error sum of squares. This is important because the experimenter needs to demonstrate the treatments effects at one place. Therefore, from experimenters view point, it is essential to get a resolvable block design with OFS, full efficiency on all the main effects and controlled efficiency on interactions. These designs have Property *K*. A catalogue of resolvable designs with factors up to a maximum of 12 levels is prepared. Designs with factors at more than 12 levels can also be obtained easily from the unified approach. The designs along with the efficiencies of various factorial effects are available at www.iasri.res.in/design/factorial/factorial 20files/catalogues.htm. We begin with some preliminaries in Section 2.

2. SOME PRELIMINARIES

We begin by giving some definitions that would be used throughout. We shall denote matrices by bold capital letters and vectors by bold small letters. Unless otherwise stated, a vector would be a column vector.

We shall denote by $\mathbf{0}_t$ a t-component vector with all elements zero and by $\mathbf{1}_t$ a t-component vector with all elements one. $\mathbf{J}_{pq} = \mathbf{1}_p\mathbf{1}'_q$ would denote a $p \times q$ matrix with all elements one. Further, $\mathbf{I}_t$ would denote an identity matrix of order t. We shall omit the order of matrices and vectors where the orders are obvious.

Let $\mathbf{A} = (a_{ij})$ and $\mathbf{B} = (b_{ij})$ be two matrices of order $m \times n$ and $p \times q$ respectively.

Definition 2.1. The *Kronecker product* (or *tensor product*) of **A** and **B,** denoted by an $mp \times nq$ matrix **F** = $\mathbf{A} \otimes \mathbf{B}$, is defined by $\mathbf{F} = (a_{ij}\mathbf{B})_{1 \le i \le m;\, 1 \le j \le n}$. In general, if $\mathbf{A}_i$ ($i = 1, 2, \ldots, m$) are $m_i \times n_i$ matrices, their Kronecker product is written as $\prod_{i=1}^{m} \otimes \mathbf{A}_i$ and has the order $\prod_{i=1}^{m} m_i \times \prod_{i=1}^{m} n_i$.

We now define *Symbolic Direct Product* (SDP). It may be noticed that the SDP is an operation on symbolic quantities and not on numerical quantities. Let $\mathbf{a}'_i = [a_i(1), a_i(2), ..., a_i(m_i)]$ be an m_i component vector, $i = 1, 2, \ldots, m$. Then the SDP of $\mathbf{a}_p$ and $\mathbf{a}_q$ is defined to be

$$\mathbf{a}_p \times \mathbf{a}_q = \begin{bmatrix} a_p(1) \\ a_p(2) \\ \vdots \\ a_p(m_p) \end{bmatrix} \times \begin{bmatrix} a_q(1) \\ a_q(2) \\ \vdots \\ a_q(m_q) \end{bmatrix} = \begin{bmatrix} a_{pq}(1,1) \\ a_{pq}(1,2) \\ \vdots \\ a_{pq}(1,m_q) \\ a_{pq}(2,m_q) \\ \vdots \\ a_{pq}(m_p,m_q) \end{bmatrix}$$

The extension of the SDP to more than two vectors is straight forward.

Let $\mathbf{A} = (a_{ij})$ and $\mathbf{B} = (b_{ij})$ be two rectangular matrices of the same order $m \times n$.

Definition 2.2. The special product of **A** and **B,** denoted by an $m \times n$ matrix $\mathbf{F} = \mathbf{A} \bullet \mathbf{B}$ is defined as symbolic product of the symbols in the columns of **A** and **B**.

Let the n-component i^{th} column vectors of **A** and **B** be $\mathbf{a}'_i = [a_i(1), a_i(2), \ldots a_i(n)]$ and $\mathbf{b}'_i = [b_i(1), b_i(2), ..., b_i(n)]$, $i = 1, 2, \ldots, m$. Then the n-component i^{th} column vector of **F** is defined to be

$$\mathbf{a}_i \bullet \mathbf{b}_i = \begin{bmatrix} a_i(1) \\ a_i(2) \\ \vdots \\ a_i(n) \end{bmatrix} \bullet \begin{bmatrix} b_i(1) \\ b_i(2) \\ \vdots \\ b_i(n) \end{bmatrix} = \begin{bmatrix} a_i(1)b_i(1) \\ a_i(2)b_i(2) \\ \vdots \\ a_i(n)b_i(n) \end{bmatrix}$$

Suppose that the totality of v treatment combinations written as m-tuples is arranged in a lexicographic order. Let $S'_j = (0, 1, 2, ..., q_j - 1)$ be a vector whose elements represent the levels of the factor F_j. The SDP $S_1 \times ... \times S_j \times ... \times S_m$ is used to order lexicographically the v treatments. The v treatments are arranged in a block design with b blocks of size k each. The block design is represented by a $v \times b$ incidence matrix $\mathbf{N} = (n_{ij})$.

