Chapter 27

LINEAR PROGRAMMING: CONCEPT AND ITS APPLICATION IN AGRICULTURE

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INTRODUCTION

Linear programming (LP) is a mathematical modeling technique designed to optimize (maximize or minimize) the usage of limited resources (Hadley, 1997). To define "LP is a mathematical technique of studying wherein we consider maximization (or minimization) of a linear expression (called the objective function) subjected to a number of linear equalities and inequalities (called linear restrictions)".

History and Application of LP

In 1939, during World War II, a Soviet economist Leonid Kantorovich used LP to plan expenditures and returns in order to reduce costs of the army and to increase losses incurred to the enemy.

LP has wide applications in various fields like military, industry, agriculture, transportation, health system, economics and behavioral sciences etc., and is also utilized for some engineering problems. Transportation, energy, telecommunications, and manufacturing are the major industries that use linear programming models. LP has proven useful in modeling diverse types of problems in planning, routing, scheduling, assignment, and design.

Assumptions of Linear Programming

1. The LP models are "***deterministic***" in nature: Assumes everything is certain and equation is mathematical in nature.
2. The LP models are "***proportional***" in nature: This condition follows directly from linearity assumptions for objective function and constraints. This means that the objective function and constraints expand and contract proportionately to the level of each activity. This condition represents constant returns to scale rather than economies or diseconomies of scale.
3. The LP models are "***additive***" in nature: That is the Left Hand Side (LHS) should be equal to Right Hand Side (RHS). The assumption of proportionality guarantees linearity if and only if the joint effects or interactions are non-existent. That means the total contribution of all activities is identical to sum of the constraints per each activity individually.
4. The decision variables are ***"divisible"***: That is the fractional levels for decision variables are permissible, the objective function and constraints are continuous function.

5. Non-negativity: The value of variables must be zero or positive but not negative.

So LP is a special case of mathematical programming to achieve the best outcome (such as maximum profit or minimum cost) in a mathematical model whose requirements are represented by linear relationships (Dorfman, 1996).

Example

Reddy Mikks (R-M) company produce both interior and exterior paints from two raw materials M_1 and M_2 (Taha, 2007). The following Table 1 provides the basic data of the problem.

Table 1: Data for the illustration

Particulars	**Tonnes of raw material required per tonne of**		**Maximum availability with R-M (tones)**
	Exterior paint	**Interior paint**	
Raw material M_1	6	4	24
Raw material M_2	1	2	6
Profit per tonne ($ 000's)	5	4	-

- The market survey restricts maximum daily demand of interior paints to 2 tonnes.
- Additionally the daily demand for interior paint cannot exceed that of exterior paint more than 1 tonne.
- The R-M company wants to determine the optimum product mix of interior and exterior paints that maximizes total daily profit.

Let us formulate the problem

LP model includes three basic elements

1) **Decision variables** that we seek to determine
   X_1=Production of exterior paints (in tonnes)
   X_2=Production of interior paints (in tonnes)
2) **Objective function** that we aim to optimize
   Main objective is to maximize total daily profit
   Let Z represents total daily profit (in $ 000's)
   Max $Z = 5X_1+4X_2$
3) **The constraints** that we need to satisfy

   a) Restriction on raw material usage: The usage of raw material for production of both paints should not exceed raw material availability.
      - **Usage of raw material M_1: $6X_1+4X_2 \leq 24$**
      - **Usage of raw material M_2: $X_1+2X_2 \leq 6$**

   b) Demand restrictions

- **Maximum daily demand of interior paint is limited to 2 tonnes: $X_2 \leq 2$**

- **Excess of daily production (**daily demand for interior paint cannot exceed that of exterior paint more than 1 tonne**):** $\mathbf{X_2 - X_1 \leq 1}$

So the LP model for above optimization problem looks like below

$$\text{Max } Z = 5X_1+4X_2$$

Subject to,

$$6X_1+4X_2 \leq 24$$
$$X_1+2X_2 \leq 6$$
$$X_2 \leq 2$$
$$X_2 - X_1 \leq 1$$
$$X_1 \ \& \ X_2 \geq 0$$

STANDARD FORM OF LP MODEL

To solve LP problem manually it must be put in a common form which we call as standard form.

