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ORIGINAL ARTICLE

MULTIFACTOR MIXTURE EXPERIMENTS : CONSTRUCTION AND ANALYSIS

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Abstract : An experiment in which there are two or more than two factors, each factor is a mixture of its components is called multifactor mixture experiment. Two methods of construction of multifactor mixture experiments have been developed. First method uses the algorithmic construction of efficient designs in less number of design points in comparison to existing designs obtained as a Kronecker product of single factor mixture designs. In the second method, Kronecker sum of matrices has been utilized for construction of multifactor designs where all the factors have same number of components. It has been illustrated with data that the designs developed allow the fitting of second order model.

Key words : Multifactor mixture experiments, *G*-efficiency, *A*-efficiency, Kronecker product, Kronecker sum, Simplex lattice and Simplex centroid.

1. Introduction

An experiment in which the response is a function only of the proportions of the components (constituents) present in the mixture and is not a function of the total amount of the mixture is called mixture experiment. Scheffé (1958, 1963) did a pioneering work in the mixture experiments by introducing simplex lattice and simplex centroid designs. For a *q*-component mixture experiment if x_i denotes the proportions of *ith* component, then

$$0 \le x_i \le 1 \text{ and } \sum_{i=1}^{q} x_i = 1, \text{ for } i = 1, 2, ..., q. \qquad (1)$$

Thus, the design factor space is a q-1dimensional simplex.

The methods of analysis of experiments with mixtures seem to be relevant and useful in many areas of agricultural research such as nutrient management of crops, cropping system, disease and pest management of crops. Many experiments has been undertaken for studying the optimum time of split application of fixed quantity of nutrients to crop during its crop growth stages. Batra *et al.* (1999) have viewed these experiments as mixture experiments and have given the procedure of its analysis.

Designs for mixture experiments introduced by Scheffe (1958, 1963) investigates only one factor with its components at a time, as the components of any mixture have to be some or other kind of a single factor. For instance, in fertilizer experiments involving applications of fixed quantity of nitrogen, where the nutrients are supplied from different sources of nitrogenous fertilizers, sources of fertilizer are the components of mixture experiments. But situations arise when proportions of components of two or more factors are to be studied. These types of mixture of experiments are called multifactor mixture experiments. This has been illustrated with an experiment available in Agricultural Field Experiments Information System.

Example 1.1 (Agricultural Field Experiments Information System) : An experiment was undertaken to study the effect of nitrogen and phosphorus on hybrid cotton plant in an attempt to maximize the yield. In this experiment fixed doses of the two fertilizers *viz.* nitrogen (150 kg/ha) and phosphorus (75 kg/ha) were applied at different proportions in three crop growth stages of the crop. The details of the doses applied at different crop growth stages are given in Table 1.

This experiment is an example of two factor mixture experiment where the two factors are nitrogen and

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Table 1 : Treatment structure of two factor mixture experiment.

Factor I (Nitrogen)			Factor II (Phosphorus)		
Basal	**45 DAS**	**90 DAS**	**Basal**	**45 DAS**	**90 DAS**
150.00	0.00	0.00	75.00	0.00	0.00
150.00	0.00	0.00	37.50	37.50	0.00
150.00	0.00	0.00	37.50	0.00	37.50
150.00	0.00	0.00	18.75	18.75	37.50
75.00	75.00	0.00	75.00	0.00	0.00
75.00	75.00	0.00	37.50	37.50	0.00
75.00	75.00	0.00	37.50	0.00	37.50
75.00	75.00	0.00	18.75	18.75	37.50
75.00	37.50	37.50	75.00	0.00	0.00
75.00	37.50	37.50	37.50	37.50	0.00
75.00	37.50	37.50	37.50	0.00	37.50
75.00	37.50	37.50	18.75	18.75	37.50
37.50	75.00	37.50	75.00	0.00	0.00
37.50	75.00	37.50	37.50	37.50	0.00
37.50	75.00	37.50	37.50	0.00	37.50
37.50	75.00	37.50	18.75	18.75	37.50

DAS : Days after sowing.

phosphorus and the three growth stages *viz*. Basal, 45 DAS and 90 DAS as three components of each of the factors.