Let $\mathbf{P}_j$ be $(q_j - 1) \times q_j$ matrix such that $(q_j^{-1/2}\mathbf{1}_j, \mathbf{P}'_j)$ is orthogonal. For any $\mathbf{x} = (x_1, ..., x_m)$, $x_j = 0, 1 \,\forall\, j$, $\mathbf{x} \ne 0$, let $\mathbf{P}^{\mathbf{x}} = \mathbf{P}_1^{x_1} \times \ldots \times \mathbf{P}_m^{x_m}$, $\varepsilon^{\mathbf{x}} = \varepsilon_1^{x_1} \times \ldots \times \varepsilon_m^{x_m}$, where

$$\mathbf{P}_j^{x_j} = \begin{cases} q_j^{-1/2}\mathbf{1}'_j & \text{if } x_j = 0 \\ \mathbf{P}_j & \text{if } x_j = 1 \end{cases}, \quad \boldsymbol{\varepsilon}_j^{x_j} = \begin{cases} \mathbf{1}_j & \text{if } x_j = 0 \\ \mathbf{I}_j & \text{if } x_j = 1 \end{cases}$$

Definition 2.3. A *proper matrix* is a square matrix with all row sums and all column sums equal.

Definition 2.4. (Mukerjee 1986) A $v \times v$ matrix **A** is said to have *structure K* if it be expressible as a linear combination of Kronecker products of proper matrices of orders $q_1, ..., q_m$ (taken in order), i.e., if $\mathbf{A} = \sum_{g=1}^{w} \xi_g (\mathbf{V}_{g1} \times ... \times \mathbf{V}_{gm})$, where w is a positive integer, $\xi_1, ..., \xi_w$, are some real numbers and for each g, $\mathbf{V}_{gj}$ is some proper matrix of order q_j, $1 \le j \le m$.

Structure K will always be with respect to a particular factorization of $v = \prod_{j=1}^{m} q_j$, with the factors occurring in a particular order.

For an equi-replicate factorial experiment in a block design with common replication number r, constant block size k and incidence matrix **N**, the following theorems were proved by Mukherjee (1979, 1980a, 1980b):

Theorem 2.1. A sufficient condition for inter-effect-orthogonality to hold is that the matrix $\mathbf{NN}'$ has structure K. In the connected case this is also a necessary condition for inter-effect-orthogonality.

Theorem 2.2. Given that $\mathbf{NN'}$ has structure K, any main effect $F_j (1 \leq j \leq m)$ is balanced if $\boldsymbol{\varepsilon}^{\mathbf{x}'}\mathbf{NN'}\boldsymbol{\varepsilon}^{\mathbf{x}} = u_1\mathbf{I}_j + u_2\mathbf{1}_j\mathbf{1}'_j$ where u_1, u_2 are real numbers and $\mathbf{x} = (x_1, ..., x_m)$, $x_j = 1$, $x_{j'} = 0 \,\forall\, j' \neq j$. In this case the loss of information on F_j is given by $L(F_j) = (rkv)^{-1}q_j u_1$.

Theorem 2.3. Full information is retained on any main effect if and only if in each block the levels of the corresponding factor occur equal number of times.

Theorem 2.4. Given that $\mathbf{NN'}$ has structure K and the design is connected, the average loss of information on a complete set of orthonormal contrasts belonging to any factorial effect $F_1^{x_1} F_2^{x_2} \ldots F_m^{x_m}$ ($x_j = 0, 1 \,\forall\, j$, $\mathbf{x} = (x_1, ..., x_m) \neq \mathbf{0}$) is given by

$$1 - r^{-1}\{\prod (q_j - 1)^{x_j}\}[\text{Trace}\{\mathbf{P^x C P^{x'}}\}^{-1}]^{-1}$$

where $\mathbf{C}$ is the usual $\mathbf{C}$-matrix of the design.

3. METHODS OF CONSTRUCTION

A large number of methods are available in the literature for the construction of factorial designs. These methods essentially make use of cyclic or generalized cyclic designs or use Kronecker or Kronecker-type products. These designs have OFS and, if appropriately used, are capable of ensuring high efficiencies with respect to the interactions of interest. For an excellent review on this topic, a reference may be made to Puri and Nigam (1976, 1978), Nigam *et al*. (1988), Gupta and Mukerjee (1989a) and Mukerjee and Wu (2006). Parsad *et al*. (2007) generated a large number of extended group divisible designs, which have orthogonal factorial structure with balance. They also generated a catalogue of designs giving efficiency of the factorial effects. But all these designs do not ensure that the main effects would be estimated with full precision, although their efficiencies are too high. Parsad *et al*. (2007) gave a series of EGD designs with three factors in which it is possible to generate designs that estimate the main effects with full efficiency. In most of these designs, however, the levels of first factor are two.

Mukerjee (1981, 1982) also gave some methods of constructing factorial designs having OFS and main effects balance. The designs retain full information on at least one main effect. Some designs are also balanced with full efficiency on main effects [see also, Mukerjee 1984, 1986].

We propose a unified but a very simple method of generating block designs with OFS and full efficiency on main effects. The designs have structure K. The designs are resolvable and can be obtained for any given replication number.

3.1 Designs with all Main Effects Balanced

For two factor factorial experiments with $v = s_1 \times s_2$, where s_1 and s_2 are the number of levels of the two factors, respectively and v is the total number of treatments, some methods of construction of factorial designs with all main effects balanced and estimated without loss of information are available in the literature when

(I) s_2 is an integral multiple of s_1; block size is a multiple of both s_1 and s_2 and $s_2/s_1 = t$ (≥ 1), an integer

(II) s_1 and s_2 have a common factor, like $s_1 = g_1 f$ and $s_2 = g_2 f$; block size is a multiple of both s_1 and s_2 and $s_2 > s_1$.

