# त्रि-पथ एवं चार-पथ क्रॉस प्रयोगों के लिए सांख्यिकीय तकनीकें

# Statistical Techniques for Triallel and Tetra-allele Cross Experiments

A Thesis

by

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# Statistical Techniques for Triallel and Tetra-allele Cross Experiments

by


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## CERTIFICATE

This is to certify that the work incorporated in the thesis entitled **Statistical Techniques for Triallel and Tetra-allele Cross Experiments** submitted in partial fulfillment of the requirement for the degree of **Doctorate of Philosophy** in **Agricultural Statistics** of the **Post-Graduate School, Indian Agricultural Research Institute, New Delhi,** is a record of *bona fide* research carried out by **Mohd Harun (Roll No-10412)** under my guidance and supervision and no part of this dissertation has been submitted for any other degree or diploma.

All assistance and help received during the course of this investigation has been duly acknowledged.

Cini Varghese

*New Delhi*
Date: 21.10.2020

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# Chapter-1 INTRODUCTION

## 1.1 Introduction

Breeding techniques are used as a tool for the development of commercial hybrids for which a major objective of plant and animal breeders is to raise the genetic potential. Any breeding experiment centres on acquiring information regarding the general combining ability (gca) effects of the individual lines involved as parents and the specific combining ability (sca) effects of the crosses. The information collected on gca and sca forms a basis of making correct choice of the best parental lines.

One of the most common and rigorously used breeding techniques is mating design, of which diallel crosses are the simplest and easily manageable. However, higher order crosses like triallel and tetra-allele cross based hybrids are genetically more viable, stable and consistent in performance than diallel cross hybrids. Triallel and tetra-allele cross hybrids have wider genetic base which gives them strong buffering mechanism as individual or when constituting a population.

There are many cases of crops (like maize or corn) and animals (like swine and chicken) where triallel and tetra-allele crosses are the commonly used breeding techniques of producing commercial hybrids (Shunmuguthai and Srinivasan, 2012). Triallel crossbred chickens show better egg traits than diallel crossbred chickens and are also having lower mortality (Khawaja *et al.*, 2013). Triallel and tetra-allele crossing scheme is very much acceptable and practiced in pig farming. The resultant product is also economical and of good quality. The silkworm production industry is also practicing the triallel and tetra-allele crosses for exploitation of heterosis.

## 1.2 Diallel cross

Diallel cross, also known as two-way or two-line or single cross, is a type of mating design which involves crossing between two parental lines to produce an offspring. A common diallel cross involving two inbred lines A and B can be symbolically represented in many ways like (A×B) or (A, B) or (A B). Diallel crosses can be categorized into complete diallel crosses (CDC) and partial diallel crosses (PDC).

### 1.2.1 Complete diallel cross

A CDC can be defined as the set of all possible matings between several genotypes (individuals, clones, homozygous lines, *etc*). The CDC among $N$ lines gives rise to $N^2$ progenies which can be further divided into three categories. They are crosses among $N$ inbred lines, a set of $\frac{N(N-1)}{2}$ $F_1$ hybrids and a set of $\frac{N(N-1)}{2}$ $F_1$ reciprocal hybrids. Depending upon the group that is considered, there are four models known as Griffing's models (I, II, III and IV):

I. Including all the $N^2$ possibilities
II. Including N parents and $\frac{N(N-1)}{2}$ $F_1$ hybrids
III. Including $\frac{N(N-1)}{2}$ $F_1$ hybrids and $\frac{N(N-1)}{2}$ $F_1$ reciprocal hybrids
IV. Including $\frac{N(N-1)}{2}$ $F_1$ hybrids only

Analysis of diallel crosses is described in Griffing (1956 a, b).

An example of complete diallel cross (Method I of Griffing) consisting of 36 crosses that can be made from 6 lines (U, V, W, X, Y and Z) is given below:

| | | | | | |
|---|---|---|---|---|---|
| (U×U) | (U×V) | (U×W) | (U×X) | (U×Y) | (U×Z) |
| (V×U) | (V×V) | (V×W) | (V×X) | (V×Y) | (V×Z) |
| (W×U) | (W×V) | (W×W) | (W×X) | (W×Y) | (W×Z) |
| (X×U) | (X×V) | (X×W) | (X×X) | (X×Y) | (X×Z) |
| (Y×U) | (Y×V) | (Y×W) | (Y×X) | (Y×Y) | (Y×Z) |
| (Z×U) | (Z×V) | (Z×W) | (Z×X) | (Z×Y) | (Z×Z) |

### 1.2.2 Partial diallel cross

Even if one excludes the $\frac{N(N-1)}{2}$ reciprocal crosses and $N$ parental inbreds then also there are $\frac{N(N-1)}{2}$ crosses consisting of a diallel among a set of $N$ lines. This number of crosses is directly related with the number of lines and hence increases rapidly as $N$ increases. For example, when there are $N$ = 4 lines, there are only 6 crosses constituting the diallel but when $N$ = 10, the number of crosses becomes 45. With scarce resources, a full diallel cross may not be possible when the number of inbred lines is large. As an alternate if one chooses not to include all the inbred lines then it may not give fruitful results as the randomly excluded lines may constitute the potent set. Thus, it is not advisable to leave any line out of the experiment. In this situation, a small sample taken from a complete diallel cross, which has at least that minimum representation, can be used to estimate the gca effects of all inbred lines. Such a fraction of CDC is known as PDC.

An example of PDC plan consisting of 36 crosses is given below for 12 lines (A, B,…, L).

| | | | | | |
|---|---|---|---|---|---|
| (A×E) | (A×F) | (A×H) | (A×I) | (A×K) | (A×L) |
| (B×D) | (B×I) | (B×J) | (B×F) | (B×I) | (B×L) |
| (C×D) | (C×I) | (C×J) | (C×E) | (C×H) | (C×K) |
| (D×H) | (D×K) | (D×I) | (D×L) | (E×G) | (E×J) |
| (E×I) | (E×L) | (F×G) | (F×J) | (F×H) | (F×K) |
| (G×K) | (G×L) | (G×J) | (G×L) | (H×J) | (H×K) |

### 1.3 Triallel cross

Triallel crosses, often referred as three-way crosses, are those type of mating designs in which each cross is obtained by crossing three inbred lines. A triallel cross can be obtained by crossing the resultant of a diallel cross with an unrelated inbred line. A common triallel cross involving three inbred lines A, B and C can be symbolically represented as (A×B)×C or (A, B, C) or simply (A B C). Unlike diallel cross, the three lines involved in the triallel cross do not contribute equally and thus, it is important to differentiate amongst them. The two lines A and B which are used first to produce a diallel cross contribute half as much as that of the third line C used to obtain the triallel cross. Hence, lines A and B are also referred as half parents whereas line C as full parent. Triallel crosses can be broadly categorized as complete triallel cross ($CT_rC$) and partial triallel crosses ($PT_rC$).

#### 1.3.1 Complete triallel cross

The set of all possible three-way matings between several genotypes (individuals, clones, homozygous lines, *etc*) leads to a $CT_rC$. If there are $N$ number of inbred lines involved in a $CT_rC$, the total number of crosses is $T = \frac{N(N-1)(N-2)}{2}$. Here is an example of complete triallel cross consisting of 30 crosses that can be made for 5 lines (P, Q, R, S and T).

| | | |
|---|---|---|
| (P×Q)×R | (P×R)×Q | (Q×R)×P |
| (P×Q)×S | (P×S)×Q | (Q×S)×P |
| (P×Q)×T | (P×T)×Q | (Q×T)×P |
| (P×R)×S | (P×S)×R | (R×S)×P |
| (P×R)×T | (P×T)×R | (R×T)×P |
| (P×S)×T | (P×T)×S | (S×T)×P |
| (Q×R)×S | (Q×S)×R | (R×S)×Q |
| (Q×R)×T | (Q×T)×R | (R×T)×Q |
| (Q×S)×T | (Q×T)×S | (S×T)×Q |
| (R×S)×T | (R×T)×S | (S×T)×R |

### 1.3.2 Partial triallel cross

When the number of lines increases, the total number of crosses in $CT_rC$ also increases. It is almost impossible for the investigator to handle it with limited available resources. This situation lies in taking a fraction of $CT_rC$ with certain underlying properties, known as $PT_rC$. In 1965, Hinkelmann defined $PT_rC$ as a set of triallel matings in which every line occurs $r_h$ and $r_f$ times as half-parent and full-parent, respectively and each cross of the type $(i \times j) \times k$ {alongwith $(i \times k) \times j$ and $(j \times k) \times i$, to maintain the Structural Symmetry Property (SSP)} occurs either once or not at all. The total number of crosses is $N$ times $r_f$ . Here is an example of $PT_rC$ consisting of 63 crosses that can be made for 7 lines (T, U, V, W, X, Y and Z) with $r_f$ = 9, $r_h$ = 18, $f$ = 3/5 (Note that, the degree of fractionation $f$ is defined as the ratio of crosses in a $PT_rC$ to a $CT_rC$ for a given number of lines):

| | | | | | |
|---|---|---|---|---|---|
| (T×U)×V | (T×V)×U | (U×V)×T | (T×U)×X | (T×X)×U | (U×X)×T |
| (T×U)×Z | (T×Z)×U | (U×Z)×T | (T×V)×X | (T×X)×V | (V×X)×T |
| (T×V)×Y | (T×Y)×V | (V×Y)×T | (T×W)×X | (T×X)×W | (W×X)×T |
| (T×W)×Y | (T×Y)×W | (W×Y)×T | (T×W)×Z | (T×Z)×W | (W×Z)×T |
| (T×Y)×Z | (T×Z)×Y | (Y×Z)×T | (U×V)×W | (U×W)×V | (V×W)×U |
| (U×V)×Y | (U×Y)×V | (V×Y)×U | (U×W)×Y | (U×Y)×W | (W×Y)×U |
| (U×W)×Z | (U×Z)×W | (W×Z)×U | (U×X)×Y | (U×Y)×X | (X×Y)×U |
| (U×X)×Z | (U×Z)×X | (X×Z)×U | (V×W)×X | (V×X)×W | (W×X)×V |
| (V×W)×Z | (V×Z)×W | (W×Z)×V | (V×X)×Z | (V×Z)×X | (X×Z)×V |
| (V×Y)×Z | (V×Z)×Y | (Y×Z)×V | (W×X)×Y | (W×Y)×X | (X×Y)×W |
| (X×Y)×Z | (X×Z)×Y | (Y×Z)×X | | | |

### 1.4 Tetra-allele cross

Tetra-allele cross often referred as four-way cross or double cross or four-line cross are those type of mating designs in which every cross is obtained by mating amongst four inbred lines. A tetra-allele cross can be obtained by crossing the resultant of two unrelated diallel crosses. A common triallel cross involving four inbred lines A, B, C and D can be symbolically represented as (A×B)×(C×D) or (A, B, C, D) or (A B C D) *etc*. Tetra-allele cross can be broadly categorized as Complete Tetra-allele Cross ($CT_eC$) and Partial Tetra-allele Crosses ($PT_eC$).

