RASHI 2 (1) : 34-38 (2017)

Trend free design under two-way elimination of Heterogeneity

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ABSTRACT

In this paper, we have considered a model under two-way elimination of heterogeneity setup in the presence of linear systematic trend component. The necessary and sufficient condition for design under two-way elimination of heterogeneity setup to be linear trend free has been obtained. A class of linear trend free designs under two-way elimination of heterogeneity setup has also been discussed.

Keywords : Heterogeneity, Information matrix, Systematic trend and Trend-free designs

1. Introduction

Heterogeneity in the experimental material is the most important problem to be taken care of while designing of scientific experiments. Block designs are the most commonly used designs when heterogeneity is present only in one direction or due to one source as in such situations local variation over the experimental material can be controlled by dividing the whole experimental material into groups/blocks such that the experimental units are homogeneous within a block than within the experimental material as a whole. However, in agricultural and allied experiments, we may come across some situations when there is evidence of heterogeneity in the experimental material in two directions or due to two sources. Thus, instead of using block designs for such situations, one should preferably use designs under two source of heterogeneity.

In agricultural experiments, many situations arise in which apart from the known source of variations, the response may also depend on the spatial or temporal position of the experimental unit, *i.e.* systematic trend effects may affect the experimental units. This situation is very common under block design setup. For example, in agricultural experiments when plots occur in strips in a field, it is often the case that differing contiguous sets of plots (which form blocks) within the same strip have different fertility gradient. Trend may occur in greenhouse experiments, where the source of heat is located on sides of the house and experimental units are kept in lines. In poultry experiments, where the source of heat is at the centre of the shed and chicks of early age are in cages. In orchard and vineyard experiments on undulating topography, where, response variable is affected by slowly migrating insects entering the area from one side to other. In industrial experiments, where blocks correspond to a time period and treatments are applied sequentially to experimental units, trends often occur within blocks as a result of different rates of aging or equipment wear out. One way to account for the presence of trends is to use analysis of covariance, treating trend values as covariate. Alternatively, for these situations, one can think of suitable designs which are orthogonal to trend effects. Such design may be called as trend-free block designs (Bradley and Yeh, 1980). A good number of literatures are available which deal with different aspect of block designs in the presence of systematic trend. For example one may refer to Bradley and Yeh (1980), Yeh and Bradley (1983), Jacroux *et al.* (1997), Majumdar and Martin (2002), Lal *et al*. (2007), Bhowmik *et al.* (2012), Bhowmik (2013) *etc*. Recently, Bhowmik *et al*. (2014) studied block model with neighbour effects from adjacent experimental units incorporating trend component. They have obtained the necessary and sufficient condition for a block design with neighbour effects to be trend free. Series of trend free totally balanced block designs have also been obtained. Bhowmik *et al.* (2015) also studied block model with second order neighbor effects in the presence of systematic trend. Trend resistance designs have also been obtained for this situation. Sarkar *et al.* (2017) obtained trend resistant neighbour balanced bipartite block designs. A good number of work though available in literature under block design setup, the literature related to experimental designs under two-way elimination of heterogeneity setup in the presence of systematic trend does not seem to be available. Following is an experimental situation under two-way elimination of heterogeneity setup when there are evidences of systematic trend affecting the response.

1.1 Experimental situation: In an animal experiment with an object of comparing the effect of different feeds (treatments), let different cows be the experimental units with their milk yield over different time interval of a particular lactation as the variate under study. Suppose breed and age of cows are two factors

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that correspond to two sources of variation apart from the treatment. Thus, both of them are actually the controlled factors and it is intended to eliminate the variation due to breeds and age of the cows. Here, as there are two sources of variation to be eliminated *viz*. breed and age, thus, **designs under** two-way elimination of heterogeneity setup is most preferable for this situation.

In animal experiments, many situations arise in which apart from the known source of variations, the response may also depend on the temporal effect, *i.e.*, the experimental units may get affected by one or more **systematic trend** present in the experimental material. For the above experimental situation, one can also identify systematic trend component which can affect the milk yield if a fact regarding milk yield can be taken into consideration as it is well known in advance that the milk yield will decrease as lactation of an animal progress over weeks. Therefore, the experimental output will be more précised within the limited available resources if the above mentioned fact can be included in the model in the form of systematic trend component.