It may happen that both s_1 and s_2 are composite numbers with $s_1 = f_1 \times f_2 \times \ldots \times f_p$ and $s_2 = h_1 \times h_2 \times \ldots \times h_q$. Using replacement and collapsing of levels, one can generate designs for $v = f_1 \times f_2 \times \ldots \times f_p \times h_1 \times h_2 \times \ldots \times h_q$ factorial experiment. The resulting design would have all the properties of the original design.

Most of the methods available fall as a particular case of the proposed unified approach of constructing designs for factorial experiments with structure K. But the designs generated by the proposed method are resolvable and can be obtained for any given replication number.

3.1.1 Unified Method of Construction

We now describe a unified method of construction of resolvable factorial design with OFS and full efficiency on main effects. The designs have Property K. Suppose $v = s_1 \times s_2 = f_1 \times f_2 \times \ldots \times f_p \times h_1 \times h_2 \times \ldots \times h_q$. For an $s_1 \times s_2$ design with $s_2 > s_1$, $s_1 = g_1 f$ and $s_2 = g_2 f$, to retain full information on both the main effects in an equi-replicated block design, the block size must be a multiple of least common multiple of s_1 and s_2. Here $g_1 \le g_2$ (≥1) are integers and f, a positive integer, is the highest common factor of s_1 and s_2. The minimum block size for this situation would $k = g_1g_2f$. The parameters of the design will be $v = s_1 \times s_2$, $b = rf$, replication = r, $k = g_1g_2f$.

Let the levels of the two factors be denoted as 0, 1, 2, ..., $s_1 - 2$, $s_1 - 1$ and 0, 1, 2, ..., $s_2 - 2$, $s_2 - 1$, respectively.

Step 1 : Write the s_2 levels of the second factor as a $k = g_1g_2f$-component vector $\mathbf{d} = (0, 1, 2, ..., s_2 - 1, 0, 1, 2, ..., s_2 - 1, ..., 0, 1, 2, ..., s_2 - 1)'$, where the symbols 0, 1, 2, ..., $s_2 - 1$ are repeated g_1 times each as a set. Obtain a $k \times b$ matrix $\mathbf{D}_2 = \mathbf{d} \times \mathbf{1}'_b$.

Step 2 : Let $\theta_1 = (0, 0, ..., 0, 1, 1, ..., 1, ..., s_1 - 1, s_1 - 1, ..., s_1 - 1)'$ be a k-component vector obtained by repeating g_2 times each of the symbols $(0, 1, 2, ..., s_1 - 1)$. Using θ_1 generate r distinct k-component vectors $\theta_2, \theta_3, . . . , \theta_r$, by cyclically permuting the elements in θ_1.

Step 3 : Using θ_u, generate a matrix $\mathbf{R}_u = \left[\theta_u^1\ \theta_u^2 \ldots \theta_u^f\right]$ of order $k \times f$, $u = 1, 2, ..., r$, where $\theta_u^{\alpha} = \{\theta_u + (\alpha - 1)\,\mathbf{1}\} \bmod (s_1)$, $\alpha = 1, 2, ..., f$.

Step 4 : Let $\mathbf{D}_1 = [\mathbf{R}_1\ \mathbf{R}_2 \ldots \mathbf{R}_r]$ be an $k \times rf$ matrix. The required design is $\mathbf{F} = \mathbf{D}_1 \bullet \mathbf{D}_2$. The order of $\mathbf{F}$ is $k \times b$. The columns of $\mathbf{F}$ are the blocks. The design is resolvable.

Example 3.1 Suppose an experimenter is interested in running a 4 × 6 factorial experiment in a resolvable incomplete block design with $r = 3$ replications so that all the main effects are estimated from the design with full efficiency. Here $s_1 = 4$, $s_2 = 6$, $f = 2$, $g_1 = 2$, $g_2 = 3$. The number of blocks is $b = 6$.

The $k = g_1g_2f = 12$-component vector $\mathbf{d} = (0, 1, 2, 3, 4, 5, 0, 1, 2, 3, 4, 5)'$. The 12-component vector θ_1 is, $\theta_1 = (0, 0, 0, 1, 1, 1, 2, 2, 2, 3, 3, 3)'$. Since $r = 3$, from θ_1 we obtain, $\theta_2 = (0, 0, 1, 1, 1, 2, 2, 2, 3, 3, 3, 0)'$; $\theta_3 = (0, 1, 1, 1, 2, 2, 2, 3, 3, 4, 0, 0)'$.

We now obtain the following:

	0	0	0	0	0	0
	1	1	1	1	1	1
	2	2	2	2	2	2
	3	3	3	3	3	3
	4	4	4	4	4	4
	5	5	5	5	5	5
	0	0	0	0	0	0
$\mathbf{D_2} = \mathbf{d} \times \mathbf{1}'_b =$	1	1	1	1	1	1
	2	2	2	2	2	2
(12 × 6)	3	3	3	3	3	3
	4	4	4	4	4	4
	5	5	5	5	5	5

$\mathbf{R}_1 =$		$\mathbf{R}_2 =$		$\mathbf{R}_3 =$	
0	1	0	1	0	1
0	1	0	1	1	2
0	1	1	2	1	2
1	2	1	2	1	2
1	2	1	2	2	3
1	2	2	3	2	3
2	3	2	3	2	3
2	3	2	3	3	0
2	3	3	0	3	0
3	0	3	0	3	0
3	0	3	0	0	1
3	0	0	1	0	1