#### 1.4.1 Complete tetra-allele cross

The set of all possible four-way mating between several genotypes (individuals, clones, homozygous lines, *etc.*) leads to a $CT_eC$. If there are $N$ number of inbred lines involved in a $CT_eC$, the the total number of crosses, $T = \frac{N(N-1)(N-2)(N-3)}{8}$. Here is an example of complete tetra-allele cross consisting of 15 crosses that can be made for 5 lines (P, Q, R, S and T).

| | | | | |
|---|---|---|---|---|
| (P×Q)×(R×S) | (P×R)×(S×T) | (P×S)×(Q×T) | (P×T)×(R×S) | (Q×T)×(R×S) |
| (P×Q)×(R×T) | (P×R)×(Q×T) | (P×S)×(R×T) | (P×T)×(Q×R) | (S×T)×(Q×R) |
| (P×Q)×(S×T) | (P×R)×(Q×S) | (P×S)×(Q×R) | (P×T)×(Q×S) | (Q×S)×(R×T) |

#### 1.4.2 Partial tetra-allele cross

When more number of lines are to be considered, the total number of crosses in $CT_eC$ also increases. Thus, it is almost impossible for the investigator to carry out the experimentation with limited available resource material. This situation lies in taking a fraction of $CT_eC$ with certain underlying properties, known as $PT_eC$. An example of $PT_eC$ consisting of 21 crosses that can be made for 7 lines (T, U, V, W, X, Y and Z) with $f$ = 1/5 is given below:

| | | |
|---|---|---|
| (T×U)×(V×W) | (T×V)×(X×Z) | (T×W)×(Z×V) |
| (U×V)×(W×X) | (U×W)×(Y×T) | (U×X)×(T×W) |
| (V×W)×(X×Y) | (V×X)×(Z×U) | (V×Y)×(U×X) |
| (W×X)×(Y×Z) | (W×Y)×(T×V) | (W×Z)×(V×Y) |
| (X×Y)×(Z×T) | (X×Z)×(U×W) | (X×T)×(W×Z) |
| (Y×Z)×(T×U) | (Y×T)×(V×X) | (Y×U)×(X×T) |
| (Z×T)×(U×V) | (Z×U)×(W×Y) | (Z×V)×(Y×U) |

### 1.5 Triallel and tetra-allele cross plans incorporating sca effects

Diallel cross is the simple and easily manageable mating design. However, triallel and tetra-allele cross based hybrids are found to be genetically more viable, stable and consistent in performance. These techniques help the breeders to improve the quantitative traits which are of economical as well as nutritional importance in crops and animals. Triallel and tetra-allele cross hybrids are more stable than pure lines as well as compared to diallel cross hybrids because they are having strong buffer mechanism because of the wider genetic base. Further, the sca effects are also of much importance for breeders besides the gca effects. In case of diallel crosses, only a single first order sca effect can be studied whereas in case of triallel crosses and tetra-allel crosses, second and even third order sca effects can be studied.

### 1.6 Random effects model approach

Most of the studies involving mating designs are done presuming the assumptions of a standard linear model where the gca effects are considered to be fixed in nature. But, there are situations where the idea of taking gca effects as fixed may not be justifiable. Consider a situation of dairy farm, where instead of mating with own bulls to the owned cows, the cows are inseminated by calling a technician. The technician is having bull semen from an esteemed artificial breeding industry. The industry owns their bulls that generally sire daughter cows that are high-yielding. It can be achieved by each year buying some young bulls that are considered to be a random sample from the population of bulls. As the bulls are considered random, their effect is also a random effect. When a sample of inbred lines from a possibly large hypothetical population is considered to predict the gca effects or to estimate the variance components and the variances of the obtained estimates, considering the gca effects as random effects is advisable.

### 1.7 Prediction of combining ability effects

There are many situations where one wants to quantify the realization of an unobservable random variable. Consider an example of measuring intelligence in human beings. Each one has some intelligence level, usually quantified and referred as IQ. It can never be measured exactly and thus some sort of test scores is used as a substitute for giving a value to anybody’s IQ. An example from breeding sector is that of predicting the genetic merit of a dairy bull from the milk yield capacity of his daughters. Same is the situation for predicting the yielding capacity of the cross from the sample of inbred lines. In all these examples, from observations on some random variables the value of some other related random variables that cannot be observed are to be predicted. The concept of Best Linear Unbiased Predictor (BLUP) is used for unbiased prediction of the yielding capacity of the cross from the sample of inbred lines under mixed effects model.

### 1.8 Variance component estimates

Both plant and animal breeders are interested to increase the production of economically important products from farm animals (e.g., eggs, milk, butter, wool, meat *etc*.), or plants. One way of achieving this objective is to use BLUP of the unobserved line effects for ranking the value of inbred lines which will increase the productivity of future generation. But the prediction of line effects depends on good estimates of variance components related to line effects and error. Thus, variance components are of much interest for breeders. Besides this, the genetic parameter heritability on which the breeding policies depends is also a function of variance components. Thus, under a random effect model for breeding experiments, assuming the gca effects as random, the variance components related to line and error effects can be estimated. Henderson Method III, which incorporates both fixed and random effect components in the model, can be adopted for this study and unbiased estimate of different variance components can be obtained.

### 1.9 Designs for breeding trials

Statistical techniques related to designing and analyses of experiments are uninterruptedly used by the breeders and are having indispensable role while going for any breeding related study or experimentation. Breeding programmes are conducted using suitable designed experiments. When the size of experiment is large, then in order to handle the variability in the experimental material it is necessary to group it in various categories which are maximum possible homogenous within themselves and heterogeneous to each other. This concept leads to blocking concept in the field of design of experiments. Higher order crosses, like triallel and tetra-allele cross experiments involve larger number of crosses which makes it difficult to handle as a single group. Even if we use block designs, it may be possible that a complete block is of very large in size and thus becomes heterogeneous, which is not acceptable. In this situation incomplete block designs with smaller block sizes are desirable. Hence, it is important to have small and efficient designs for mating designs of higher order.

### 1.10 Robust design for breeding experiments

An optimal or efficient design may not remain so and may become disconnected and all the contrasts pertaining to combining ability effects may not be estimable or may become inefficient if the underlying assumptions are not fulfilled. Hence, it is much important that the design used is robust against such disturbances like missing observation(s), outlying observation(s), exchange or interchange of crosses, inadequacy of assumed model, *etc*. In case of breeding experiments the loss of observation is much prevalent because an observation may not sprout or may not survive till the time of measurement. Besides this, any human error regarding tagging may also result in loss of observation. Robust designs involving triallel and tetra-allele crosses against missing observation would be obtained using the robustness criteria of connectedness and efficiency. These connected and efficient designs will be helpful for the breeders to estimate the gca effects of lines even if an observation is missing.

### 1.11 Motivation and objectives

Higher order crosses like triallel and tetra-allele cross based hybrids are not frequently used by the breeders as they require extra resources as compared to diallel crosses. But, triallel and tetra-allele crosses can provide information on gca and sca effects which may not be captured by diallel crosses. Majority of the work on mating-environmental designs has been done in diallel and ignoring the sca effects. Apart from inferring about gca effects often breeders are interested in getting information on gca effects after adjusting for sca effects, which is possible only if sca effects are also included in the model. Developing general methods for constructing designs will not only attract the breeders to use them but the information obtained on the higher order sca along with gca effects may also help in developing hybrids with important traits.

The use of mixed effects model is more reliable because only a sample of inbred lines from a possibly large hypothetical population is considered to predict the gca effects or to estimate the variance components and the variances of the obtained estimates in triallel or tetra-allele cross hybrids. The concept of BLUP would be helpful for unbiased prediction of the yielding capacity of the cross from the sample of inbred lines under mixed effects model.

Estimation of genetic components of variances is important in plant and animal breeding experiments and can be used further for the estimation of heritability. Estimation of the variance components and variances of the estimates under mixed effects model is important as the BLUP of the unobserved line effects depends on these estimates.

Since, large number of crosses is to be made in case of higher order crosses, there is a chance that some human error may creep in due to loss of tags or labels or some of the observations may be lost after making the crosses. Robust designs involving triallel and tetra-allele crosses would help the breeders to get information on gca effects from their experiments without much loss of precision, even if an observation is missing. A catalogue of robust designs using the concept of connectedness and efficiency criteria would be beneficial to the experimenters to choose a robust design.