In an n-factor mixture experiment if the ith factor is having p_i components and x_{ij} represents the proportion of jth component of the ith factor, j = 1, 2,...,p_i and i = 1, 2, …, n, then multifactor mixture experiment can be defined as:

$$0 \le x_{ij} \le 1 \text{ and } \sum_{j}^{p_i} x_{ij} = 1,\ \forall i = 1,2,...,n; j=1,2,...,p_i.$$

Now the problem is to obtain efficient designs for multifactor mixture experiments.

Nigam (1973) observed that the designs for multifactor mixture experiment can be constructed such that for each combination of components of one factor, all the combinations of components of the other factor must occur. Kumari and Mittal (1986) gave a method of construction of designs for two factor mixture experiments for fitting linear model. Murthy and Murty (1989) gave the method of construction of two factor mixture experiments by transforming symmetric factorial designs.

The designs for multifactor mixture experiments available in literature are obtained by Kronecker product and require a large number of runs for the experiment. Further, the analysis of data generated in multifactor experiment has been discussed for fitting of models of first degree only. Some new methods of construction of designs for multi-factor mixture experiments in smaller number of runs that allow fitting of second order model needs to be developed.

2. Design Evaluation Criteria

Design efficiency criteria are often used to evaluate a proposed experimental design. The design efficiency measures that have often been used in response surface studies to compare different designs are G- efficiency and relative A –efficiency and are given by

$$\text{G-efficiency} = \frac{p}{n \times d} \tag{2}$$

$$\text{Relative A-efficiency} = \left(\frac{\text{trace}\left(\mathbf{X}_*'\mathbf{X}_*\right)^{-1}}{\text{trace}\left(\mathbf{X}'\mathbf{X}\right)^{-1}} \right) \tag{3}$$

Per-point Relative A-efficiency

$$= \left(\frac{n \times \text{trace}\left(\mathbf{X}_*'\mathbf{X}_*\right)^{-1}}{n_* \times \text{trace}\left(\mathbf{X}'\mathbf{X}\right)^{-1}} \right) \tag{4}$$

Where, n = number of design points in the design; p = number of parameters in the model and $\text{d} = \max\left\{ v = \mathbf{x}\left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{x}' \right\}$ over a specified set of design points (the row vector) **x** in **X**, where **X** is the extended design matrix depending on model to be fitted, $\mathbf{X}_*$ is the sub-matrix of **X** having same number of columns as **X** and n_* is the number of runs for the design matrix $\mathbf{X}_*$.

As a practical rule of thumb, Wheeler (1972) suggested that any design with a G-efficiency ≥ 50% could be called a “good” design for practical purposes and showed that pursuit of higher efficiencies is not generally justified in practice.

3. Model for Multifactor Mixture Experiment

It has been observed that in agricultural experiments, the behavior of components of different factors may be quadratic in nature, therefore, multifactor designs for experiments with mixtures need to be obtained so as to fit the second order polynomial model.

Let x_{1i} denotes the proportion of the ith component of the first factor and x_{2j} denote the proportion of the jth component of the second factor, i = 1, 2, ..., p_1 and j = 1, 2, …, p_2. The uth point in the two factor mixture experiment is denoted by (x_{11u}, x_{12u}, …, $x_{1p_{1u}}$; x_{21u}, x_{22u}, …, $x_{2p_{2u}}$). For two factor mixture experiment, Nigam (1973) has given the following model

$$Y=\sum_{i}^{p_1}\alpha_i x_{1i}+\sum_{i<i'}^{p_1}\alpha_{ii'}x_{1i}x_{1i'}+\sum_{j}^{p_2}\alpha_j x_{2j}+\sum_{j<j'}^{p_2}\alpha_{jj'}x_{2j}x_{2j'}+\sum_{i,j}^{p_1p_2}\alpha_{ij}x_{1i}x_{2j}+\varepsilon \qquad (5)$$

But as the sum of the components in different runs for each factor is constant, it has been seen that the design matrix is not of full column rank, *i.e.* the design matrix is singular and consequently, it is not possible to estimate the parameters uniquely. For this reason, we will transform the mixture model (5) so that the transformed design matrix is of full column rank.