The 12 × 6 matrix $\mathbf{D}_1 = [\mathbf{R}_1\ \mathbf{R}_2 \ldots \mathbf{R}_r]$ is

$\mathbf{D}_1$						
	0	1	0	1	0	1
	0	1	0	1	1	2
	0	1	1	2	1	2
	1	2	1	2	1	2
	1	2	1	2	2	3
	1	2	2	3	2	3
	2	3	2	3	2	3
	2	3	2	3	3	0
	2	3	3	0	3	0
	3	0	3	0	3	0
	3	0	3	0	0	1
	3	0	0	1	0	1

The required design is $\mathbf{F} = \mathbf{D}_1 \bullet \mathbf{D}_2$. The order of $\mathbf{F}$ is $(k \times b)$ 12×6. The resolvable design, written with rows as blocks, and parameters $v = 6 \times 4$, $b = 6$, $r = 3$, $k = 12$ is given below

Replication I		Replication II		Replication III	
Block-1	Block-2	Block-3	Block-4	Block-5	Block-6
00	10	00	10	00	10
01	11	01	11	11	21
02	12	12	22	12	22
13	23	13	23	13	23
14	24	14	24	24	34
15	25	25	35	25	35
20	30	20	30	20	30
21	31	21	31	31	01
22	32	32	02	32	02
33	03	33	03	33	03
34	04	34	04	04	14
35	05	05	15	05	15

The 24×24 concurrence matrix $\mathbf{NN'}$ for this design can be written as

3	2	1	0	1	2	0	1	2	3	2	1	3	2	1	0	1	2	0	1	2	3	2	1
2	3	2	1	0	1	1	0	1	2	3	2	2	3	2	1	0	1	1	0	1	2	3	2
1	2	3	2	1	0	2	1	0	1	2	3	1	2	3	2	1	0	2	1	0	1	2	3
0	1	2	3	2	1	3	2	1	0	1	2	0	1	2	3	2	1	3	2	1	0	1	2
1	0	1	2	3	2	2	3	2	1	0	1	1	0	1	2	3	2	2	3	2	1	0	1
2	1	0	1	2	3	1	2	3	2	1	0	2	1	0	1	2	3	1	2	3	2	1	0
0	1	2	3	2	1	3	2	1	0	1	2	0	1	2	3	2	1	3	2	1	0	1	2
1	0	1	2	3	2	2	3	2	1	0	1	1	0	1	2	3	2	2	3	2	1	0	1
2	1	0	1	2	3	1	2	3	2	1	0	2	1	0	1	2	3	1	2	3	2	1	0
3	2	1	0	1	2	0	1	2	3	2	1	3	2	1	0	1	2	0	1	2	3	2	1
2	3	2	1	0	1	1	0	1	2	3	2	2	3	2	1	0	1	1	0	1	2	3	2
1	2	3	2	1	0	2	1	0	1	2	3	1	2	3	2	1	0	2	1	0	1	2	3
3	2	1	0	1	2	0	1	2	3	2	1	3	2	1	0	1	2	0	1	2	3	2	1
2	3	2	1	0	1	1	0	1	2	3	2	2	3	2	1	0	1	1	0	1	2	3	2
1	2	3	2	1	0	2	1	0	1	2	3	1	2	3	2	1	0	2	1	0	1	2	3
0	1	2	3	2	1	3	2	1	0	1	2	0	1	2	3	2	1	3	2	1	0	1	2
1	0	1	2	3	2	2	3	2	1	0	1	1	0	1	2	3	2	2	3	2	1	0	1
2	1	0	1	2	3	1	2	3	2	1	0	2	1	0	1	2	3	1	2	3	2	1	0
0	1	2	3	2	1	3	2	1	0	1	2	0	1	2	3	2	1	3	2	1	0	1	2
1	0	1	2	3	2	2	3	2	1	0	1	1	0	1	2	3	2	2	3	2	1	0	1
2	1	0	1	2	3	1	2	3	2	1	0	2	1	0	1	2	3	1	2	3	2	1	0
3	2	1	0	1	2	0	1	2	3	2	1	3	2	1	0	1	2	0	1	2	3	2	1
2	3	2	1	0	1	1	0	1	2	3	2	2	3	2	1	0	1	1	0	1	2	3	2
1	2	3	2	1	0	2	1	0	1	2	3	1	2	3	2	1	0	2	1	0	1	2	3

The matrix $\mathbf{NN'}$ can easily be written as a linear combination of Kronecker product of proper matrices as per Definition 2.4. Therefore this matrix has structure K. Again the matrix $\boldsymbol{\varepsilon}^{\mathbf{x}'}\mathbf{NN'}\boldsymbol{\varepsilon}^{\mathbf{x}}$ for both factors has the structures $0\mathbf{I}_4 + 54\mathbf{1}_4\mathbf{1}_4'$ and $0\mathbf{I}_6 + 24\mathbf{1}_6\mathbf{1}_6'$, respectively and, therefore, all main effects of the design are balanced by Theorem 2.2.

Remark 5.1 In this unified method if $g_1 = 1$, then one gets designs for the experimental setting I given in Section 3.1. We give below another unified method of construction of designs for this setting when $g_1 = 1$. However, the design obtained may have different block contents.