Keeping these points in view following objectives have been considered:

- To obtain efficient triallel and tetra-allele cross designs including sca effects in the model
- To obtain Best Linear Unbiased Predictor (BLUP) for predicting the unobserved combining ability effects together with general mean effect in triallel and (or) tetra-allele cross designs
- To obtain estimates of variance components under mixed effects triallel and (or) tetra-allele cross model
- To study the robustness of designs involving triallel and tetra-allele crosses against missing observation(s)

### 1.12 Scope of the present investigation

Mating designs find an important place in the area of research done in the field of Genetics as well as Statistics. A large number of breeders and statisticians working in this area and volume of research work published in this area show the importance of this topic. In the present investigation, an attempt has been made to supplement and carry forward the work done in the area of designs involving higher order crosses, *i.e.*, triallel and tetra-allele crosses. A general introduction to the topic is given in this chapter.

In Chapter II of the thesis, a critical review of the research work related to triallel and tetra-allele cross designs incorporating specific combining ability effects, prediction of combining ability effects, variance components estimate and robust design for breeding experiments has been given.

Chapter III focuses on various existing fixed effects and random effects models for triallel and tetra-allele cross experiments. Various methodologies related to prediction of combining ability effects; variance components and robustness of designs have been discussed. Various definitions, designs, formulae and details of programs used in this investigation have also been reported in this chapter.

The first section of Chapter IV dedicates to the development of general methods of constructing designs involving triallel and tetra-allele crosses. The joint information matrix for combining ability effects have been derived including both gca and sca effects in the model and the expression for estimated variance of elementary contrasts pertaining to gca effects have been also derived. Characterization properties of the designs obtained have been studied. Efficiencies of the developed designs have been calculated.

In second section of Chapter IV, considering a random effects model for tetra-allele crosses, the best linear unbiased predictor (BLUP) for predicting the unobserved combining ability effects together with general mean effect in tetra-allele cross design has been obtained. Lower bound of mean square error (MSE) has been also derived to characterize efficient classes of designs.

In third section of Chapter IV, variance components along with their large sample variance, using mixed linear model approach in tetra-allele crosses has been obtained using Henderson Method III.

In last section of Chapter IV, robustness of newly developed and previously available designs for triallel and tetra-allele cross experiments have been investigated using connectedness and efficiency criteria, against missing observation(s). Programs have been written in Statistical Software Package (SAS) to calculate the efficiencies of the designs to choose a robust design and a catalogue has also been prepared consisting of parameters of robust designs along-with efficiency factor.

Chapter V deals with a general discussion of various methodologies developed and designs obtained for triallel and tetra-allele cross experiments.

Finally, the thesis concludes with a brief summary, abstracts (in English and Hindi), references and Annexures consisting of SAS programs.

# Chapter-2 REVIEW OF LITERATURE

## 2.1 Introduction

Breeding techniques are evolving day by day and have emerged as a major tool for the development of present day commercial hybrids. This technique exploits the hybrid vigour of the cross which is normally reflected as improvement in desirable characteristics.

Schmidt (1919) has defined diallel crossing as a technique which can be used to compare the breeding value of ancestors. The terms gca and sca were originally defined by Sprague and Tatum (1942). Griffing (1956 a, b) has given the analytic procedure of diallel crossing systems. Since then, a lot of literature is available on diallel crosses.

Although diallel cross is the simplest and easily manageable hybridization method, triallel and tetra-allele cross hybrids are genetically more stable, viable and consistent in performance. There are various examples of the application of triallel and tetra-allele crosses in plant as well as animal breeding. Triallel and tetra-allele crosses are the common mating designs used in background for the production of maize hybrids on commercial scale (Shunmugathai and Srinivasan, 2012).

Triallel crossbred chickens are proved to have better egg traits than diallel crossbred chickens with lower mortality (Khawaja *et al.* 2013). The area of piggery is very much accountable for higher order crosses as they are used in cultivating and harvesting products with higher qualities. The silkworm production industry is also using the triallel and tetra-allele crosses for the purpose of harvesting heterosis.

In this chapter, a rigorous account of the work done related to statistical techniques involving triallel and tetra-allel crosses has been given.

## 2.2 Triallel and tetra-allele cross designs

Little research is available on designs for tetra-allele crosses. So these are reviewed with triallel crosses and presented here. Triallel and tetra-allele crosses are considered to be higher order crosses as they involve more number of lines as compared to simple diallel cross. Triallel crosses are often referred as three-way crosses wherein every cross is obtained by crossing three inbred lines. Tetra-allele cross often referred as four-way cross or double cross or four-line cross in which every cross is obtained by mating amongst four inbred lines.

Triallel cross has been defined by Rawlings and Cockerham (1962 a) as a set of all possible three-way matings among a group of lines. The definition given by them is also applicable for full or complete triallel crosses. Rawlings and Cockerham (1962 b) gave the method of analysis for tetra-allele cross hybrids using the analysis method of single cross hybrids under the assumption of no linkage. This analysis was meant for obtaining information related to genetic as well as non-genetic, from tetra-allele cross hybrids. Analysis of variance (ANOVA) was done under a linear model set up. The results were established in terms of dispersion properties of related hybrids. The analysis gave the basis for inferring that there is an interaction system present in tetra-allele crosses hybrids.

Hinkelmann (1965) was the first to introduce the concept of $PT_rC$. He developed $PT_rC$ involving only a fraction taken from all possible triallel crosses and gave a method of construction using Generalized Partially Balanced Incomplete Block (GPBIB) designs. The underlying model involved only gca effects under the assumption that the sca effects are insignificant or very small. The analysis for obtaining the gca effects of line was also described. It was also established that $PT_rC$ are related to incomplete block designs and this property was used for constructing designs and analysing appropriate plans.

Ponnuswamy (1971) considered the research problem of constructing incomplete block designs for triallel crosses. He gave a method of constructing designs for triallel cross based on a three associate PBIB design. The method gives a 3 associate design for triallel crosses in two replicates. Some more methods of construction using Latin Squares and Graeco Latin Squares have also been obtained. The method of analysis is also given providing full information for gca effects. A good review of complete triallel cross and $PT_rC$ can be seen in the review paper by Hinkelmann (1975).

Arora and Aggarwal (1984) discussesd the applications of confounded triallel experiments which is nothing but directly related to the $PT_rC$. The concept of extended triangular designs has been used to get $PT_rC$ designs. In their study, they have assigned the sum total degrees of freedom to three orthogonal components, which belong to gca effects of the lines, two-line sca effects and three-line sca effects, respectively. Venkatesan (1985) has considered the multi-way crosses and various aspects of multi-way crosses have been studied.

Arora and Aggarwal (1989) extended their research work on triallel crosses including reciprocal effects in the model. They derived a four class association scheme from a three associate class partially balanced design and then assigned the sum total degrees of freedom to four orthogonal components, which belong to gca effects of the lines, two-line sca effects, three-line sca effects and reciprocal effects, respectively.

Ceranka *et al.* (1990) have worked regarding the estimation of parameters involved in the model for triallel crosses under the blocked set up.

Ponnuswamy and Srinivasan (1991) used a new class of balanced incomplete block (BIB) designs known as Partially Doubly Balanced Incomplete Block (PDBIB) designs for the construction of a class of $PT_rC$. The method gives a $PT_rC$ design which reciprocates the property of a $CT_rC$ design. They also provided a list of PDBIB designs which can be further used for the purpose of constructing suitable triallel crosses. Subbarayan (1992) studied the applications of pure cyclic triple system for experiments involved in breeding.

Das and Gupta (1997) worked in the area of optimality of block designs for triallel crosses. In their study, a new class of designs known as Nested Balanced Block (NBB) designs have been introduced and used for obtaining optimal designs for triallel crosses. A fixed effects model has been considered for the study without including sca effects in the model. It is shown that there is a one to one correspondence between NBB designs and optimal designs for triallel crosses. Several classes of such designs which can lead to optimal triallel cross designs have been reported.

Dharmalingam (2002) gave a method of construction of $PT_rC$ using Trojan square designs and the method is more general as compared to others. Degree of fractionation of their designs is considerably small.

Parsad *et al.* (2005) worked on optimality aspect of diallel and tetra-allele cross designs. The problem of finding an optimal class of designs has been considered in three types of model set-up, that is, zero-way, one-way and two-way elimination of heterogeneity. In each case, only gca effects is considered in the model and sca effects are presumed to be absent. Method of construction of universally optimal block designs for diallel cross experiments using a NBB design is given. Method of construction of universally optimal row column design for diallel cross experiments using NBB designs is also described. MS-optimal designs under all the three set ups are given and are claimed to have a high A- and D-efficiency. Subbarayan

(2009) has considered two associate class PBIB designs for the construction of designs involving $PT_rC$.

Some classes of mating designs have been obtained using generalized incomplete Trojan type designs by Varghese and Jaggi (2011) and various aspects of designs and analysis of genetic cross experiments were explained in detail by Singh *et al.* (2012).

Sharma *et al.* (2012) considered the problem of investigating optimal class of designs involving $PT_rC$. The model considered includes gca effects of first and second kind but not t the sca effects. A method of construction of designs involving triallel crosses has been proposed for prime or prime power number of lines based on Mutually Orthogonal Latin Squares (MOLS). Method of analysis has also been illustrated. Shunmugathai and Srinivasan (2012) discussed that triallel has helped both plant as well as animal breeders to improve various nutritionally and economically important traits.

Khawaja *et al.* (2013) have studied various parameters like production performance, egg quality, *etc.* of triallel cross hybrid chickens in comparison to reciprocal $F_1$ crossbred chickens and found that triallel cross hybrid chickens are having better egg qualities than diallel chickens with lower cases of mortality.

Harun (2014) discussed various methods of constructing designs for triallel cross experiments using MOLS, association schemes and PBIB designs. The variance factor of contrasts pertaining to estimated gca effects of first and second kind has been given.

Harun *et al.* (2016 a) developed some methods of constructing designs involving complete and PTrC using MOLS and PBIB designs. Further, Harun *et al.* (2016 b & c) developed methods for constructing various classes of $PT_rC$ designs for test line versus single control comparisons. Harun *et al.* (2017) developed $PT_rC$ plans based on BIB designs.