It is clear from the above experimental situations that, in agricultural, animal or allied experiments under two-way elimination of heterogeneity setup apart from the known sources of variation, systematic trend may also affect the response of a particular experimental unit and hence it should be incorporated into the model for proper model specification. In this article, we have defined the model under two-way elimination of heterogeneity setup incorporating systematic trend. The condition for a design under two-way elimination of heterogeneity setup to be trend free has been obtained. A class of trend free designs under two-way elimination of heterogeneity setup has also been discussed.

2. EXPERIMENTAL SETUP AND MODEL

Consider a class of designs under two-way elimination of heterogeneity setup with v treatments and n experimental units. Let both the sources of heterogeneity are having p levels. Let Y_{ijk} be the response obtained due to i^{th} treatment (i = 1,2,...,v), j^{th} level of first source of heterogeneity and k^{th} level of second source of heterogeneity (j,k =1,2,...,p).

It is also assumed that response from a particular experimental unit is also affected by systematic trend component and the trend effect is represented by orthogonal polynomial of degree one. Based on the above experimental setup, following fixed effects additive model in matrix notations, can be considered for capturing the effect of systematic trend for designs under two-way elimination of heterogeneity setup:

$$Y = \mu 1 + \Delta' \tau + D_1' \rho + D_2' \chi + Z\theta + e \qquad (1)$$

where, Y is a vector of observations of order n × 1, μ is the general mean, 1 is a vector of unity of order n × 1, Δ' is an n × v matrix of observations versus treatments, τ is a v × 1 vector of treatment effects, D'_1 is an n × p incidence matrix of observations versus one source of heterogeneity, ρ is a p × 1 vector corresponding to one source of heterogeneity, D'_2 is an n × p incidence matrix of observations versus another source of heterogeneity, χ is a q × 1 vector corresponding to another source of heterogeneity, θ is a u × 1 vector representing the trend effects. The matrix Z, of order n × u, is the matrix of coefficients which is given by $Z = 1_p \otimes F$ where F is a p × 1 matrix with columns representing the (normalized) orthogonal polynomials. Here, e is an n ×1 vector of errors where errors are normally distributed random variable with E(e) = 0 and D(e) = $\sigma^2 I_n$. Since F is a k × p matrix with columns representing the (normalized) orthogonal polynomials, thus $1' F = 0,\ F'F = I_p$ and hence $Z' Z = pI_p$. Here, $D_1Z = 0$ and $D_2Z = 0$.

Rewriting the model as follows by writing parameter of interest first:

$$Y = \Delta' \tau + \mu 1 + D_1' \rho + D_2' \chi + Z\theta + e \qquad (2)$$

Equation (2) can also be written as:

$$Y = X_1\theta_1 + X_2\theta_2 + e \qquad (3)$$

where,

$$X_1 = \Delta';\ x_2 = \left[1\ D'_1\ \ D'_2\ \ Z\right];$$

$$\theta_1 = \tau;\ \theta_2 = \left[1\ \rho'_1\ \ \chi'\ \ \theta'\right]' \qquad (4)$$

Let,

$$R = \Delta\Delta',\ N_1 = \Delta D'_1,\ \ N_2 = \Delta D'_2,$$

$$D_1D'_1 = Q,\ D_2D'_2 = P,\ D_1D'_2 = N$$

and $r = \Delta 1, q = D_1 1, p = D_2 1$

where $r = \left[r_1\ \ r_2\ \ ....\ r_v\right]$ is the replication vector of treatments of order v × 1 with r_i as the number of times i^{th} treatment appears in the design,

Therefore,

$$X_1'X_1 = X'_1X_2 = \left[\Delta 1\ \ \Delta D'_1\ \ \Delta D'_1\ \ \Delta Z\ \right] = \left[r\ q\ p\ \Delta Z\right]$$

and

$$X_2'X_2 = \begin{bmatrix} 1'1 & 1'D_1' & 1'D_2' & 1'Z \\ D_1 1 & D_1D_1' & D_1D_2' & D_1Z \\ D_2 1 & D_2D_1' & D_2D_2' & D_2Z \\ Z'1 & Z'D_1' & Z'D_2' & Z'Z \end{bmatrix} = \begin{bmatrix} n & q' & p' & 0 \\ q & Q & N_3 & 0 \\ p & N_3' & P & 0 \\ 0 & 0 & 0 & pI_p \end{bmatrix}$$