We construct a resolvable design for $v = s_1 \times s_2 = f \times g_2 f$ in $b = rf$ blocks of size $k = g_2 f = s_2$ and replication r. The number of blocks within a replication is f.

Let the levels of the two factors be denoted as 0, 1, 2, ..., $s_1 - 2$, $s_1 - 1$ and 0, 1, 2, ..., $s_2 - 2$, $s_2 - 1$, respectively.

Step 1: Write the s_2 levels of the second factor as a vector $\mathbf{d}_2' = (0\ 1\ 2\ \ldots\ s_2 - 2\ s_2 - 1)$. Obtain an $s_2 \times b$ matrix $\mathbf{D}_2 = \mathbf{d}_2\mathbf{1}_b'$.

Step 2: Write the s_1 levels of the first factor as a vector $\mathbf{d}_1^{*\prime} = (0\ 1\ 2\ \ldots\ s_1 - 2\ s_1 - 1)$. Let $\mathbf{P}^g$ be a permutation matrix of order $s \times s$. The matrix $\mathbf{P}^g$ is a (0, 1) matrix and satisfies $\mathbf{P}^g\mathbf{1} = \mathbf{1}$ and $\mathbf{1}'\mathbf{P}^g = \mathbf{1}'$. One can get $\Lambda = s_1!$ such matrices. Obtain an s-component vector $\mathbf{d}_1^g = \mathbf{P}^g\mathbf{d}_1^*$, g = 1, 2, ..., Λ.

Step 3: Define a $g_2 s_1 \times r$ matrix $\mathbf{D}_1^* = \left[\mathbf{d}^1\ \mathbf{d}^2 \ldots \mathbf{d}^i \ldots \mathbf{d}^r\right]$, $\mathbf{d}'^i = \left[\mathbf{d}_1^{\prime i_1} \ldots d_1^{\prime i_u} \ldots d_1^{\prime i_t}\right]$, $i_u \in \{1, 2, \ldots, \Lambda\}$ and $u = 1, 2, \ldots, g_2$. Further. $\mathbf{d}^i - \mathbf{d}^{i'} \neq \mathbf{0}\ \forall\ i \neq i' = 1, 2, \ldots, r$.

Step 4: Obtain an $s_1 \times s_1$ matrix $\mathbf{U}_1^{i_j}$ from $\mathbf{d}_1^{i_j}$ by taking cyclic permutations of the columns of $\mathbf{d}_1^{i_j}$, $j = 1, 2, \ldots, g_2$. Let $\mathbf{U}^i = \left[\mathbf{U}_1^{\prime i_1}\ \mathbf{U}_1^{\prime i_2} \ldots \mathbf{U}_1^{\prime i_{g2}}\right]'$ be a $g_2 s_1 \times s_1$ matrix.

Then define a $g_2 s_1 \times b$ matrix $\mathbf{D}_1 = \left[\mathbf{U}^1\ \mathbf{U}^2 \ldots \mathbf{U}^i \ldots \mathbf{U}^r\right]$.

Step 5: The required design is $\mathbf{F} = \mathbf{D}_1 \bullet \mathbf{D}_2$. The order of $\mathbf{F}$ is $s_2 \times b$. The columns of $\mathbf{F}$ are the blocks. This design is resolvable in r replications with s_1 blocks per replication.

Step 6: Use replacement of levels technique to generate designs for $v = f_1 \times f_2 \times ... \times f_p \times h_1 \times h_2 \times ... \times h_p$ factorial experiment. For instance, replace the s_1 symbols by $f_1 \times f_2 \times ... \times f_p$ level combinations written in a lexicographic order and the s_2 symbols by $h_1 \times h_2 \times ... \times h_q$ level combinations written in a lexicographic order to get the required design.

This replacement scheme enables us to estimate the interaction effects among the factors which are replaced.

Example 3.2. A factorial experiment for $3^2 \times 2$ factorial experiment is to be run in incomplete blocks with parameters $b = 9$, $k = 6$ and $r = 3$. For constructing this factorial design, we first construct a design for $v = 3 \times 6$, $b = 9$, $k = 6$ and $r = 3$. The following design is obtained.

In this Example, $s_1 = 3$, $s_2 = 6$, $t = 2$, $\mathbf{d}_2' = (0\ 1, 2, 3, 4, 5)$, $\mathbf{d}_1'^{*} = (0\ 1, 2)$, $\mathbf{d}_1'^{1} = (0, 1, 2, 1, 2, 0)$, $\mathbf{d}_1'^{2} = (0, 2, 1, 2, 1, 0)$, $\mathbf{d}_1'^{3} = (2, 0, 1, 1, 0, 2)$.

$$\text{Further, } \mathbf{D}_2 = \mathbf{d}_2\mathbf{1}_b' = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\ 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 \end{bmatrix}$$

$$\mathbf{D}_1 = \begin{bmatrix} 0 & 2 & 1 & 0 & 1 & 2 & 2 & 1 & 0 \\ 1 & 0 & 2 & 2 & 0 & 1 & 0 & 2 & 1 \\ 2 & 1 & 0 & 1 & 2 & 0 & 1 & 0 & 2 \\ 1 & 0 & 2 & 2 & 0 & 1 & 1 & 2 & 0 \\ 2 & 1 & 0 & 1 & 2 & 0 & 0 & 1 & 2 \\ 0 & 2 & 1 & 0 & 1 & 2 & 2 & 0 & 1 \end{bmatrix}$$

and $\mathbf{F} = \mathbf{D}_1 \bullet \mathbf{D}_2$ gives the following required design with full efficiency on main effects