Properties of standard form

- **All the constraints should be expressed as equations by adding slack or surplus and or artificial variables.**

A constraint of the type $\leq$ ($\geq$) can be converted to an equation by adding ***slack*** variable to (subtracting ***surplus*** variable from) the left side of the constraint.

Ex 1: $3X_1+2X_2 \leq 6$

$3X_1+2X_2 +S_1=6$, where S_1 is a slack variable represents the unused amount of resources

Ex 2: $2X_1+X_2 \geq 6$

$2X_1+X_2 - S_2=6$, where S_2 is a surplus variable represents the excess amount of resources

Note: The introduction of slack and surplus variables alters neither the nature of the constraint nor the objective function. Accordingly such variables are incorporated into objective function with zero coefficient.

- **The right hand side of each constraint should be made non-negative (if not).**

The RHS of the equation can always be made non-negative by multiplying both the sides by -1.

Ex: $2X_1+3X_2-7X_3=-5$ can be written as $-2X_1-3X_2+7X_3=5$

$2X_1-X_2 \leq -5$ can be written as $-2X_1+X_2 \geq 5$ (the direction of inequality is reversed when both sides are multiplied by -1).

- **The objective function must be maximization type**

Solving of maximization problem is easier than solving of minimization problem. So we can convert minimization form to maximization form for easy calculation and later we can interpret it as minimization solution. The maximization of a function is equivalent to minimization of a negative of the same function and vice-versa.

For a given set of constraints,

Max $Z=5X_1+2X_2+3X_3$ is mathematically **equivalent** to Min (-Z) = $-5X_1-2X_2-3X_3$.

Equivalence means that for the same set of constraints the optimal values of X_1, X_2 and X_3 are the same in both cases. The only difference is that the values of the objective function, although equal numerically, will appear with opposite signs

Table 2: General and standard form of LP model involving only less than or equal constraints (≤)

General form	Standard form
Max $Z = 5X_1+4X_2$ subject to; $6X_1+4X_2 \leq 24$ $X_1+2X_2 \leq 6$ $X_2 \leq 2$ $X_2 - X_1 \leq 1$ X_1 & $X_2 \geq 0$	Max $Z = 5X_1+4X_2+0S_1+0S_2+0S_3+0S_4$ subject to; $6X_1+4X_2 +S_1 =24$ $X_1+2X_2 + S_2 =6$ $X_2 +S_3=2$ $X_2 - X_1 + S_4=1$ X_1 , X_2, S_1, S_2, S_3 & $S_4 \geq 0$

Artificial Variable (AV)

In case of problems with infeasible solution artificially we introduce a variable into objective function to obtain feasible solution. We use AV only to start solution and subsequently force them to be zero in the solution otherwise the resulting solution will be infeasible. To guarantee such assignments in the optimal solution, AVs are incorporated into objective function with very large positive co-efficient in minimization problem or very large negative co-efficient in maximization problem.

AVs do change the nature of constraint since they are added only to one side of inequality. That is if the original constraint is an equation (=) or of the type greater than or equal to (≥), then we have no longer basic starting feasible solution.

Table 3: General and standard form of LP model involving all kind of constraints (≤, =, ≥)

General form	Standard form
Max $Z = 5X_1+2X_2$ subject to; $6X_1+X_2 \leq 6$ $4X_1+3X_2 \geq 12$ $X_1+X_2 =1$ X_1 & $X_2 \geq 0$	Max $Z = 5X_1+2X_2+0S_1+0S_2-MA_1-MA_2$ subject to; $6X_1+X_2 +S_1 =6$ $4X_1+3X_2 - S_2+A_1 =12$ $X_1+X_2 +A_2=2$ X_1 , X_2, S_1, S_2, A_1 & $A_2 \geq 0$

Solution to LP Problem

There are two approaches for solving LP problems.

1) Graphical approach and
2) Simplex technique

1) Graphical approach: LP problems which involve only two decision variables can be solved graphically. Since it is not possible to display the set of feasible solution for more than two variables in a graph for locating best optimal solution. There are two graphical solution methods namely, extreme point solution method and iso-profit (Cost) function line method. Of these, extreme point solution method is most commonly used method for solving LP problem involving two decision variables.