### 2.3 Mating plans with models incorporating sca effects

Diallel cross is the simplest and easily manageable mating design but triallel and tetra-allele cross based hybrids are found to be genetically more viable, stable and consistent in performance than diallel cross based hybrids. The sca effects are also of much importance for breeders besides the gca effects. In case of diallel cross, different aspects of designs considering both gca and sca effects have been addressed but higher order mating designs are mostly studied for gca effects only. Since sca effects are of prime importance and the major research including sca effects in the model is done only in the area of diallel crosses, there is a need to investigate the problem of considering sca effects in the model alongwith gca effects for triallel and tetra-allele cross experiments.

Chai and Mukerjee (1999) considered the problem of including sca effects while investigating the problem of diallel cross designs. They have considered a model which includes sca effects along with gca effects in addition to the block effects. PBIB designs based on triangular association scheme were investigated for optimality aspects and it has been concluded that these designs are universally optimal for estimating gca effects in the presence of sca effects. Choi *et al.* (2002) investigated orthogonal designs for diallel crosses in blocked set up and studied the optimality of designs including sca effects in the model.

Das and Dey (2004) investigated designs involving diallel crosses with sca effects. They considered a model that incorporates both types of combining ability effects and under such model, conditions have been derived for blocked setup so that the gca effects are estimated free from sca effects.

Varghese and Varghese (2013) included sca effects in the model of mating-environmental designs for diallel crosses under two-way blocking setup. Further, Varghese *et al.* (2015) considered the usual linear fixed effect model under the row column set up including sca effects in the model alongwith the gca effects and information matrix related to gca effects has been derived eliminating sca effects.

Varghese *et al.* (2016) discussed a methodology for estimating gca effects free from sca effects in case of Type III CDC considering a row-column set-up. They also obtained a class of variance balanced row column designs.

Varghese and Varghese (2017) investigated the problem of comparing test lines with a control line including sca effects in the model. Various classes of designs under two-way blocking and variance balanced structure have been obtained for estimating the contrasts related to gca effects orthogonal to sca effects.

**2.4 Prediction of combining ability effects and variance component estimates**

Most of the studies involving mating designs are done under the assumptions of a standard linear model where the gca effects are considered to be fixed in nature. But, there are situations where the idea of taking gca effects as fixed may not be always justifiable and one can also consider them as random effects according to the situation. In this investigation too, only a small fraction of inbreds is taken from a large population in order to predict the gca effects or to estimate the variance components and the variances of the obtained estimates.

Cockerham (1961) established the relationship between the design properties and the genetic components of variance in case of triallel crosses. Rawlings and Cockerham (1962 a) proposed an orthogonal model for triallel cross designs. Rawlings and Cockerham (1962 b) have presented the analysis of tetra-allele cross hybrids. They performed an orthogonal analysis of variance and the interpretation was done in terms of variances of the effects based on the underlying linear model and also in terms of variance-covariance structure of relatives.

Hinkelmann (1965) gave an alternative model for the analysis of triallel cross hybrids, which as compared to the previous work of Rawlings and Cockerham (1962 a) was non orthogonal and thus, resulted in a class of estimators.

Ponnuswamy (1971) worked in the area of estimation of genetic component of variances for triallel crosses based on the model proposed by Hinkelmann in 1965. The determination of relationship of genetic component with the design was done through studying covariance between relatives. Rather than considering the full model which involves all possible number of loci, a restricted model was considered. The model consisted of only two loci and, the higher order interactions were considered to be negligible. Thus, an explicit relation was established for the variance components with the involved design when the number of loci considered was exactly two.

Srinivasan and Ponnuswamy (1993) considered the problem of estimating variance components taking into account the epistatic effects. A systematic and purely mathematical approach has been suggested based on a new triallel cross model. Using a non orthogonal model, a possible solution for variance components have been obtained. Considering the unimaginable complexity while considering the methods like ANOVA, maximum likelihood (ML), restricted maximum likelihood (REML) *etc*., they have used a different approach which is based on quadratic unbiased estimation in mixed linear model.

Higher order mating designs like triallel and tetra-allele crosses can be used to study and harvest the epistatic properties of gene action but the limited resources availability is the main hurdle to use them. Srinivasan and Ponnuswamy (1995) considered the idea of using $PT_rC$ with certain basic properties for the estimation of genetic and design component of variance.

The $PT_rC$ constructed by Ponnuswamy and Srinivasan (1991) using PDBIB designs have been considered and the design components of variance estimates have been obtained by superimposing them on completely randomized design. The method of analysis used was based on quadratic unbiased estimation under general mixed linear model set-up.

Zhu and Weir (1996) gave a bio-model for diallel analysis and suggested the use of minimum norm quadratic unbiased estimation (MINQUE) for estimating different variance and covariance components. Venkateswarlu and Ponnuswamy (1998) considered the problem of estimation of variance components in diallel model. They used the approach of Srinivasan and Ponnuswamy (1993).

Ghosh and Das (2003) investigated the problem of estimation of heritability and optimality aspects of diallel cross designs. A random effect model, where the gca effects of the lines have been considered as random is proposed for the estimation of variance components and the variances of estimates. The unbiased estimates and their variances has been worked out both under unblocked and blocked set up. The existence of an A-optimal incomplete block design for diallel cross experiments along with the existence of a NBIB designs has been established. They have also proved that any known MS-optimal design under the fixed effect model remains valid under A- and D-optimal designs for random effect model.

Ghosh *et al.* (2005) considered the mixed effects model for the estimation of variance components and characterization of optimal diallel cross designs. They worked out the unbiased estimates of variance components and their large sample variances. The given estimator is having one to one correspondence with heritability. The optimality of designs under unblocked and blocked set up has been studied and optimal partial diallel cross designs have been characterized. Based on simulation study, they claimed that for small samples also the large sample variances are nearly equal to the exact variances.

Ghosh and Das (2005) considered the problem of predicting the yield capacity of crosses through diallel cross experiments. Under this investigation, they considered a random effects model for studying the optimality of diallel cross designs for best linear unbiased prediction. The BLUP for gca effects has been given and various properties of the predictor have been studied. The characterization for A-optimality of designs for BLUP has been done. Some efficient classes of PDC designs have been reported.

The problem of diallel analysis and separation of genetic variance components has been studied using eight faba bean genotypes by Zeinab and Helal (2014).

### 2.5 Robust designs for breeding experiments

An optimal/efficient design may not remain so if the underlying assumptions are violated. Sometimes, situation may arise such that all the contrasts pertaining to combining ability effects may not be estimable. Hence, it is much important to obtain robust designs against disturbances like missing observation(s), outlying observation(s), exchange/interchange of crosses, inadequacy of assumed model, *etc*.

The concept of connectedness criterion of robustness was introduced by Ghosh (1978), with respect to robustness of BIB designs. The criterion advocates that a design which is connected in the sense that it allows the estimation of all the elementary treatment contrasts through the design, remains connected even though some disturbances have crept in.

Panda (2000) considered the problem of interchange of a pair of crosses while conducting a complete diallel experiment. The problem considered involved both the situations where the interchanged crosses are totally different or differ in one of either lines involved in the diallel cross. He studied the robustness of block designs for CDC experiments using two criteria, connectedness and efficiency. The study reveals that randomized complete block designs (RCBD) used for conducting CDC experiments and binary balanced block designs are robust against interchange between two crosses in two of its blocks. Some non-binary balanced block designs are also reported to be robust against interchange between two crosses in two of its blocks.

Panda *et al*. (2001) have considered optimal block designs for triallel cross experiments for investigating robustness against an exchanged cross. They have taken all the three possible conditions in which the two exchanged crosses are totally different, different in two lines and different in one line. They have considered the connectedness and efficiency criteria against the cross interchange and concluded that any universally optimal design for triallel cross experiment is robust if the number of lines involved is greater than 9.

Dey *et al.* (2001) considered the problem of missing observations in diallel cross experiments. The two cases of diallel cross experiments where a single observation was missing and another with a complete block missing was investigated for robustness using connectedness and efficiency criteria. Robust block designs against one missing observation and robust proper binary balanced block designs are reported for diallel cross experiments.

Bhar and Gupta (2002) considered the problem of missing observations in diallel cross experiments. They studied the robustness of variance balanced designs under row column set up against missing observations. Lal and Jeisobers (2002) investigated the problem of missing crosses from a block to study the robustness of diallel cross designs under blocked set up.

Prescott and Mansson (2004) investigated the problem of loss of one or more observations in a diallel cross design. Theoretical results have been given for loss of one cross from CDC based on BIB designs and PDC based on PBIB designs. The effect of missing observation is seen as A-efficiency which is based on average variances of estimates of elementary contrasts of gca effects of lines and the study reveals that the designs are fairly robust. Robustness of block designs for CDC experiments have been investigated using the connectedness and efficiency criteria against interchange of a pair of crosses by Panda *et al.* (2005).

Shunmugathai and Srinivasan (2012) studied robustness of NBIB designs under the situation of interchange of a pair of crosses in triallel crosses. A-efficiency has been calculated by taking the ratio of harmonic means of the non-zero eigenvalues of the information matrices. The study suggested that NBIB designs are fairly robust.

## 3.1 Introduction

Higher order mating designs like triallel and tetra-allele cross based hybrids are found to be better than diallel cross based hybrids, when viability and consistency in performance is considered. The sca effects are of much importance for breeders besides the gca effects. In case of diallel crosses, only first order sca effect can be studied whereas in triallel crosses three first order and one second order sca effect and in case of tetra-allele crosses first, second and even third order sca effects can be studied.

## 3.2 Triallel cross

Triallel crosses are intermediate between diallel and tetra-allele crosses with respect to number of lines used, complexity of handling the crosses and the amount of information regarding combining abilities.