Replication I			Replication II			Replication III		
B1	B2	B3	B1	B2	B3	B1	B2	B3
00	20	10	00	10	20	20	10	00
11	01	21	21	01	11	01	21	11
22	12	02	12	22	02	12	02	22
13	03	23	23	03	13	13	23	03
24	14	04	14	24	04	04	14	24
05	25	15	05	15	25	25	05	15

The 18×18 concurrence matrix $\mathbf{NN}'$ for this design can be written as

3	0	0	1	0	2	0	2	1	1	1	1	0	1	2	1	2	0
0	3	0	2	1	0	1	0	2	1	1	1	2	0	1	0	1	2
0	2	3	0	2	1	2	1	0	1	1	1	1	2	0	2	0	1
1	2	0	3	0	0	1	1	1	0	1	2	1	0	2	0	2	1
0	1	2	0	3	0	1	1	1	2	0	1	2	1	0	1	0	2
2	0	1	0	0	3	1	1	1	1	2	0	0	2	1	2	1	0
0	1	2	1	1	1	3	0	0	0	1	2	0	2	1	2	1	0
2	0	1	1	1	1	0	3	0	2	0	1	1	0	2	0	2	1
1	2	0	1	1	1	0	0	3	1	2	0	2	1	0	1	0	2
1	1	1	0	2	1	0	2	1	3	0	0	2	0	1	0	1	2
1	1	1	1	0	2	1	0	2	0	3	0	1	2	0	2	0	1
1	1	1	2	1	0	2	1	0	0	0	3	0	1	2	1	2	0
0	2	1	1	2	0	0	1	2	2	1	0	3	0	0	0	0	3
1	0	2	0	1	2	2	0	1	0	2	1	0	3	0	3	0	0
2	1	0	2	0	1	1	2	0	1	0	2	0	0	3	0	3	0
1	0	2	0	1	2	2	0	1	0	2	1	0	3	0	3	0	0
2	1	0	2	0	1	1	2	0	1	0	2	0	0	3	0	3	0
0	2	1	1	2	0	0	1	2	2	1	0	3	0	0	0	0	3

The matrix $\mathbf{NN}'$ can easily be written as a linear combination of Kronecker product of proper matrices as per Definition. Therefore, this matrix has structure K. Again the matrix $\boldsymbol{\varepsilon}^{\mathbf{x}'}\mathbf{NN}'\boldsymbol{\varepsilon}^{\mathbf{x}}$ for both factors has the structures $0\mathbf{I}_3 + 36\mathbf{1}_3\mathbf{1}_3'$ and $0\mathbf{I}_6 + 9\mathbf{1}_6\mathbf{1}_6'$, respectively and, therefore, all main effects of the design are balanced by Theorem 2.2.

The interaction efficiency from this design is obtained by Theorem 2.4 as F1F2 = 0.721. We now use the replacement technique to get the required design for $v = 3^2 \times 2$ experiment in $b = 9$ blocks, $k = 6$ and $r = 3$. The replacement technique is the following: $0 \to 0\ 0$; $1 \to 1\ 0$; $2 \to 2\ 0$; $3 \to 0\ 1$; $4 \to 1\ 1$; $5 \to 2\ 1$. The required design again with full efficiency on main effects is

Replication I			Replication II			Replication III		
B1	B2	B3	B1	B2	B3	B1	B2	B3
000	200	100	000	100	200	200	100	000
110	010	210	210	010	110	010	210	110
220	120	020	120	220	020	120	020	220
101	001	201	201	001	101	101	201	001
211	111	011	111	211	011	011	111	211
021	221	121	021	121	221	221	021	121

One can easily verify that **NN′** matrix has structure *K* and all main effects are balanced. The interactions efficiencies from this design are obtained using Theorem 2.4 as F1F2 = 0.825, F1F3 = 1.000 and F2F3 = 1.000, F1F2F3 = 0.656.

Example 3.3. A factorial experiment for 3 × 4 × 6 factorial experiment is to be run in incomplete blocks. In order to construct the factorial design, we first construct a design for $v = 6 \times 12$ experiment in $b = 18$ blocks of size $k = 12$ each in $r = 3$ replications. The following design with full efficiency on main effects is obtained.

Replication I					
B1	B2	B3	B4	B5	B6
0 0	1 0	2 0	3 0	4 0	5 0
1 1	2 1	3 1	4 1	5 1	0 1
2 2	3 2	4 2	5 2	0 2	1 2
3 3	4 3	5 3	0 3	1 3	2 3
4 4	5 4	0 4	1 4	2 4	3 4
5 5	0 5	1 5	2 5	3 5	4 5
2 6	5 6	4 6	3 6	1 6	0 6
0 7	2 7	5 7	4 7	3 7	1 7
1 8	0 8	2 8	5 8	4 8	3 8
3 9	1 9	0 9	2 9	5 9	4 9
4 10	3 10	1 10	0 10	2 10	5 10
5 11	4 11	3 11	1 11	0 11	2 11