Extreme point solution method: Extreme point refers to corner of the feasible region i.e. the point lies at the intersection of two constraint equations. In this method, the co-ordinates of all corner or extreme points of the feasible region are determined and then value of the objective function at each of these points is computed and compared. The co-ordinates of an extreme point where the optimal (maximum or minimum) value of the objective function is found represent the solution of the given LP problem.

Example:

$$\text{Max } Z = 5X_1 + 7X_2$$

Subjected to,

$X_1 \leq 6$

$2X_1 + 3X_2 \leq 19$

$X_1 + X_2 \leq 8$

$X_1, X_2 \geq 0$

Solution:

Here we are not going to add any slack or surplus variable but we are just putting it into equation.

$X_1 = 6$

$2X_1 + 3X_2 = 19$

$X_1 + X_2 = 8$

$X_1 + X_2 = 8$ Extreme points: a) $X_1=0$ $X_2=8$, Co-ordinates:(0,8) b) $X_2=0$ $X_1=8$, Co-ordinates:(8,0)	$2X_1 + 3X_2 = 19$ Extreme points: a) $X_1=0$ $X_2=6.33$, Co-ordinates: (0,6.33) b) $X_2=0$ $X_1=9.5$, Co-ordinates: (9.5,0)	$X_1=6$ Extreme points: a) $X_1=6$, $X_2=0$, (6,0)

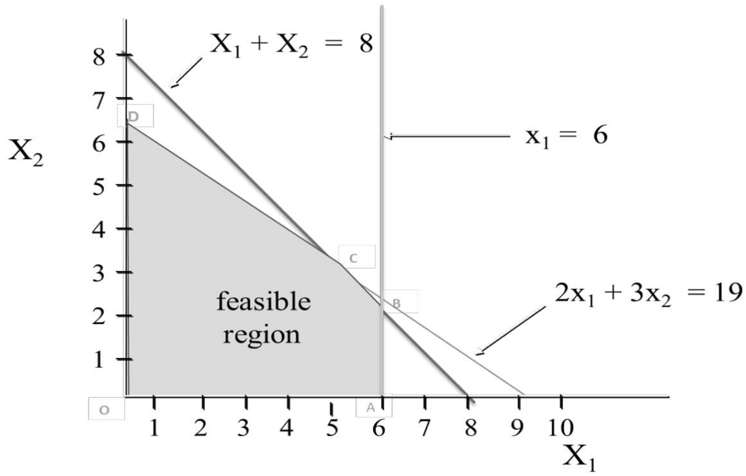

Fig 1: Combined-constraint graph showing feasible region

The shaded zone is called feasible area where all the constraints holds good or this region satisfies all constraints so it is called *feasible region* (Fig 1).

- The corners or vertices of the feasible region are referred to as the extreme points.
- An optimal solution to an LP Maximization problem can be found at an extreme point of the feasible region.
- When looking for the optimal solution, you do not have to evaluate all feasible solution points.
- Consider only the extreme points of the feasible region.

Table 4: Value of objective function at extreme points of feasible region

Extreme Point	Co-Ordinates	Z value ($Z=5X_1+7X_2$)
O	(0,0)	0
A	(6,0)	30
B	(6,2)	44
C	**(5,3)**	**46**
D	(0,6.33)	44.31

At point C all constraints are satisfied and the Z value is highest hence it is optimal point.

Solution: At $X_1=5$ and $X_2=3$, Max Z=46

2) Simplex method

It is an algorithm adopted to solve LP problem which employs an iterative procedure that starts at a feasible corner point, normally the origin and systematically moves from one feasible point to another point until it reaches optimum point.

Linear programming solvers are now part of many spreadsheet packages, such as Microsoft Excel. The leading commercial package is “LINDO”. We can solve LP problems in packages like “R” and “SAS” also.

Special Cases in Simplex Method of Application

1. Degeneracy

In case of model consisting of at least one redundant (No longer needed or not useful) constraint then the optimum value won't improve upon iterations instead same solution is generated over the iterations.

Example

$$\text{Max } Z = 3X_1+9X_2$$

Subject to,

$X_1+4X_2 \leq 8$

$X_1+2X_2 \leq 4$

$X_1 \& X_2 \geq 0$

In above case the first constraint is a redundant constraint.