### 3.2.1 Full model with sca effects

Consider a triallel cross experiment involving $N$ number of lines giving rise to $T$ number of crosses. Let a cross of type $(i \times j) \times k$ is represented as $(i, j, k)$ and the fixed effect of the triallel cross $(i, j, k)$ by $y_{ijk}$, then the following model can be used for representing cross effect:

$$y_{ijk} = \bar{y} + h_i + h_j + g_k + s_{ij} + s_{ik} + s_{jk} + s_{ijk} + e_{ijk}, \qquad (3.2.1.1)$$

where $\bar{y}$ is the average effect of the crosses, $\{h_\alpha\}, \alpha = i, j$ and $\{g_k\}$ represents the gca effects half parents and full parents respectively, $\{s_{\alpha\beta}\}, (\alpha, \beta) \in (i, j, k)$ represents the first order sca effects, $s_{ijk}$ represents the second order sca effects, $e_{ijk}$ represents the random error component with the constraints

$$\sum_{i=1}^{N} h_i = 0 \text{ and } \sum_{i=1}^{N} g_i = 0 \qquad (3.2.1.2)$$

$$\sum_{\alpha\beta} s_{\alpha\beta} = 0 \;\forall\; (\alpha, \beta) \in (i, j, k), i \neq j \neq k \;= 1,2, \dots, N \text{ and} \qquad (3.2.1.3)$$

$$\sum_{ijk} s_{ijk} = 0 \;\forall\; i \neq j \neq k = 1,2, \dots, N \qquad (3.2.1.4)$$

It is important to note here that if a cross $(i, j, k)$ is occurring in the experiment then the other two alternative forms $(i, k, j)$ and $(j, k, i)$ are also included in the experiment, to satisfy the SSP of triallel crosses.

### 3.2.2 Model without sca effects

In this approach, gca effects of first and second kind corresponding to half and full parents will be estimated for which it is assumed that the sca effects are contributing much less to the total combining ability effects as compared to gca effects and hence sca effects are negligible. The model can be written as

$$y_{ijk} = \bar{y} + h_i + h_j + g_k + e_{ijk}, \tag{3.2.2.1}$$

where $\bar{y}$ is the average effect of the treatments, $\{h_\alpha\}, \alpha = i, j$, represents the gca effects of first kind corresponding to the lines occurring as half parents, $\{g_k\}$ represents the gca effects of second kind corresponding to the lines occurring as full parents, $e_{ijk}$ is the random error component and

$$g_1 + g_2 + \cdots + g_N = 0 \text{ or } \sum_{i=1}^{N} g_i = 0 \tag{3.2.2.2}$$

$$h_1 + h_2 + \cdots + h_N = 0 \text{ or } \sum_{i=1}^{N} h_i = 0 \tag{3.2.2.3}$$

The model in matrix notation is expressed as:

$$\boldsymbol{y} = \bar{y}\,\mathbf{1}_T + \boldsymbol{W}_1'\boldsymbol{h} + \boldsymbol{W}_2'\boldsymbol{g} + \boldsymbol{e}, \tag{3.2.2.4}$$

where, $\boldsymbol{y}$ is the $T \times 1$ vector of responses due to crosses, $\bar{y}$ is the mean effect of crosses, $\boldsymbol{h}$ is the $N \times 1$ vector of gca effects due to half parent, $\boldsymbol{g}$ is the $N \times 1$ vector of gca effects due to full parent and $\boldsymbol{e}$ is the $N \times 1$ vector of random error component. $\boldsymbol{W_1}$ and $\boldsymbol{W_2}$ are $N \times T$ matrices with rows indexed by the line numbers $1, 2, \ldots N$ and columns by the three-way crosses arranged in the manner described earlier, such that the $\{t, (i, j, k)\}^{th}$entry of $\boldsymbol{W_1}$ is 0.5 if $t \in (ij)$ and zero otherwise and the $\{t, (i, j, k)\}^{th}$entry of $\boldsymbol{W_2}$ is 1 if $t \in k$ and zero otherwise. The normal equations are as

$$E(\boldsymbol{y}) = \bar{y}\,\mathbf{1}_T + \boldsymbol{W}_1'\boldsymbol{h} + \boldsymbol{W}_2'\boldsymbol{g},$$

$$\boldsymbol{W_1}E(\boldsymbol{y}) = \bar{y}\,\boldsymbol{W_1}\mathbf{1}_T + \boldsymbol{W_1}\boldsymbol{W}_1'\boldsymbol{h} + \boldsymbol{W_1}\boldsymbol{W}_2'\boldsymbol{g}, \text{ and}$$

$$\boldsymbol{W_2}E(\boldsymbol{y}) = \bar{y}\,\boldsymbol{W_2}\mathbf{1}_T + \boldsymbol{W_2}\boldsymbol{W}_1'\boldsymbol{h} + \boldsymbol{W_2}\boldsymbol{W}_2'\boldsymbol{g}.$$

On solving the three normal equations, the estimate of the gca effects of half parent is given as:

$$\widehat{\boldsymbol{h}} = \left(\boldsymbol{W_1}\boldsymbol{W}_1'\right)^{-}(\boldsymbol{W_1}\boldsymbol{y} - \boldsymbol{W_1}\bar{y}\,\boldsymbol{1}_T)$$

$$= [\left(\boldsymbol{W_1}\mathbf{W}_1'\right)^{-}\boldsymbol{W_1} - \left(\boldsymbol{W_1}\mathbf{W}_1'\right)^{-}\boldsymbol{W_1}\mathbf{J}_{TT}/T)]\boldsymbol{y} = \boldsymbol{G}_1\boldsymbol{y} \text{ (say)}, \qquad (3.2.2.5)$$

and the estimate of gca effects of full parent is given as:

$$\widehat{\boldsymbol{g}} = \left(\boldsymbol{W_2}\boldsymbol{W}_2'\right)^{-}(\boldsymbol{W_2}\boldsymbol{y} - \boldsymbol{W_2}\bar{y}\,\boldsymbol{1}_T)$$

$$= [\left(\boldsymbol{W_2}\boldsymbol{W}_2'\right)^{-}\boldsymbol{W_2} - \left(\boldsymbol{W_2}\boldsymbol{W}_2'\right)^{-}\boldsymbol{W_2}\boldsymbol{J}_{TT}/N)]\boldsymbol{y} = \boldsymbol{G_2}\boldsymbol{y} \text{ (say)}. \qquad (3.2.2.6)$$

The restrictions being imposed in order to estimate the gca effects of half parents free from gca effects of full parents are as:

$$\boldsymbol{1}'\widehat{\boldsymbol{h}} = \boldsymbol{1}'\widehat{\boldsymbol{g}} = \boldsymbol{G}_1\boldsymbol{1} = \boldsymbol{G}_2\boldsymbol{1} = \boldsymbol{G}_1'\boldsymbol{G}_2 = \boldsymbol{0} \text{ and}$$

$$rank(\boldsymbol{G_1}) = rank(\boldsymbol{G_2}) = (N-1). \qquad (3.2.2.7)$$

Now, considering the usual setup of a block design $d$, the joint information matrix regarding $\begin{pmatrix}\boldsymbol{G}_1\\ \boldsymbol{G}_2\end{pmatrix}\boldsymbol{y}$ is given by:

$$\boldsymbol{C}_{\boldsymbol{d\_gca}} = \begin{bmatrix}\boldsymbol{G}_1\boldsymbol{C}_d\boldsymbol{G}_1' & \boldsymbol{G}_1\boldsymbol{C}_d\boldsymbol{G}_2'\\ \boldsymbol{G}_2\boldsymbol{C}_d\boldsymbol{G}_1' & \boldsymbol{G}_2\boldsymbol{C}_d\boldsymbol{G}_2'\end{bmatrix}, \qquad (3.2.2.8)$$

where $\boldsymbol{C}_d = \boldsymbol{R}_d - \frac{1}{k}\boldsymbol{N}_d\boldsymbol{N}_d'$, $\boldsymbol{R}_d = diag(r_1, r_2, \ldots, r_T)$ is the diagonal matrix of replications of the crosses under the design $d$ and $\boldsymbol{N}_d$ is the incidence matrix of crosses versus blocks. Here, $\boldsymbol{C}_d$ is the information matrix of the general block design $d$ where treatments are nothing but the $T$ number of tri-allele crosses, hence we have $\boldsymbol{C}_d\boldsymbol{1}_T = \boldsymbol{0}$. As discussed earlier regarding orthogonality, in order to estimate $\boldsymbol{G}_1\boldsymbol{C}_d\boldsymbol{G}_1'$ and $\boldsymbol{G}_2\boldsymbol{C}_d\boldsymbol{G}_2'$ orthogonally the off diagonal components must vanish and we must have $\boldsymbol{G}_2\boldsymbol{C}_d\boldsymbol{G}_1' = \boldsymbol{G}_1\boldsymbol{C}_d\boldsymbol{G}_2' = \boldsymbol{0}$. Thus we have $\boldsymbol{C}_{\text{gca\_half}} = \boldsymbol{G}_1\boldsymbol{C}_d\boldsymbol{G}_1'$ and $\boldsymbol{C}_{\text{gca\_full}} = \boldsymbol{G}_2\boldsymbol{C}_d\boldsymbol{G}_2'$.

### 3.3 Tetra-allele cross

Tetra-allele cross experiment provides us more information regarding the combining abilities and the hybrids developed based on them are found to be more stable and consistence in performance due to broad genetic base. Based on nature of gca effects, fixed or random, in the model, two approaches are described.