The dimensionality of the model (5) is reduced by making substitution

$$x_{1p_1}=\left(1-\sum_{i=1}^{p_1-1}x_{1i}\right) \text{ and } x_{2p_2}=\left(1-\sum_{j=1}^{p_2-1}x_{2j}\right) \qquad (6)$$

Where, x_{1p_1} is the p_1th component of the first factor (X_1) and x_{2p_2} is the p_2th component of the second factor (X_2).

After substitution and algebraic simplification, the resulting model takes the form

$$Y=\beta_0+\sum_{i=1}^{p_1-1}\beta_i x_{1i}+\sum_{j=1}^{p_2-1}\gamma_j x_{2j}+\sum_{i=1}^{p_1-1}\beta_{ii}x_{1i}^2+\sum_{j=1}^{p_2-1}\gamma_{jj}x_{2j}^2+\sum_{i<i'}^{p_1-1}\beta_{ii'}x_{1i}x_{1i'}+\sum_{j<j'}^{p_2-1}\gamma_{jj'}x_{2j}x_{2j'}+\sum_{i=1}^{p_1-1}\sum_{j=1}^{p_2-1}\delta_{ij}x_{1i}x_{2j}+\varepsilon \qquad (7)$$

Which is similar to model used for fitting in context of response surface designs with $\sum_{i=1}^{n}(p_i-1)$ factors.

4. Construction of Designs for Multifactor Mixture Experiments

The method of construction of multifactor mixture experiments given by Nigam (1973) requires a large number of runs in comparison to the number of parameters to be fitted. It is desirable to obtain efficient designs that economize on experimental resources. In this paper an algorithm has been developed for obtaining efficient designs multifactor mixture experiments with lesser number of runs.

Method 4.1 : In this method, an algorithm has been developed for computer aided generation of designs for multifactor experiments with mixtures. The designs obtained are evaluated on G-efficiency (2), relative A-efficiency criterion (3) and per-point relative A-efficiency (4) keeping in view the requirements of model under investigation.

Algorithm to construct a n-factor mixture experiment, where the ith factor has p_i components, i = 1, 2, …, n is described in the sequel.

Algorithm 4.1

Step 1 : Input n and p_i and required design **D**.

{D is a $\left(m\times\sum_{i=1}^{n}p_i\right)$ matrix with blank cells, $m \geq p$, the number of parameters to be estimated in the model through the design; {n is the number of factors and p_i is the number of components of the ith factor, i = 1, 2, ..., n}.

Step 2 : Input standard mixture design $\mathbf{D}_i$ for each of n-factors involving p_i components and each having n_i runs.

{Standard mixture design may be simplex lattice design, simplex centroid design or central axial design as per user requirements and model to be fitted}.

Step 3 : Obtain design matrix $\mathbf{D}^*$ following the method given by Nigam (1973) as Kronecker product of $\mathbf{D}_i$ *i.e.*

$$\mathbf{D}^*=\mathbf{D_1}\otimes\mathbf{D_2}\otimes\ldots\otimes\mathbf{D_n} \qquad (8)$$

of order $\prod_{i=1}^{n}n_i\times\prod_{i=1}^{n}p_i$. The design is having $N=\prod_{i=1}^{n}n_i$ runs.

Step 4 : Select model for describing the relationship between response and input variables of mixture and evaluate the number of parameters say 'p' to be estimated for the second order model described in Section 3.

Step 5 : Evaluate norm of each of the run of the design $\mathbf{D'}$, obtained in Step 3 and arrange the runs in different groups such that runs within a group have same norm.

{The norm of an n-component vector $[x_1\ x_2 .... x_n]'$

Table 2 : Two factor mixture design obtained through Kronecker product.

x_{11}	x_{12}	x_{21}	x_{22}	x_{23}
1	0	1	0	0
1	0	0	1	0
1	0	0	0	1
1	0	0.50	0.50	0
1	0	0.50	0	0.50
1	0	0	0.50	0.50
1	0	0.33	0.33	0.34
0	1	1	0	0
0	1	0	1	0
0	1	0	0	1
0	1	0.50	0.50	0
0	1	0.50	0	0.50
0	1	0	0.50	0.50
0	1	0.33	0.33	0.34
0.5	0.5	1	0	0
0.5	0.5	0	1	0
0.5	0.5	0	0	1
0.5	0.5	0.50	0.50	0
0.5	0.5	0.50	0	0.50
0.5	0.5	0	0.50	0.50
0.5	0.5	0.33	0.33	0.34