2. Alternative optima

Alternative optima exists when objective function running parallel to one of the constraints. Then the objective function will assume same optimal value at more than one solution point.

Example

$$\text{Max } Z = 2X_1+4X_2$$

Subject to,

$X_1+2X_2 \leq 5$

$X_1+X_2 \leq 4$

$X_1 \& X_2 \geq 0$

In above case the objective function runs parallel to first constraint.

3. Unbounded solutions

The solution to a maximization LP problem is unbounded if the value of the solution may be made indefinitely large without violating any of the constraints. Sometimes feasible solution for the given LP problem exists and this has infinite values for the objective function. For real problems, this is the result of improper formulation.

4. Infeasible or non-existent solutions

No unique solution to the LP problem satisfies all the constraints, including the non-negativity conditions. Graphically, this means a feasible region does not exist. Causes includes formulation error, too high expectations by management or too many restrictions have been placed on the problem (i.e. the problem is over-constrained).

INTEGER PROGRAMMING

In case of linear programming, the decision variables considered are supposed to take any real value. However in practical situations it makes no sense in assigning a real value to a variable where it has meaning only when it takes only integer values (Rao, 2007). To be clear let us consider a practical problem like optimum size of herd in a dairy project, it makes no sense if our optimal value from LP solution is 5.8.

In such situations, we naturally tend to round-off the optimal value to the nearest integer value say "6" in above example. However, the round-off may have following fundamental problems,

a) The round-off solution may not be feasible.
b) The objective function value given by the rounded-off solutions (even if some are feasible) may not be the optimal one.
c) Even if some of the rounded-off solutions are optimal, checking all the rounded-off solutions is computationally expensive.

So integer programming deals with the solution of mathematical programming problems in which some or all the variables can assume non-negative integer values only.

Types of integer programming problems

1) Pure integer programming problem: An integer programming problem in which all variables are required to be integers.
2) Mixed integer programming problem: If some variables are restricted to be integer and some are not restricted i.e. can be continuous or fractional.
3) Binary integer programming problem/ 0-1 programming problems: If some or all variables are restricted to be either "0" or "1". It can be pure or mixed.

The general form of integer programme is as below:

$$\text{Max } Z = 7X_1+9X_2$$

subject to,

$-X_1+3X_2 \leq 6$

$7X_1+X_2 \leq 35$

X_1 & X_2 are non-negative integers.

Applications of Linear Programming in Agriculture

Case-1: Naidu Dairy farm uses at least 800 Kg of *Special feed* daily. The *Special feed* is a mixture of corn silage and soybean meal with the following composition.

Table 5: Constituents of special feed and their unit cost

Feed stuff	In terms of kg per every kg of feed stuff		
	Protein	Fiber	Cost (Rs./Kg)
Corn silage	0.09	0.02	20
Soybean meal	0.60	0.06	62

The dietary requirements of *Special feed* must have at least 30 per cent protein and at most 5 per cent fiber. Now the Naidu Dairy farm wishes to determine the daily minimum cost of feed mix?

Solution:

Decision variables:

X_1= Quantity of corn silage to be used in feed mix (Kg)

X_2=Quantity of soybean meal to be used in feed mix (Kg)

Objective function

$$\text{Min } Z=20X_1+62X_2$$

Constraints

Demand constraint (Daily requirement): $X_1+X_2 \geq 800$

Protein constraint: $0.09X_1 +0.60X_2 \geq 0.30(X_1+X_2)$ on simplification $-0.21\ X_1+0.30\ X_2 \geq 0$

Fiber constraint: $0.02X_1 +0.06X_2 \leq 0.05\ (X_1+X_2)$ on simplification $-0.03\ X_1+0.01\ X_2 \leq 0$