### 3.3.1 Fixed effects model with sca effects

Consider a tetra-allele cross experiment in which the $T$ crosses are regarded as treatments. Let a cross of type $(i \times j) \times (k \times l)$ be represented as $(i, j, k, l)$, and the fixed effect of the cross $(i, j, k, l)$is denoted by $y_{ijkl}$, then the following model can be used:

$$y_{ijkl} = \bar{y} + g_i + g_j + g_k + g_l + s_{ij} + s_{ik} + s_{il} + s_{jk} + s_{jl} + s_{kl}$$
$$+ s_{ijk} + \ s_{ijl} + s_{ikl} + s_{jkl} + s_{ijkl} + e_{ijkl}, \qquad (3.3.1.1)$$

where $\bar{y}$ is the average effect of the crossess, $\{g_\alpha\}, \alpha = i, j, k, l,$ represents the gca effects, $\{s_{\alpha\beta}\}, (\alpha, \beta) \in (i, j, k, l)$ represents the first order sca effects, $\{s_{\alpha\beta\gamma}\}, (\alpha, \beta, \gamma) \in i, j, k, l,$ represents the second order sca effects, $s_{ijkl}$ represents the third order sca effects, $e_{ijkl}$ represents the random error, and with

$$\sum_{i=1}^{N} g_i = 0, \sum_{\alpha\beta} s_{\alpha\beta} = 0, \sum_{\alpha\beta\gamma} s_{\alpha\beta\gamma} = 0 \text{ and}$$
$$\sum_{ijkl} s_{ijkl} = 0, i \neq j \neq k \neq l, i < j, k < l\ , i, j, k, l\ = 1,2, \dots, N, \qquad (3.3.1.2)$$

for every $(\alpha, \beta, \gamma) \in (i, j, k, l)$, for $i \neq j \neq k \neq l,\ i < j, k < l,\ i, j, k, l\ = 1,2, \dots, N$.

Since complete tetra-allele crosses are considered here, whenever a cross $(i, j, k, l)$ is involved, it means that the other two alternative crosses of the types $(i, k, j, l)$ and $(i, l, j, k)$ are also included in the experiment, simultaneously.

### 3.3.2 Random effects model for tetra-allele cross

Consider a mating design involving tetra-allele cross experiment performed to analyse the observations taken from $T$ crosses based on $N$ lines. If the effect of the cross $(i, j, k, l)$ is denoted by $y_{ijkl}$ then we can have the following representation:

$$y_{ijkl} = \bar{y} + f_i + f_j + f_k + f_l + e_{ijkl}, \qquad (3.3.2.1)$$

where $\bar{y}$ is the average effect of the crosses, $\{f_\alpha\}, \alpha = i, j, k, l,$ represents the gca effects with $E\ (f_\alpha) = 0$ or $\sum f_\alpha = 0$, $Var(f_\alpha) = \sigma_\alpha^2 \geq 0$, $Cov(f_\alpha, f_\beta) = 0$, $e_{ijkl}$ is the random error component uncorrelated with $f_\alpha$, with $E\ (e_{ijkl}) = 0$ and $Var(e_{ijkl}) = \sigma_e^2 \geq 0$, $(\alpha, \beta) \in (i, j, k, l), i, j, k, l\ = 1,2, \dots, N$.

The model can be expressed in matrix notation as

$$\boldsymbol{y} = \boldsymbol{X\beta} + \boldsymbol{e} \text{ or}$$

$$\boldsymbol{y} = \bar{y}\mathbf{1}_N + \boldsymbol{W}_4'\boldsymbol{f} + \boldsymbol{e}, \tag{3.3.2.2}$$

such that $\boldsymbol{X} = [\boldsymbol{X}_1 \quad \boldsymbol{X}_2] = [\mathbf{1}_N \quad \boldsymbol{W}_4']$ and $\boldsymbol{\beta} = \begin{bmatrix}\boldsymbol{\beta}_1\\ \boldsymbol{\beta}_2\end{bmatrix} = \begin{bmatrix}\bar{y}\\ \boldsymbol{f}\end{bmatrix}$, where $\boldsymbol{y}$ is an $T \times 1$ vector of observations, $\boldsymbol{f}$ is a $N \times 1$ vector of gca effects with $E(\boldsymbol{f}) = \mathbf{0}$ and $D(\boldsymbol{f}) = \boldsymbol{V_f} = \sigma_f^2\boldsymbol{I}_N$, $\boldsymbol{e}$ is a $T \times 1$ vector of random error with $E(\boldsymbol{e}) = \mathbf{0}$ and $D(\boldsymbol{e}) = \sigma_e^2\boldsymbol{I}_T$, and $\boldsymbol{W_4}$ is an $N \times T$ incidence matrix with rows indexed by the line numbers $1,2, \dots m, \dots N$ and columns by the $T$ crosses such that the $\{m, (i, j, k, l)\}^{\text{th}}$ entry of $\boldsymbol{W_4}$ takes a value 1 if $m \in (i, j, k, l)$ and 0, otherwise. Let $\boldsymbol{p} = (p_1, p_2, \dots, p_N)'$, where $p_i$ is the replication of $i^{th}$ line. Also, $\boldsymbol{W_4}\boldsymbol{W}_4' = \boldsymbol{G}$ is a matrix whose elements $\{g_{ij}\}$ is the number of times a pair $(i, j)$ appears in the design. Thus, we have $\boldsymbol{W}_4\mathbf{1}_T = \boldsymbol{p}$ . Also, we have $rank(\boldsymbol{W_4}) = N$, $\boldsymbol{G}$ is a symmetric matrix with $rank(\boldsymbol{G}) = N$ and $trace(\boldsymbol{G}) = 4T$.

### 3.4 Prediction of combining ability effects

There are many situations where one wants to quantify the real value of a random variable which is directly unobservable in nature. An example from breeding sector is that of predicting the genetic merit of a dairy bull from the milk yield capacity of his daughters. In this study too, the yielding capacity of the crosses from a sample of inbred lines are to be predicted. When we have observations on some random variables from which we have to predict the value of some other related random variable that cannot be observed, the concept of Best Linear Unbiased Predictor (BLUP) is used for unbiased prediction of the yielding capacity of the cross from the sample of inbred lines under mixed effect model.

#### 3.4.1 Model and experimental set up

Consider the usual fixed effects linear model set up expressed as

$$\boldsymbol{y} = \boldsymbol{X\theta} + \boldsymbol{e},$$

where $\boldsymbol{y}$ is a $T \times 1$ vector of observations, $\boldsymbol{\theta}$ is a $t \times 1$ vector of parameters which are of fixed effect nature, $\boldsymbol{X}$ is the $T \times t$ incidence matrix and $\boldsymbol{e}$ is an $T \times 1$ error vector such that $E(\boldsymbol{e}) = \mathbf{0}$ and $D(\boldsymbol{e}) = \boldsymbol{\sigma_e^2 I_T}$. Consider that the random effects of the model can be represented by $\boldsymbol{Zb}$ which are having properties parallel to $\boldsymbol{X\theta}$, where $\boldsymbol{b}$ is a vector of random effects that occurs in the model and $\boldsymbol{Z}$ is the corresponding incidence matrix. Now, the vector $\boldsymbol{b}$ can be further poartitioned into $r$ sub-vectors as per the numbers of random effect parameters involved in the experimentation. Thus, incorporating $\boldsymbol{b}$ in the fixed effects model, a general mixed effects model can be obtained and expressed as

$$\boldsymbol{y} = \boldsymbol{X\theta} + \boldsymbol{Zb} + \boldsymbol{e}, \tag{3.4.1.2}$$

where $\boldsymbol{\theta}$ and $\boldsymbol{b}$ represents the fixed and random effects respectively such that $D(\boldsymbol{b}_i) = \sigma_i^2 \boldsymbol{I}_{q_i}$, $i = 1,2,\dots,r$, $Cov(\boldsymbol{b}_i, \boldsymbol{b}_J) = 0$, $i \neq j, i,j = 1,2,\dots,r$ and $Cov(\boldsymbol{b},\boldsymbol{e}) = \boldsymbol{0}.$ We can get the forms related to expectation and dispersion of vector $\boldsymbol{b}$ as $\overline{\boldsymbol{b}} = E(\boldsymbol{b}) = 0$, and

$$\boldsymbol{V}_b = D(\boldsymbol{b}) = \begin{bmatrix} \sigma_1^2 \boldsymbol{I}_{q_1} & 0 & \dots & \boldsymbol{0} \\ 0 & \sigma_2^2 \boldsymbol{I}_{q_2} & & \boldsymbol{0} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & & \sigma_r^2 \boldsymbol{I}_{q_r} \end{bmatrix}.$$

Now, the matrix $\boldsymbol{Z}$ can also be partitioned in $r$ sub-matrices which are conformable with vector $\boldsymbol{b}$ as $\boldsymbol{Z} = [\boldsymbol{Z}_1 \quad \boldsymbol{Z}_2 \quad \dots \quad \boldsymbol{Z}_r]$. The model can be redefined as

$\boldsymbol{y} = \boldsymbol{X\theta} + \sum_1^r \boldsymbol{Z}_i \boldsymbol{b}_i + \boldsymbol{e}$, such that

$E(\boldsymbol{y}) = \boldsymbol{X\theta}$,

$\boldsymbol{V_y} = D(\boldsymbol{y}) = D(\boldsymbol{X\theta} + \sum_1^r \boldsymbol{Z}_i \boldsymbol{b}_i + \boldsymbol{e})$

$= \sum_1^r \sigma_i^2 \boldsymbol{Z}_i \boldsymbol{Z}_i' + \sigma_e^2 \boldsymbol{I}$ and

$$C_{by} = Cov(\boldsymbol{b},\boldsymbol{y}) = D(\boldsymbol{b})\boldsymbol{Z}' = \begin{bmatrix} \sigma_1^2 \boldsymbol{Z}_1' & 0 & \dots & \boldsymbol{0} \\ 0 & \sigma_2^2 \boldsymbol{Z}_2' & & \boldsymbol{0} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & & \sigma_r^2 \boldsymbol{Z}_r' \end{bmatrix}.$$

### 3.4.2 BLUP and Properties of BLUP

Under the above defined mixed effects model, the problem of prediction can be considered. For some known matrix $\boldsymbol{L}$, the function $\boldsymbol{L}'\boldsymbol{\theta}$ remains an estimable function, the predictor is taken as

$\boldsymbol{w} = \boldsymbol{L}'\boldsymbol{\theta} + \boldsymbol{b}$, where the joint distribution is given as

$$\begin{bmatrix} \boldsymbol{w} \\ \boldsymbol{y} \end{bmatrix} \sim \left\{ \begin{bmatrix} \boldsymbol{L}'\boldsymbol{\theta} \\ \boldsymbol{X\theta} \end{bmatrix}, \begin{bmatrix} \boldsymbol{V_b} & \boldsymbol{C_{by}} \\ \boldsymbol{C_{yb}} & \boldsymbol{V_y} \end{bmatrix} \right\}.$$

Since the predictor involves both fixed and random effects, the procedure must be first predicting $\boldsymbol{w}$ and then choosing the best predictor $\widetilde{\boldsymbol{w}}$ having the following properties:

- ***Best,*** as it will be minimizing $E(\boldsymbol{w}-\widetilde{\boldsymbol{w}})'\boldsymbol{A}(\boldsymbol{w}-\widetilde{\boldsymbol{w}})$ for some positive definite and symmetric matrix $\boldsymbol{A}$.
- ***Linear,*** as it can be expressed as some linear form of $\boldsymbol{y}$, $\widetilde{\boldsymbol{w}}=\boldsymbol{a}+\boldsymbol{B}\boldsymbol{y}$, such that the vector and matrix are not related to $\boldsymbol{\theta}$.
- ***Unbiased,*** as $E(\widetilde{\boldsymbol{w}})=E(\boldsymbol{w})$.