Replication II					
B1	B2	B3	B4	B5	B6
4 0	5 0	3 0	00	2 0	1 0
1 1	4 1	5 1	31	0 1	2 1
2 2	1 2	4 2	52	3 2	0 2
0 3	2 3	1 3	43	5 3	3 3
3 4	0 4	2 4	14	4 4	5 4
5 5	3 5	0 5	25	1 5	4 5
3 6	5 6	4 6	06	1 6	2 6
2 7	3 7	5 7	47	0 7	1 7
1 8	2 8	3 8	58	4 7	0 8
0 9	1 9	2 9	3 9	5 9	4 9
4 10	0 10	1 10	2 10	3 10	5 10
5 11	4 11	0 11	1 11	2 11	3 11

Replication III					
B1	B2	B3	B4	B5	B6
5 0	2 0	4 0	1 0	0 0	3 0
3 1	5 1	2 1	4 1	1 1	0 1
0 2	3 2	5 2	2 2	4 2	1 2
1 3	0 3	3 3	5 3	2 3	4 3
4 4	1 4	0 4	3 4	5 4	2 4
2 5	4 5	1 5	0 5	3 5	5 5
3 6	5 6	2 6	1 6	4 6	0 6
0 7	3 7	5 7	2 7	1 7	4 7
4 8	0 8	3 8	5 8	2 8	1 8
1 9	4 9	0 9	3 9	5 9	2 9
2 10	1 10	4 10	0 10	3 10	5 10
5 11	2 11	1 11	4 11	0 11	3 11

The interaction efficiency is F1F2 = 0.868. We now use the replacement technique to get the required design for $v = 3 \times 4 \times 6$ experiment in $b = 12$ blocks, $k = 12$ and $r = 3$. The replacement technique is the following: 0 → 0 0; 1 → 0 1; 2 → 0 2; 3 → 0 3; 4 → 1 0; 5 → 1 1; 6 → 1 2; 7 → 1 3; 8 → 2 0; 9 → 2 1; 10 → 2 2; 11 → 2 3. The required design with full efficiency on main effects is:

Replication I					
B1	B2	B3	B4	B5	B6
0 0 0	1 0 0	2 0 0	3 0 0	4 0 0	5 0 0
1 0 1	2 0 1	3 0 1	4 0 1	5 0 1	0 0 1
2 0 2	3 0 2	4 0 2	5 0 2	0 0 2	1 0 2
3 0 3	4 0 3	5 0 3	0 0 3	1 0 3	2 0 3
4 1 0	5 1 0	0 1 0	1 1 0	2 1 0	3 1 0
5 1 1	0 1 1	1 1 1	2 1 1	3 1 1	4 1 1
2 1 2	5 1 2	4 1 2	3 1 2	1 1 2	0 1 2
0 1 3	2 1 3	5 1 3	4 1 3	3 1 3	1 1 3
1 2 0	0 2 0	2 2 0	5 2 0	4 2 0	3 2 0
3 2 1	1 2 1	0 2 1	2 2 1	5 2 1	4 2 1
4 2 2	3 2 2	1 2 2	0 2 2	2 2 2	5 2 2
5 2 3	4 2 3	3 2 3	1 2 3	0 2 3	2 2 3

Replication II					
B1	B2	B3	B4	B5	B6
4 0 0	5 0 0	3 0 0	0 0 0	2 0 0	1 0 0
1 0 1	4 0 1	5 0 1	3 0 1	0 0 1	2 0 1
2 0 2	1 0 2	4 0 2	5 0 2	3 0 2	0 0 2
0 0 3	2 0 3	1 0 3	4 0 3	5 0 3	3 0 3
3 1 0	0 1 0	2 1 0	1 1 0	4 1 0	5 1 0
5 1 1	3 1 1	0 1 1	2 1 1	1 1 1	4 1 1
3 1 2	5 1 2	4 1 2	0 1 2	1 1 2	2 1 2
2 1 3	3 1 3	5 1 3	4 1 3	0 1 3	1 1 3
1 2 0	2 2 0	3 2 0	5 2 0	4 2 0	0 2 0
0 2 1	1 2 1	2 2 1	3 2 1	5 2 1	4 2 1
4 2 2	0 2 2	1 2 2	2 2 2	3 2 2	5 2 2
5 2 3	4 2 3	0 2 3	1 2 3	2 2 3	3 2 3

Replication III					
B1	B2	B3	B4	B5	B6
5 0 0	2 0 0	4 0 0	1 0 0	0 0 0	3 0 0
3 0 1	5 0 1	2 0 1	4 0 1	1 0 1	0 0 1
0 0 2	3 0 2	5 0 2	2 0 2	4 0 2	1 0 2
1 0 3	0 0 3	3 0 3	5 0 3	2 0 3	4 0 3
4 1 0	1 1 0	0 1 0	3 1 0	5 1 0	2 1 0
2 1 1	4 1 1	1 1 1	0 1 1	3 1 1	5 1 1
3 1 2	5 1 2	2 1 2	1 1 2	4 1 2	0 1 2
0 1 3	3 1 3	5 1 3	2 1 3	1 1 3	4 1 3
4 2 0	0 2 0	3 2 0	5 2 0	2 2 0	1 2 0
1 2 1	4 2 1	0 2 1	3 2 1	5 2 1	2 2 1
2 2 2	1 2 2	4 2 2	0 2 2	3 2 2	5 2 2
5 2 3	2 2 3	1 2 3	4 2 3	0 2 3	3 2 3