Overall the LP model looks like

$$\text{Min } Z=20X_1+62X_2$$

Subject to,

$X_1+X_2 \geq 800$

$-0.21\ X_1+0.30\ X_2 \geq 0$

$-0.03\ X_1+0.01\ X_2 \leq 0$

$X_1 \& X_2 \geq 0$

<u>R code for the above LP problem</u>

```
library(lpSolve)
obj=c(20,62)
mat=matrix(c(1,1,-0.21,0.3,-0.03,0.01), nrow=3, byrow=TRUE)
rhs=c(800,0,0)
```

```
dir=c(">=",">=","<=")
prod.sol= lp("min", obj, mat, dir, rhs, compute.sens = TRUE)
prod.sol$status
prod.sol$objval
prod.sol$solution
prod.sol$duals
prod.sol$duals.from
prod.sol$duals.to
prod.sol$sens.coef.from
prod.sol$sens.coef.to
```

A) Optimal solution

Z	29835.29
X1	470.58
X2	329.41

The daily minimum cost of feed mix by using 470.58 Kg of corn silage and 329.41 Kg of soybean meal is Rs. 29835.29.

B) Sensitivity analysis

a) Maximum change in resource availability (RHS of binding constraints)

Binding Constraint	Shadow price	RHS	Sensitivity (Range)
Special feed	37.29	800	0 to $1*10^{30}$
Protein	82.35	0	-168 to 138

b) Maximum change in marginal cost (Co-efficients of DV's in objective function)

Variable	Value of DV's	Unit price	Sensitivity (Range)
Corn silage (X1)	470.58	20	-43.40 to 62.00
Soybean meal (X2)	329.41	62	20 to $1*10^{30}$

Case 2: Venkatesh, a Crop+Dairy farming system based farmer wishes to maximize the total revenue with the available resources. The below table provides the information on the resource availability and the information on resource requirement for the enterprises from his past experience. Ragi being the regular diet of Venkatesh's family he needs minimum 1 acre of his land to be under the same which also serves the fodder security of his dairy. Since the dairy is earning him the regular income for family maintenance he insists at least one cross breed (CB) cow in his farming system.

Resources	Availability	Per unit requirement			
		Tomato	Cabbage	Ragi	CB Cow
Land (Acres)	4	-	-	-	-
Labour (Man days)	350	180	65	32	38
Capital (Rs.)	250000	125000	65000	12500	33000
Water (acre inches)	100	24.5	17.8	9.4	0.5
Returns (Rs.)	-	280000	135000	19000	65000

Solution:

Decision variables:

X_1= Area under Tomato crop to be taken (Acres)
X_2= Area under Cabbage crop to be taken (Acres)
X_3= Area under Ragi crop to be taken (Acres)
X_4= Number of cross breed cows to be considered in his farming system

Objective function

Max $Z=280000X_1+135000X_2+19000\,X_3+65000\,X_4$

Constraints

Land constraint (Overall): $X_1+X_2+X_3\leq 4$
Labour constraint: $180X_1+65X_2+32X_3+38X_4\leq 350$
Capital constraint: $125000X_1+65000X_2+12500X_3+33000X_4\leq 250000$
Water constraint: $24.5X_1+17.8X_2+9.4X_3+0.5X_4\leq 100$
Constraint for Ragi mandate: $X_3\geq 1$
Constraint for Dairy mandate: $X_4\geq 1$

Overall the LP model looks like

$$\text{Max } Z=280000X_1+135000X_2+19000\,X_3+65000\,X_4$$

Subject to,

$X_1+X_2+X_3\leq 4$
$180X_1+65X_2+32X_3+38X_4\leq 350$
$125000X_1+65000X_2+12500X_3+33000X_4\leq 250000$
$24.5X_1+17.8X_2+9.4X_3+0.5X_4\leq 100$
$X_3\geq 1$
$X_4\geq 1$
$X_1+X_2+X_3\geq 0$ & X_4 is a non-negative integer

R code for the above LP problem

```
library(lpSolve)
obj=c(280000,135000,19000,65000)
mat=matrix(c(1,1,1,0,180,65,32,38,125000,65000,12500,33000,24.5,17.8,9.4,
0.5,0,0,1,0,0,0,0,1), nrow=6, byrow=TRUE)
rhs=c(4,350,250000,100,1,1)
dir=c("<=","<=","<=","<=",">=",">=")
```

```
prod.sol= lp("max", obj, mat, dir, rhs, int.vec=4, compute.sens = TRUE)
prod.sol$status
prod.sol$objval
prod.sol$solution
prod.sol$duals
prod.sol$duals.from
prod.sol$duals.to
prod.sol$sens.coef.from
prod.sol$sens.coef.to
```

A) Optimal solution

Z	536713.28
X1	1.37
X2	0.50
X3	1
X4	1

The maximum total revenue that the farmer can achieve is Rs. 5,36,713.3/- by cultivating Tomato, Cabbage and Ragi in 1.37, 0.50 and 1 acre respectively, along with 1 CB cow.