Since, the unbiasedness of $\widetilde{\boldsymbol{w}}$ is achieved only if $\boldsymbol{a}+\boldsymbol{B}\boldsymbol{X}\boldsymbol{\theta}=\boldsymbol{L}'\boldsymbol{\theta}$ for all $\boldsymbol{\theta}$. Also, the vector $\boldsymbol{a}$ must be independent of $\boldsymbol{a}$, which is possible only if $\boldsymbol{a}=\boldsymbol{0}$ and $\boldsymbol{B}\boldsymbol{X}=\boldsymbol{L}'$.

The best linear unbiased predictor for $\boldsymbol{w}$ is given as

$$\text{BLUP}(\boldsymbol{w})=\widetilde{\boldsymbol{w}}=\boldsymbol{L}'\boldsymbol{\theta}^0+\boldsymbol{C}_{\boldsymbol{by}}\boldsymbol{V}_{\boldsymbol{y}}^{-\mathbf{1}}(\boldsymbol{y}-\boldsymbol{X}\boldsymbol{\theta}^0),$$

where $\boldsymbol{X}\boldsymbol{\theta}^0$ is the BLUE($\boldsymbol{X}\boldsymbol{\theta}$) such that

$$\boldsymbol{\theta}^0=(\boldsymbol{X}'\boldsymbol{V}_{\boldsymbol{y}}^{-\mathbf{1}}\boldsymbol{X})^-\boldsymbol{X}'\boldsymbol{V}_{\boldsymbol{y}}^{-\mathbf{1}}\boldsymbol{y}.$$

### 3.5 Robust design for breeding experiments

An optimal or efficient design may not remain so if the underlying assumptions are violated. Sometimes, situation may arise such that all the contrasts pertaining to combining ability effects may not be estimable. Hence, it is much important to obtain robust designs against disturbances like missing observation(s), outlying observation(s), exchange/interchange of crosses, inadequacy of assumed model, *etc*. Missing observation(s) is the most commonly occurring disturbance in breeding experiments since large number of crosses is to be made in case of higher order crosses and there is a chance that some human error may creep in due to loss of tags or labels, or some of the observations may be lost or die after making the crosses. There are many criteria to check robustnesss, of which the connectedness and efficiency criteria of robustness have been considered here.

**Connectedness Criterion for robustness:** Consider a connected design $d\in\mathcal{D}$ which allows the estimation of all elementary treatments contrasts pertaining to gca effects of lines and let $d^*$ be the residual design which we get due to $\xi$ *i.e.* the disturbance caused by missing observation. Then connectedness criterion of robustness against $\xi$ demands that the design $d^*$

must remain connected even after the disturbance so that all the elementary contrasts are estimable.

**Efficiency criterion for robustness:** The connectedness criterion alone cannot serve the purpose of robustness against a disturbance because a design may remain connected but may become inefficient due to $\xi$. Thus, it is necessary to take care of efficiency of the residual design, for which the efficiency criterion of robustness is studied. This criterion involves the calculation of efficiency of the design $d^*$, which is the ratio of the harmonic mean of the non-zero eigenvalues of information matrices related to the designs $d$ and $d^*$ respectively. If $\mathbf{C}_d$ is the information matrix of the original design $d$ and $\mathbf{C}_{d^*}$ of the residual design, then the efficiency $(E)$ of the residual design can be calculated relative to the original one as

$$E = \frac{\text{Harmonic mean of non−zero eigenvalues of } \mathbf{C}_{d^*}}{\text{Harmonic mean of non−zero eigenvalues of } \mathbf{C}_d}.$$

### 3.6 Definitions

**Degree of fractionation:** The value of degree of fractionation $(f)$ can be obtained for any mating design for given number of lines. In order to obtain the value of $f$, one has to find out the ratio of total number of crosses involved in the partial mating design to the total number of crosses in the complete mating design. Let $T_{CT_rC}$ and $T_{CT_eC}$ be the number of crosses involved in $CT_rC$ and $CT_eC$ designs, respectively, whereas $T_{PT_rC}$ and $T_{PT_eC}$ represents the number of crosses involved in $PT_rC$ and $PT_eC$ designs respectively, for a given number of lines $N$. Then the degree of fractionation $f_{PT_rC}$ and $f_{PT_eC}$ related to designs involving $PT_rC$ and $PT_eC$, respectively, are calculated as

$$f_{PT_rC} = \frac{T_{PT_rC}}{T_{CT_rC}} = \frac{2T_{PT_rC}}{N(N-1)(N-2)} \text{ and}$$

$$f_{PT_eC} = \frac{T_{PT_eC}}{T_{CT_eC}} = \frac{8T_{PT_eC}}{N(N-1)(N-2)(N-3)}.$$

**Canonical efficiency factor:** The canonical efficiency factor of a design with lines replicated $r$ times, is calculated relative to an orthogonal design with the same number of lines, by working out the harmonic mean of $(1/r)$ times the non-zero eigenvalues of the information matrix of the design, assuming that the error variance is same for both situations.

**Canonical efficiency factor of $\text{PT}_\text{r}\text{C}$ designs:** Consider a design $d$ used for performing a $\text{PT}_\text{r}\text{C}$ experiment. Let $\mathbf{C}_{d_{gca-half}}$ and $\mathbf{C}_{d_{gca-full}}$ are the information matrices related to the half parents and related to the full parents under the design $d$. Let $r_h$ and $r_f$ are the replications of lines as half-parent and full-parent, respectively and let $E_h$ and $E_f$ denote the canonical efficiency factors pertaining to gca effects of half parents and full parents, respectively, then we can calculate the efficiencies as:

$$E_h = \frac{1}{r_h}\text{(harmonic mean of non-zero eigenvalues of } \mathbf{C}_{d_{gca-half}}\text{) and}$$

$$E_f = \frac{1}{r_f}\text{(harmonic mean of non-zero eigenvalues of } \mathbf{C}_{d_{gca-full}}\text{).}$$

**AM-HM inequality:** Let $a_1, a_2, \dots, a_i, \dots, a_n$ be a set of $n$ real positive numbers. Then, the Arithmetic Mean (AM) and Harmonic Mean (HM) of $a_i$s $(i = 2, \dots, n)$ are $\frac{1}{n}\sum_{i=1}^{n} a_i$ and $\frac{n}{\sum_{i=1}^{n}(\frac{1}{a_i})}$ respectively. Then according to AM-HM inequality we have

$$\frac{1}{n}\sum_{i=1}^{n} a_i \geq \frac{n}{\sum_{i=1}^{n}(\frac{1}{a_i})} \quad \Rightarrow \sum_{i=1}^{n}(\frac{1}{a_i}) \geq \frac{n^2}{\sum_{i=1}^{n} a_i},$$

where equality is attained when all $a_i$'s $(i = 2, \dots, n)$ are equal.

**Henderson Method III (**Searle *et al.*, 1992**):** The three methods of Henderson (I, II and III) are just three of many possible variations of the ANOVA method which can be used for handling unbalanced data in the process of estimating variance components. Out of the three different ways of using unbalanced data, from random or mixed models, Method III is based on the concept of borrowing sums of squares from the analysis of fixed effects models. The sums of squares used are the reductions in sums of squares due to fitting one model and various sub-models of it. In this method, the results of both fixed effects model and mixed effects model is used. The merits of Method III include that the estimates are unbiased, it can be applied to all mixed models, and interactions between random factors can be included in the model. For balanced data all the three methods are same.

### 3.7 Balanced incomplete block (BIB) designs

A BIB design (Dey, 1986) can be defined as an incomplete block design in which $v$ treatments are arranged in $b$ blocks of size $k$ each, such that the block contents are distinct from each other and are less in number than $v$, with each treatment and pair of treatments replicated in $r$ and $\lambda$ blocks, respectively. Here, the positive integers $v$, $b$, $r$, $k$ and $\lambda$ are usually referred in the literature as the parameters of the BIB design. These designs are binary, equireplicated, proper and variance balanced.