Table 3 : Groups formed by different treatment combinations.

	x_{11}	x_{12}	x_{21}	x_{22}	x_{23}	Norm	Group	Size
Combinations of Type	1	0	1	0	1	1.414	G1	6
	1	0	1/2	1/2	0	1.225	G2	9
	1/2	1/2	1	0	0			
	1	0	1/3	1/3	1/3	1.155	G3	2
	1/2	1/2	1/2	1/2	0	1	G4	3
	1/2	1/2	1/3	1/3	1/3	0.913	G5	1

Note : Group size is obtained by taking factor wise permutation of various distinct combinations occurring in the group.

is given by: $\text{norm} = \sqrt{\sum_{i=1}^{n} x_i^2}\ \}$ (9)

Let number of distinct group formed be 'g' with k*th* group having m_k elements after arranging groups in decreasing order of magnitude of norm. Let the groups are $[\mathbf{G}_1\ \ \mathbf{G}_2\ \ \ldots\mathbf{G}_k\ldots\ \ \mathbf{G}_g]$.

Step 6 : Form a class of designs **C** consisting of $2^g - 1 = M$ (say) designs by including a group or excluding a group and retaining those designs where number of runs > p. Obviously $\mathbf{D} \in \mathbf{C}$.

Step 7 : For each design in **C** obtain **X** matrix for the model (5). If the matrix is non-singular then evaluate its G-efficiency (2) and/ or relative A-efficiency (3), otherwise reject the design.

Step 8 : If the design have the efficiency measure more than the value desired by the experimenter, then select the design, else reject the design.

Remark 4.1 : Algorithm (8) is used for obtaining designs of two factor mixture experiments, where first factor is having two component and the second factor having three components.

Example 4.1 : At the first instance input the number of factors (n) and number of components for each factor (p_i). In this present case, take $n = 2$ and $p_1 = 2$; $p_2 = 3$.

Let us take the standard mixture design ($\mathbf{D}_i$), $i = 1, 2$, as simplex centroid design [Cornell (2002)]. By following the method given by Nigam (1973), we obtain the design for two factor mixture experiment as Kronecker product of $\mathbf{D}_i$'s *i.e.* $\mathbf{D}^* = \mathbf{D_1} \otimes \mathbf{D_2}$ of order 21×5 as in Table 2.

For the situation (Table 2) to fit the second order model (7) one needs ≥ 10 runs.

The norm of each row of **D** has been calculated and the runs in different groups are arranged such that runs within a group have same norm. Five groups are formed after arranging norms in decreasing order of magnitude and are shown in Table 3.

After taking all possible combinations of different groups, a class of designs **C** has been obtained so that each member of **C** has number of runs more than 10. Each member of class **C** is the candidate design for the situations under consideration. The design matrix for the model (7) and the G-efficiency (2), relative A-efficiency (3) and per-point relative A-efficiency (4) of the each element of **C** has been calculated. The G-efficiency, relative A-efficiency, total number of runs and also the % loss of runs for different designs have been given in Table 4.

From Table 4, it is seen that algorithm generates large number of designs with G-efficiency > 0.50 for the situations under investigation. The designs have been arranged in decreasing order of % reduction of runs vis-â-vis Kronecker product type designs. One can choose the design as per available resources and interest of the experimenter in testing various types of mixture blends. However, there is a need to study the geometric properties of the generated designs.

Visual Basic code and SAS code has been

Table 4 : Two factor mixture experiment ($p_1 = 2$; $p_2 = 3$).

Groups Constituting Design	A-efficiency		G-efficiency	Run	% of Runs Reduced
	A_{e1}	A_{e2}			
1, 2	0.998	0.713	0.980	15	28.57
2, 3	1.026	0.537	0.909	11	47.62
2, 4	1.023	0.585	0.833	12	42.86
2, 5	1.236	0.589	0.983	10	52.38
1, 2, 3	1.002	0.811	0.813	17	19.05
1, 2, 4	0.999	0.856	0.781	18	14.29
1, 2, 5	1.268	0.966	0.876	16	23.81
1, 3, 4	0.987	0.517	0.909	11	47.62
1, 4, 5	1.654	0.788	0.951	10	52.38
2, 3, 5	0.980	0.560	0.841	12	42.86
2, 4, 5	0.780	0.483	0.775	13	38.10

Table 5 : Design for two factor mixture experiments obtained through Kronecker sum.