B) Sensitivity analysis

a) Maximum change in resource availability (RHS of binding constraints)

Binding Constraint	Shadow price	RHS	Sensitivity (Range)
Labour	370.62	350	283.20 to 364.48
Capital	1.70	250000	239944.4 to 284847.8
Ragi	-14188.81	1	0 to 1.98
CB cow	-5391.60	1	0 to 2.52

b) Maximum change in marginal profit (Co-efficients of DV's in objective function)

Variable	Value of DV's	Unit price	Sensitivity (Range)
Tomato (X1)	1.37	280000	259615.4 to 373846.15
Cabbage (X2)	0.50	135000	118802.5 to 145600
Ragi (X3)	1	19000	$-1*10^{30}$ to 33188.81
CB cow (X4)	1	65000	$-1*10^{30}$ to 70391.61

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Chapter 28

MULTI-OBJECTIVE PROGRAMMING

Chandra Sen

INTRODUCTION

Multi-objective programming (MOP) is the field of applied mathematics that studies problems of achieving multiple conflicting objectives over a domain of mathematical relations. Multi-objective programming is also termed as multi-objective optimization, vector optimization, multi-criteria optimization, multi-attribute optimization or Pareto optimization. The MOP methods are increasingly used for making decisions in achieving multiple conflicting objectives simultaneously. Many a times, the decision maker fails to make good decisions under a complex situation. A natural explanation for such a situation is that the applied mathematics is not sufficiently realistic to incorporate all the criteria and constraints. Several MOP methods have been developed to help in such a complex situation. These methods are getting more and more importance in applications in engineering, agriculture and management.

Multi-objective programming methods

$\mathcal{E}$-Constraint method

The method was introduced by Haimes *et al.* (1971). In this method the one objective is selected for max./min. keeping remaining objectives within user specified values. The problem is formulated as:

$$\text{Max./Min. } Z_r(x)$$

Subject to,

$$Z_m(x) \leq / \geq / = \mathcal{E}_m \qquad n=1,...N \quad r \neq n$$
$$a_i x_i \leq / \geq / = b_i \qquad i= 1.....K$$

The choice of the objective to be optimized and the values of the remaining objectives kept as constraints are subjective. Hence, the solution may not be more acceptable.

Lexicographic method

In Lexicographic method, the objective functions are arranged in order of importance and each objective is optimized as per its priority. The optimal value of first objective is put as constraint for achieving the second objective. The process ends in optimizing the last objective. The method can be explained as:

$$\text{Max./Min. } Z_r(x)$$

Subject to,

$Z_m (x) \leq / \geq / = \mathcal{E}_m \quad n=1,...N \quad r \neq n$

$a_i x_i \leq / \geq / = b_i \quad i= 1.....K$

The solution obtained using this method is also not much preferable because of the subjective prioritization the objectives.

Weighted-sum method

The weighted sum method was suggested by Gass and Satty (1955). This is the most commonly used in solving multi-objective optimization problems. The method is described as:

$$\text{Max./ Min. } \sum_{j=1}^{k} wjZ\text{j}$$

Subject to,

$AX = b$ and $X \geq 0$

Where wj is the weight assigned to j^{th} objective function.

It is very easy to formulate and optimize the combined objective function. However the method cannot generate efficient solution always due to following limitations:

(i) The weights (w_j) assigned to different objectives are subjective.

(ii) In most of the MOP applications, the objectives are non commensurable. Hence the addition of the values of different dimensions is not logical.

(iii) The method becomes biased to the dominating objective/s.