The interactions efficiencies are F1F2 = 0.9409, F1F3 = 0.8932 and F2F3 = 1.000, F1F2F3 = 0.8781. From this design one can also obtain a design for $v = 2 \times 3^2 \times 4$, $b = 18$, $k = 12$ and $r = 3$ by replacing the 6 levels of first factor by two factors at 2 and 3 levels respectively. The replacement scheme is $0 \rightarrow 00$; $1 \rightarrow 01$; $2 \rightarrow 02$; $3 \rightarrow 10$; $4 \rightarrow 11$; $5 \rightarrow 12$. The new design with full efficiency on main effects is

Replication I					
B1	B2	B3	B4	B5	B6
0000	0100	0200	1000	1100	1200
0101	0201	1001	1101	1201	0001
0202	1002	1102	1202	0002	0102
1003	1103	1203	0003	0103	0203
1110	1210	0010	1100	0210	1010
1211	0011	0111	0211	1011	1111
0212	1212	1112	1012	0112	0012
0013	0213	1213	1113	1013	0113
0120	0020	0220	1220	1120	1020
1021	0121	0021	0221	1221	1121
1122	1022	0122	0022	0222	1222
1223	1123	1023	0123	0023	0223

Replication II					
B1	B2	B3	B4	B5	B6
1100	1200	1000	0000	0200	0100
0101	1101	1201	1001	0001	0201
0202	0102	1102	1202	1002	0002
0003	0203	0103	1103	1203	1003
1010	0010	0210	0110	1110	1210
1211	1011	0011	0211	0111	1111
1012	1212	1112	0012	0112	0212
0213	1013	1213	1113	0013	0113
0120	0220	1020	1220	1120	0020
0021	0121	0221	1021	1221	1121
1122	0022	0122	0222	1022	1222
1223	1123	0023	0123	0223	1023

Replication III					
B1	B2	B3	B4	B5	B6
1200	0200	1100	0100	0000	1000
1001	1201	0201	1101	0101	0001
0002	1002	1202	0202	1102	0102
0103	0003	1003	1203	0203	1103
1110	0110	0010	1010	1210	0210
0211	1111	0111	0011	1011	1211
1012	1212	0212	0112	1112	0012
0013	1013	1213	0213	0113	1113
1120	0023	1020	1220	0220	0120
0121	1121	0021	1021	1221	0221
0222	0122	1122	0022	1022	1222
1223	0223	0123	1123	0023	1023

The interactions efficiencies from this design are F1F2 = 1.000, F1F3 = 0.9350, F1F4 = 0.8966, F2F3 = 0.9367, F2F4 = 0.9178, F3F4 = 1.000, F1F2F3 = 0.9512, F1F2F4 = 0.8838, F1F3F4 = 0.8941, F2F3F4 = 0.8816 and F1F2F3F4 = 0.8970.

Note: In all these designs derived under Example 3.3, the main effects are balanced and can easily be verified using Theorem 2.2.

Remark 3.2. This unified approach does not allow the construction of designs when s_1 and s_2 are co-prime and block size is a multiple of both s_1 and s_2. This needs to be investigated further.

We give below a catalogue of designs that can be generated from the unified method. The catalogue restricts to designs in which the maximum number of levels of any factor is smaller than 20. Some designs with efficiencies of factorial effects are given in the CD.

Using the above described method of construction, a catalogue of designs is prepared and is given in Table 1 in the Appendix. These designs permit the estimation of all main effects with full efficiency. The parameters of the designs are also given in the table. These designs along with the two-factor interactions efficiencies are available at Design Resources Server being maintained at IASRI. The URL is www.iasri.res.in/design/factorial/factorial.htm.

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APPENDIX

Table 1. Catalogue of resolvable designs obtainable from the general method for a maximum of 12 levels of any factor in 3 replications

2^3	4×3^2	6^2	9^2
2^4	4×3^3	6×2	9×3
2^5	$4^2\times2$	6×3	9×6
2^6	$4^2\times2^2$	6×4	9×3^2
3^3	$4^2\times3$	6×2^2	$9\times4\times3$
3^4	$4\times3\times2$	6×3^2	$9\times6\times2$
3×2^2	$4\times3\times2^2$	$6^2\times2$	10^2
3×2^3	$4\times3\times2^3$	$6\times3\times2$	10×2
3×2^4	$4\times3^2\times2$	$6\times4\times2$	10×4
3×2^5	$4^2\times3\times2$	$6\times4\times3$	10×5
$3^2\times2$	5^2	$6\times5\times2$	10×6
$3^2\times2^2$	5×2^2	$6\times4\times2^2$	10×8
$3^2\times2^3$	5×2^3	$6\times5\times2^2$	$10\times5\times2$
$3^3\times2$	5×2^4	7^2	11^2
$3^3\times2^2$	$5^2\times2$	8^2	12^2
4^2	$5^2\times 2^2$	8×2	12×2
4^3	$5\times3\times2^2$	8×4	12×3
4×2	$5\times3\times2^3$	8×6	12×4
4×2^2	$5\times4\times2$	$8\times4\times2$	12×6
4×2^3	$5\times4\times2^2$	$8\times5\times2$	12×8
4×2^4	$5\times4\times3\times2$	$8\times6\times2$	12×9
			12×10