Factor 1			Factor 2		
x_{11}	x_{12}	x_{13}	x_{21}	x_{22}	x_{23}
0.00	0.33	0.67	0.00	0.33	0.67
0.00	0.67	0.33	0.00	0.67	0.33
0.33	0.67	0.00	0.33	0.67	0.00
0.33	0.00	0.67	0.33	0.00	0.67
0.67	0.00	0.33	0.67	0.00	0.33
0.67	0.33	0.00	0.67	0.33	0.00
0.00	0.33	0.67	0.33	0.67	0.00
0.00	0.67	0.33	0.33	0.00	0.67
0.33	0.67	0.00	0.67	0.00	0.33
0.33	0.00	0.67	0.67	0.33	0.00
0.67	0.00	0.33	0.00	0.33	0.67
0.67	0.33	0.00	0.00	0.67	0.33
0.33	0.67	0.00	0.00	0.33	0.67
0.33	0.00	0.67	0.00	0.67	0.33
0.67	0.00	0.33	0.33	0.67	0.00
0.67	0.33	0.00	0.33	0.00	0.67
0.00	0.33	0.67	0.67	0.00	0.33
0.00	0.67	0.33	0.67	0.33	0.00
0.67	0.00	0.33	0.00	0.33	0.67
0.67	0.33	0.00	0.00	0.67	0.33
0.00	0.33	0.67	0.33	0.67	0.00
0.00	0.67	0.33	0.33	0.00	0.67
0.33	0.67	0.00	0.67	0.00	0.33
0.33	0.00	0.67	0.67	0.33	0.00

G-efficiency of the above design is 0.625.
Catalogue of designs under different experimental situations has been prepared using the above method and is available with the first author.

developed and the catalogue of designs has been prepared using the above method and is available with the first author.

4.2 Design for multifactor mixture experiments obtained through Kronecker sum

The designs for multifactor mixture experiments can also be constructed using Kronecker sum of matrices. For the sake of completeness, we first define Kronecker sum.

Kronecker Sum : Let $\mathbf{A} = (a_{ij})_{mn}$; $\mathbf{B} = (b_{ij})_{pq}$ be two matrices with entries from a finite additive abelian group G of order s with elements as $(0, 1, 2, \ldots, s-1)$.

Table 6 : Two factor mixture experiments each factor having two components *viz.* x_{11}, x_{12}, x_{21} and x_{22} and response (y).

Factor 1		Factor 2		Response
x_{11}	x_{12}	x_{21}	x_{22}	y
0.146	0.854	0.146	0.854	856
0.146	0.854	0.854	0.146	689
0.854	0.146	0.146	0.854	726
0.854	0.146	0.854	0.146	789
0	1.000	0.500	0.500	799
1.000	0	0.500	0.500	750
0.500	0.500	0	1	798
0.500	0.500	1.000	0.000	812
0.500	0.500	0.500	0.500	345
0.500	0.500	0.500	0.500	395

Then Kronecker sum of **A** and **B** denoted by **A** ⊗ **B** is defined as

$$\mathbf{A} \otimes \mathbf{B} = \mathbf{A} \otimes \mathbf{J} + \mathbf{J} \otimes \mathbf{B}$$

$$= \left(\mathbf{B} + a_{ij}\mathbf{J}\right)_{1\le i\le m, 1\le j\le n} \mod(s)$$

Designs for multifactor mixture in which the number of components of each factor is same can be easily obtained through the Kronecker sum of two matrices. The method of construction is as described as in sub-Section 4.3.

4.3 Construction

Existence of matrices $\mathbf{A}_{a\times n}$ and $\mathbf{B}_{b\times p}$ with $\mathbf{B}_{b\times p}\mathbf{1}_{p\times 1} = c\times\mathbf{1}$ (constant) with entries from a finite additive abelian group of order p. Then $\frac{1}{c}\left[\mathbf{A}\otimes\mathbf{B}\right]$ gives us a n-factor mixture design with each factor having p components. The number of runs of the resulting design is $a \times b$, where ⊗ denotes the Kronecker sum of matrices.