Scalarizing method

Realizing the above mentioned problems the scalarizing method for solving MOP problems was proposed by Sen (1982). The method is described as below:

$$\text{Maximize } Z = \frac{\sum_{j=1}^{r} Z\text{j}}{|\theta_r|} - \frac{\sum_{j=r+1}^{s} Z\text{j}}{|\theta_s|}$$

Subject to,

$AX = b$ and $X \geq 0$

$\Theta_r \;\&\; \Theta_s \neq 0$

Where,

Θ_r and Θ_j is the optimal value of r^{th} and j^{th} objective function and there are 'r' maximization and's-r' are minimization objective functions.

The method has been successfully applied in resource use planning in agriculture (Sen, 1982; Sen, 1983; Sen and Dubey, 1994; Singh, 2005; Kumar, 2012; Gautam, 2013; Kumari *et al.,* 2017; Maurya, 2018; Gwandi *et al.,* 2019 etc.) for increasing income and employment with lesser use of fertilizer, irrigation, environment pollution etc. The application of the method can be understood with following example:

Example 1

$$\text{Max.}Z_1 = 7000X_1 + 10000X_2 + 4000X_3 + 5000X_4 + 9000X_5$$
$$\text{Max. } Z_2 = 89X_1 + 50X_2 + 100X_3 + 79X_4 + 95X_5$$
$$\text{Min. } Z_3 = 10X_1 + 15X_2 + 14X_3 + 13X_4 + 12X_5$$
$$\text{Min. } Z_4 = 150X_1 + 120X_2 + 140X_3 + 100X_4 + 130X_5$$

Subject to,

$X_1 + X_2 + X_3 + X_4 + X_5 = 7$ (Land restriction)

$X_1 \geq 0.7$ (Home requirement)

$X_2 \geq 0.3$ (Home requirement)

$X_3 \geq 0.4$ (Home requirement)

$X_5 \geq 0.6$ (Home requirement)

where $X_{1,...}X_5$ are crops, $_{Z1}$ = Income (Rs.), Z_2 = Employment (man days), Z_3 = Plant Protection chemicals (litre) and Z_4 = Fertilizer use (kg.).

Solution

All the objectives have been optimized individually to find the optimal values of each objective. The multi-objective function was formulated as explained above. Finally the scalarized multi-objective function was optimized with the same common constraints. The results are presented in Table 1.

Table 1: Individual and multi-objective optimization

Item	Individual Optimization				Sen's MOP
	Max. Z_1	Max. Z_2	Min. Z_3	Min. Z_4	
X_i	X1=0.7, X2=5.3, X3=0.4, X4 =0, X5=0.6	X1=0.7, X2=0.3, X3=5.4, X4 =0, X5=0.6	X1=5.7, X2=0.3, X3=0.4, X4 =0, X5=0.6	X1=0.7, X2=0.3, X3=0.4, X4 =5.0 X5=0.6	X1=0.7, X2=0.3, X3=0.4, X4 =0, X5=5.6
Z_1	64900	34900	49900	39900	59900
Z_2	424.3	674.3	619.3	569.3	649
Z_3	99.3	94.3	74.3	89.3	84.3
Z_4	875	975	1025	775	925

All the individual optimizations have generated different solutions. This reveals the presence of conflicts among objectives. The maximization of income increased the income Rs. 64900 which is highest in comparison to other individual optimal solutions. The similar pattern have been observed in all the individual optimizations. However the Sen's MOP method has generated a compromising solution achieving all the objective simultaneously. The income can be increased up to Rs. 59900 which is not as high as in individual optimization but better as remaining individual optimizations. The achievements of remaining objective are also superior over individual optimizations.

Averaging method

Several averaging methods for solving MOP problems have been proposed (Sulaiman *et al.,* 2013; Nejamudin *et al.,* 2016; Akhtar *et al.,* 2017; Samsun and Abdul, 2017) during past two decades. The combined objective function was formulated by scalarizing the objective functions by various averages. The averaging methods are formulated as:

$$\text{Maximize } Z = \frac{\sum_{j=1}^{r} Zj}{|\Theta_1|} - \frac{\sum_{j=r+1}^{s} Zj}{|\Theta_2|}$$

Subject to,

$AX = b$ and $X \geq 0$

$\Theta_j \neq 0$ for J=1, 2..........s.

where

$|\Theta_1|$ = Mean, Geometric Mean and Harmonic Mean of optimal values of maximization objective functions.