Therefore, the information matrix for estimating the contrast pertaining to treatment effects can be obtained as follows:

$$C_\tau = R - N_1Q^{-1}N_1' - N_1Q^{-1}N_3E^{-1}N_3'Q^{-1}N_1' + N_2E^{-1}N_3'Q^{-1}N_1' + N_1Q^{-1}N_3E^{-1}N_2' - N_2E^{-1}N_2' - \frac{1}{p}\Delta ZZ'\Delta'$$

with $E = \left[P - N_3'Q^{-1}N_3\right]$

Based on the model, when the levels of all the factors are equal say v, the information matrix for estimating the contrast pertaining to the effect of treatments can be obtained as:

$$C = v\left(1 - \frac{11'}{v}\right) - \frac{1}{v}\Delta ZZ\Delta'$$

Following is the definition of a trend free design under two-way elimination of heterogeneity setup:

Definition 2.1: A design under two-way elimination of heterogeneity setup incorporating trend component, is called a *trend-free* design if the adjusted treatment sum of squares under the corresponding model is same as the adjusted treatment sum of squares under the usual model with two-way elimination of heterogeneity setup without trend component.

2. Conditions for the Block Design with Interference Effects to be Trend Free

The conditions for a block design under two-way elimination of heterogeneity setup have been obtained here so that the treatment effects and trend effects are orthogonal and the analysis of the design could then be done in the usual manner, as if no trend effect was present. Such designs are known as trend free designs. We now derive the necessary and sufficient condition for a design under two-way elimination of heterogeneity setup to be trend free.

Theorem 3.1 : A design under two-way elimination of heterogeneity setup and incorporating trend component is said to be trend free iff $\Delta Z = 0$, where the symbols have their usual meaning as defined earlier.

Proof : As defined in equation (4),

$X_2 = [1 \quad D'_1 \quad D'_2 \quad Z]$.

Let

$X_3 = [1 \quad D'_1 \quad D'_2\,]$.

We define,

$A_u = I_n - X_u\,(X'_u X_u)^{-} X'_u \quad (u = 2, 3)$

Then, $C_{u\tau} = \Delta A_u \Delta'$ (7)

Thus,

$$X_2'X_2 = \begin{bmatrix} n & q' & p' & 0 \\ q & Q & N_3 & 0 \\ p & N_3' & P & 0 \\ 0 & 0 & 0 & pI_p \end{bmatrix} \text{ and } X_3'X_3 = \begin{bmatrix} n & q' & p' \\ q & Q & N_3 \\ p & N_3' & P \end{bmatrix} \quad (8)$$

Therefore,

$$\begin{aligned} A_2 &= I - X_2\left(X_2'X_2\right)^{-} X_2' \\ &= I - D_1'Q^{-1}D_1 - D_1'Q^{-1}N_3E^{-1}N_3'Q^{-1}D_1 + D_2'E^{-1}N_3'Q^{-1}D_1 + D_1'Q^{-1}N_3E^{-1}D_2 - D_2'E^{-1}D_2'\ \frac{1}{p}\Delta ZZ'\Delta' \end{aligned} \quad (9)$$

and

$$\begin{aligned} A_3 &= I_n - X_3\left(X_3'X_3\right)^{-} X_3' \\ &= I - D_1'Q^{-1}D_1 - D_1'Q^{-1}N_3E^{-1}N_3'Q^{-1}D_1 + D_2'E^{-1}N_3'Q^{-1}D_1 + D_1'Q^{-1}N_3E^{-1}D_2 - D_2'E^{-1}D_2' \end{aligned} \quad (10)$$

Now, we first prove the necessary part. Let the design under two-way elimination of heterogeneity setup is a trend free design. Thus, we have to prove $\Delta Z = 0$. Let T_z and T_0 be the adjusted treatment sum of squares under Model (1) and under the usual model with two-way elimination of heterogeneity setup without trend effect respectively. Since the design is assumed to be trend free, thus by definition (2.1), we can write $T_z = T_0$ i.e.