Note : (i) Choice of a and b is arbitrary and should be chosen in such a manner as to get sufficient number of runs to fit desired model.

(ii) Matrices **A** and **B** have been obtained by trial and error.

(iii) This method is suitable in the situation where all the factors in the experiment have equal number of components.

Example 4.2 : For constructing a two factor mixture experiment with each factor having three components which allows fitting of second order model (7) at least 15 runs are needed. So one should chose a and b in such a manner that $a \times b \ge 15$. By taking **A** and **B** as

$$\mathbf{A} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 2 & 0 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 2 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \\ 2 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$$

In this case the row sum of **B** is 3.

Obtain **D** = **A** ⊗ **B** as given below

$$\mathbf{D} = \begin{bmatrix} 0 & 1 & 2 & 0 & 1 & 2 \\ 0 & 2 & 1 & 0 & 2 & 1 \\ 1 & 2 & 0 & 1 & 2 & 0 \\ 1 & 0 & 2 & 1 & 0 & 2 \\ 2 & 0 & 1 & 2 & 0 & 1 \\ 2 & 1 & 0 & 2 & 1 & 0 \\ 1 & 2 & 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 1 & 0 & 2 \\ 0 & 2 & 1 & 2 & 0 & 1 \\ 1 & 2 & 0 & 2 & 1 & 0 \\ 1 & 0 & 2 & 0 & 1 & 2 \\ 2 & 0 & 1 & 0 & 2 & 1 \\ 1 & 1 & 2 & 0 & 1 & 2 \\ 1 & 2 & 0 & 0 & 2 & 1 \\ 2 & 0 & 1 & 1 & 2 & 0 \\ 2 & 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 2 & 2 & 0 & 1 \\ 0 & 2 & 1 & 2 & 1 & 0 \\ 2 & 0 & 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 0 & 2 & 1 \\ 0 & 1 & 2 & 1 & 2 & 0 \\ 0 & 2 & 1 & 1 & 0 & 2 \\ 1 & 2 & 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 & 1 & 0 \end{bmatrix}$$

In each row, sum of first three columns and the last three columns are equal to 3. To get the mixture experiments of two factors, divide each element of **D** by 3. The final design is given in Table 5.

5. Illustration

An analysis of the multifactor mixture experiment using model (7) has been illustrated with a set of data.

Consider an experiment of two factor mixture experiment; each factor is having two components. The hypothetical data for two factor mixture experiment with the response variable is shown in Table 6.

To fit model (7) using the above data PROC RSREG of the SAS has been utilized and the results are shown below:

Response Mean	695.90
Root MSE	34.23
R-Square	0.98
Coefficient of Variation	4.92

Regression	D.F.	Sum of Squares	R-Square	F-Ratio	Pr > F
Linear	2	2119.729	0.0074	0.90	0.4741
Quadratic	2	266353	0.9301	113.66	0.0003
Crossproduct	1	13225	0.0462	11.29	0.0283
Total Model	5	281698	0.9836	48.08	0.0012

Parameter	DF	Estimate	Standard Error	t Value	Pr > |t|
Intercept	1	1330.76	54.02	24.63	<.0001
x_{11}	1	-1830.61	149.02	-12.28	0.0003
x_{21}	1	-1947.32	149.02	-13.07	0.0002
$x_{11}*x_{11}$	1	1566.12	127.96	12.24	0.0003
$x_{21}*x_{11}$	1	458.84	136.58	3.36	0.0283
$x_{21}*x_{21}$	1	1688.12	127.96	13.19	0.0002

Factor	DF	Sum of Squares	Mean Square	F-value	Pr > F
x_{11}	3	189969	63323	54.04	0.0011
x_{21}	3	218035	72678	62.03	0.0008

Factor	Critical Value
x_{11}	0.610
x_{21}	0.507

	Eigen vectors	
Eigen Values	x_{11}	x_{21}
1864.5090	0.6095	0.7928
1389.7268	0.7928	-0.6095

Stationary point is a minimum.

The optimum value of x_{11} = 0.610 and that of x_{21} = 0.507, so the optimum value for x_{12} = 1 – 0.610 = 0.39 and the optimum value of x_{22}=1 – 0.507 = 0.493.

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