$|\Theta_2|$ = Mean, Geometric Mean and Harmonic Mean of optimal values of Minimization Objective functions.

The averaging methods for solving MOP problems are not efficient (Chandra 2018a, Chandra 2018b, Chandra 2018c, Chandra 2019) in providing the appropriate solutions due the limitations with these methods similar to weighted-sum methods. These methods have been applied to solve the MOP problem of example 2.

Example 2

$\text{Max. } Z_1 = 12500X_1 + 25100X_2 + 16700X_3 + 23300X_4 + 20200X_5$

$\text{Max. } Z_2 = 21X_1 + 15X_2 + 13X_3 + 17X_4 + 11X_5$

$\text{Min. } Z_3 = 370X_1 + 280X_2 + 350X_3 + 270X_4 + 240X_5$

$\text{Min. } Z_4 = 1930X_1 + 1790X_2 + 1520X_3 + 1690X_4 + 1720X_5$

Subject to,

$X_1 + X_2 + X_3 + X_4 + X_5 = 4.5$

$2X_1 \geq 1.0$

$3X_4 \geq 1.5$

All the objectives of the above example have been optimized individually to examine the presence conflicts among the objectives. The solution of individual optimization is given in Table 2.

Table 2: Individual optimization matrix

Item	Indivaidual Optimization			
	Max. Z_1	Max. Z_2	Min. Z_3	Min. Z_4
X_i	0.5, 3.5, 0, 0.5, 0	4, 0, 0, 0.5, 0	0.5, 0, 0, 0.5, 3.5	0.5, 0, 3.5, 0.5, 0
Z_1	105750	61650	88600	76350
Z_2	71.5	92.5	57.5	64.5
Z_3	1300	1615	1160	1545
Z_4	8075	8565	7830	7130

It is clear from Table 2 that all the four objective have different and conflicting solutions. The above example has been solved using averaging methods. The Mean, Geometric mean and Harmonic mean have been used for scalarizing the objective functions. The results are presented in Table 3.

Table 3: Multi-objective optimization

Item	Existing Averaging Techniques			Improved Averaging Techniques		
	Mean	Harmonic Mean	Geometric Mean	Mean	Harmonic Mean	Geometric Mean
Xi	0.5, 3.5, 0, 0.5, 0	0.5, 3.5, 0, 0.5, 0	0.5, 3.5, 0, 0.5, 0	0.5, 0, 0, 4, 0	0.5, 0, 0, 4, 0	0.5, 0, 0, 4, 0
Z_1	105750	105750	105750	99450	99450	99450
Z_2	71.5	71.5	71.5	78.5	78.5	78.5
Z_3	1300	1300	1300	1265	1265	1265
Z_4	8075	8075	8075	7725	7725	7725

The results in Table 3 revealed that all the averaging methods have achieved the first objective only and ignored the remaining three objectives. None of these methods optimizes all the objectives simultaneously.

Improved Averaging Method

The problems with existing averaging methods have been realized and improved averaging methods for solving MOP problems have been proposed (Chandra, 2019). The improved averaging method is explained as:

$$\text{Maximize } Z = \frac{\sum_{j=1}^{r} Z\text{j}}{|\theta_i|} - \frac{\sum_{j=r+1}^{s} Z\text{j}}{|\theta_j|}$$

Subject to,

$AX = b$ and $X \geq 0$

$\theta_j \neq 0$ for J=1, 2..........s.

where

$|\theta_i|$ = Mean, Geometric Mean and Harmonic Mean of optimal and suboptimal values of i^{th} Maximization Objective function.

$|\theta_j|$ = Mean, Geometric Mean and Harmonic Mean of optimal and suboptimal values of J^{th} Minimization Objective function.

The improved averaging methods have also been applied to solve example 2 and the results are presented in Table 3. The improved averaging methods have achieved all the objectives simultaneously. The values of all the objectives are closer to their individual optima. Hence, the improved averaging methods are efficient in solving the MOP problems.

Several methods for solving MOP problems have been discussed in the chapter. There is continuous improvements in the existing methods for obtaining acceptable solutions for achieving multiple objectives at a time. It can be concluded that improved averaging methods are superior over other available methods for solving MOP problems.

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