$$Y'A_2\Delta'C^{-}_{2\tau}\Delta A_2Y = Y'A_3\Delta'C^{-}_{3\tau}\Delta A_3Y \quad (11)$$

Thus

$$A_2\Delta'C^{-}_{2\tau}\Delta A_2 = A_3\Delta'C^{-}_{3\tau}\Delta A_3$$

$$\Rightarrow \quad \Delta A_2'C^{-}_{2\tau}\Delta A_2 = \Delta A_3\Delta'C^{-}_{3\tau}\Delta A_3\Delta'$$

$$\Rightarrow \quad \Delta(A_2 - A_3)\,\Delta' = 0. \quad (12)$$

Substituting the value of A_2 and A_3 from Equation (9) and (10) into Equation (12) and then solving the corresponding equation we get $\Delta Z = 0$.

To prove the sufficiency, we assume that the condition given in the above theorem is true *i.e*. $\Delta Z = 0$.

Pre-multiplying and post-multiplying both sides of Equation (9) and (10) by Δ and Δ' respectively and using (7) we get : $C_{2\tau} = C_{3\tau}$ as $\Delta Z = 0$.

As, $C_{2\tau} = C_{3\tau}$, thus it is obvious that $T_z = T_0$. Hence, the condition given in the above theorem is both necessary and sufficient.

Corollary 3.1: For a trend free design under two-way elimination of heterogeneity setup, the information matrix for estimating the treatment effects with trend is same as the information matrix for estimating the treatment effects without trend component. Therefore, for trend free design under two-way elimination of heterogeneity setup, when the levels of all the factors are equal say v, the information matrix for estimating the contrast pertaining to the effect of treatments is:

$$C = v\left(I - \frac{11'}{v}\right)$$

4. TREND-FREE DESIGNS

In this section, a method for construction of linear trend free designs under two-way elimination of heterogeneity setup have been described. It is assumed that all the source of heterogeneity along with treatment are having same number of levels (say v) and there are evidences of linear trend component in the experimental material. Therefore, one can choose F as a $v \times 1$ vector with columns representing the (normalized) orthogonal polynomials and Z can be obtained based on F as defined earlier.

The method for obtaining a trend free designs under two-way elimination of heterogeneity setup is as follows:

Step 1: Obtain a Latin Square (LS) of order v and represent the symbols of the LS as treatment numbers.

Step 2: Choose another LS of order v from the set of MOLS of order v and represent the symbols by english alphabet.

Step 3: Superimpose the LS obtained in the step 2, on the LS obtained in step 1. The resulting design will be a trend free design under two-way elimination of heterogeneity set up where the two sources of heterogeneities are represented by row and alphabets in the design respectively.

For this class of designs,

$$R = vI_v,\ P = v\,I_v,\ Q = v\,I_v$$

$$N_1 = 11',\ N_2 = 11_2',\ N_3 = 11'$$

Example 4.1: For v = 4, following is a trend free designs under two-way elimination of heterogeneity setup:

1 A	2 D	3 B	4 C
2 B	1 C	4 A	3 D
3 C	4 B	1 D	2 A
4 D	3 A	2 C	1 B

Here, row indicate one source of heterogeneity, letters indicate another source of heterogeneity and the treatments are represented by numbers. For the above design, in order to estimate the linear trend effect, one can choose F as

$$F = \left[\frac{-3}{\sqrt{20}}\quad \frac{-1}{\sqrt{20}}\quad \frac{1}{\sqrt{20}}\quad \frac{3}{\sqrt{20}}\right]'$$

Since, the above design is a trend free design, therefore, the adjusted treatment sum of squares arising from the effects of treatments under the model (1) with trend component is same as the adjusted treatment sum of squares under the usual model without trend component.

5. Discussion

In agricultural, animal and allied experiments, where there may be evidences of two sources of heterogeneity apart from the treatment applied to the experimental material, response may also get affected by systematic trend. The effect of trend although remote, still may have high influence on response and hence should be incorporated in to the model for proper model specification. The trend-free design under two-way elimination of heterogeneity setup would nullify the effects of common trend effects and thus will increase the precision of the experiments.

ACKNOWLEDGEMENT

Authors are thankful to director, ICAR-Indian Agricultural Statistics Research Institute, Pusa, New Delhi for providing all kind of support while carrying out the work.

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