### 3.8 Partially balanced incomplete block (PBIB) designs and association schemes

**m-class association scheme** (Dey, 1986): An abstract relationship defined on $v$ symbols or treatments is called an $m$-class association scheme ($m \geq 2$) if the following conditions are satisfied:

- Any two treatments $a_1$ and $a_2$ are either 1st, 2nd, … , or, $m^{\text{th}}$ associates, the relation of association being symmetrical, *i.e.*, if $a_1$ is the $i^{\text{th}}$ associate of $a_2$, then so is $a_2$ of $a_1$.
- Given a treatment $a_1$, the number of treatments that are $i^{\text{th}}$ associates of $a_1$ is $n_i$ for $i = 1, 2, \dots, m$, where the number $n_i$ does not depend on the treatments chosen, *viz.*, $a_1$.
- Given a pair of treatments $a_1$ and $a_2$, which are mutually $i^{\text{th}}$ associates, the number of treatments which are simultaneously $j^{\text{th}}$ associate of $a_1$ and $k^{\text{th}}$ associate of $a_2$ is $p^i_{jk}$, where $p^i_{jk}$ does not depend on the pair of $i^{\text{th}}$ associates chosen, *viz.*, $a_1$ and $a_2$.
- The positive integers $v$, $n_i$, $p^i_{jk}$ $(i, j, k = 1, 2, \dots, m)$ are usually referred as the parameters of the m-class association scheme.

**Triangular association scheme:** Triangular association scheme is a two-class association scheme which holds particular interest in the construction of designs for breeding trials, due to the special properties they possess. The scheme is defined (Dey, 1986) as follows:

Let there be $t = \frac{n(n-1)}{2}$ treatments, arranged in a $n \times n$ square array of side $n$, such that the principal diagonal positions of the array are kept empty, the $\frac{n(n-1)}{2}$ positions above the main diagonal are placed with the $t$ treatment symbols and in a similar manner the positions below the main diagonal are filled up by the $t$ symbols such that the resultant arrangement is symmetrical about the main diagonal. The association rule followed in this two-class triangular association scheme is that two treatments are first associates if they are placed in the same row or same column of the array and are second associates, otherwise.

Example: Let $n$ = 5. The association scheme for $t$ = 10 treatments (K, L, M, N, O, P, Q, R, S and T) can be depicted as:

| | | | | |
|---|---|---|---|---|
| * | K | L | M | N |
| K | * | O | P | Q |
| L | O | * | R | S |
| M | P | R | * | T |
| N | Q | S | T | * |

The first and second associates of all the treatments are as follows:

| **Treatment** | **$1^{st}$ Associates** | **$2^{nd}$ Associates** |
|---|---|---|
| K | L, M, N, O, P, Q | R, S, T |
| L | K, M, N, O, R, S | P, Q, T |
| M | K, L, N, P, R, T | O, Q, S |
| N | K, L, M, Q, S, T | O, P, R |
| O | K, P, Q, L, R, S | M, N, T |
| P | K, O, Q, M, R, T | L, N, S |
| Q | K, O, P, N, S, T | L, M, R |
| R | L, O, S,M, P, T | K, N, Q |
| S | L, O, R, N, Q, T | M, K, P |
| T | M, P, R, N, Q, S | K, L, O |

**PBIB designs (**Dey, 1986**):** PBIB designs belong to the class of incomplete block designs, and are based on $m$ class ($m \geq 2$) association schemes. In these, $v$ treatments are arranged in $b$ blocks of size $k$ each, such that the block contents are distinct and are less in number than $v$, with each treatment and pair of treatments are replicated in $r$ and $\lambda_i$ blocks, respectively. If two treatments $a_1$ and $a_2$ are mutually $i^{\text{th}}$ associates in the association scheme, then $a_1$ and $a_2$ occur together in $\lambda_i$ blocks, where the integer $\lambda_i$ does not depend on the pair ($a_1$, $a_2$ ) so long as they are mutually $i^{\text{th}}$ associates, $i = 1, 2, \ldots, m$. Further, not all $\lambda_i$'s are equal. Here, the positive integers $v$, $b$, $r$, $k$ and $\lambda_i$ are usually referred as the parameters of the design. These designs are binary, equireplicated, proper but not variance balanced.

### 3.9 Latin squares

A Latin square (LS) is defined (Yates, 1940) as an arrangement of $t$ symbols in $t^2$ cells arranged in $t$ rows and $t$ columns such that each symbol occurs in each row and in each column exactly once. This $t$ is called the order of the Latin square. Two Latin squares of order 3 and 5, respectively are given below.

| | | |
|---|---|---|
| $\alpha$ | $\beta$ | $\gamma$ |
| $\beta$ | $\gamma$ | $\alpha$ |
| $\gamma$ | $\alpha$ | $\beta$ |

| | | | | |
|---|---|---|---|---|
| P | Q | R | S | T |
| Q | R | S | T | P |
| R | S | T | P | Q |
| S | T | P | Q | R |
| T | P | Q | R | S |

**Standard Form of a Latin square:** A standard Latin square has the symbols or letters constituting the rows and columns appear in natural order in the first row and first column.
The following Latin squares are in standard form:

| | | | |
|---|---|---|---|
| 0 | 1 | 2 | 3 |
| 1 | 2 | 3 | 0 |
| 2 | 3 | 0 | 1 |
| 3 | 0 | 1 | 2 |

| | | | |
|---|---|---|---|
| A | B | C | D |
| B | C | D | A |
| C | D | A | B |
| D | A | B | C |

**Orthogonal Latin Squares (OLS):** Two Latin Squares of order $t$, are said to be orthogonal to each other if one is superimposed on other then every pair of symbol is distinct and appears exactly once. Consider the following two Latin Squares of order 3:

| | | |
|---|---|---|
| $\alpha$ | $\beta$ | $\gamma$ |
| $\beta$ | $\gamma$ | $\alpha$ |
| $\gamma$ | $\alpha$ | $\beta$ |
| **LS-I** | | |

**(i)**

| | | |
|---|---|---|
| $\alpha$ | $\gamma$ | $\beta$ |
| $\beta$ | $\alpha$ | $\gamma$ |
| $\gamma$ | $\beta$ | $\alpha$ |
| **LS-II** | | |

**(ii)**

| | | |
|---|---|---|
| $\alpha\alpha$ | $\beta\gamma$ | $\gamma\beta$ |
| $\beta\beta$ | $\gamma\alpha$ | $\alpha\gamma$ |
| $\gamma\gamma$ | $\alpha\beta$ | $\beta\alpha$ |
| **LS-I superimposed on LS-II** | | |

**(iii)**

From (iii) it can be seen that LS-I and LS-II are two orthogonal Latin squares.

**Mutual Orthogonal Latin Squares (MOLS):** In a complete set of Latin squares of order $t$, if every pair of Latin squares are orthogonal to each other then the set is called MOLS of order $t$. The maximum number of MOLS possible of order $t$ is $(t-1)$. Set of such $(t-1)$ OLS is known as a complete set of MOLS. Complete set of MOLS of order $t$ exists when $t$ is either prime or prime power. A table of complete sets of MOLS for $t^2 \leq 9$ can be seen in Fisher and Yates (1963).

A simple method of generating complete set of MOLS is used in this study for number of symbols as odd prime. The elements of first row of first LS are taken in natural order and then the columns are developed by jumping one symbol for first LS, jumping two symbol for second LS and so on.

For example, a complete set of MOLS of order 5, using symbols P, Q, R, S and T is given below, where the complete set of 4 MOLS is generated through a very simple procedure:

| | | | | |
|---|---|---|---|---|
| P | Q | R | S | T |
| Q | R | S | T | P |
| R | S | T | P | Q |
| S | T | P | Q | R |
| T | P | Q | R | S |

**LS-I**

| | | | | |
|---|---|---|---|---|
| P | Q | R | S | T |
| R | S | T | P | Q |
| T | P | Q | R | S |
| Q | R | S | T | P |
| S | T | P | Q | R |

**LS-II**

| | | | | |
|---|---|---|---|---|
| P | Q | R | S | T |
| S | T | P | Q | R |
| Q | R | S | T | P |
| T | P | Q | R | S |
| R | S | T | P | Q |

**LS-III**

| | | | | |
|---|---|---|---|---|
| P | Q | R | S | T |
| T | P | Q | R | S |
| S | T | P | Q | R |
| R | S | T | P | Q |
| Q | R | S | T | P |

**LS-IV**

Now, it can be verified by superimposition of a LS on other that every possible combination of two LS chosen from these LS are OLS.

| | | | | |
|---|---|---|---|---|
| PP | QQ | RR | SS | TT |
| QR | RS | ST | TP | PQ |
| RT | SP | TQ | PR | QS |
| SQ | TR | PS | QT | RP |
| TS | PT | QP | RQ | SR |

**LS-I on LS-II**

| | | | | |
|---|---|---|---|---|
| PP | QQ | RR | SS | TT |
| QS | RT | SP | TQ | PR |
| RQ | SR | TS | PT | QP |
| ST | TP | PQ | QR | RS |
| TR | PS | QT | RP | SQ |

**LS-I on LS-III**

| | | | | |
|---|---|---|---|---|
| PP | QQ | RR | SS | TT |
| QT | RP | SQ | TR | PS |
| RS | ST | TP | PQ | QR |
| SR | TS | PT | QP | RQ |
| TQ | PR | QS | RT | SP |

**LS-I on LS-IV**

| | | | | |
|---|---|---|---|---|
| PP | QQ | RR | SS | TT |
| RS | ST | TP | PQ | QR |
| TQ | PR | QS | RT | SP |
| QT | RP | SQ | TR | PS |
| SR | TS | PT | QP | RQ |

**LS-II on LS-III**

| | | | | |
|---|---|---|---|---|
| PP | QQ | RR | SS | TT |
| RT | SP | TQ | PR | QS |
| TS | PT | QP | RQ | SR |
| QR | RS | ST | TP | PQ |
| SQ | TR | PS | QT | RP |

**LS-II on LS-IV**

| | | | | |
|---|---|---|---|---|
| PP | QQ | RR | SS | TT |
| ST | TP | PQ | QR | RS |
| QS | RT | SP | TQ | PR |
| TR | PS | QT | RP | SQ |
| RQ | SR | TS | PT | QP |

**LS-III on LS-IV**

### 3.10 Lattice Designs

Lattice designs belong to the class of incomplete block designs. They may fall in the category of either BIB or PBIB designs. In case of Lattice designs a subgroup of blocks together makes a complete replication *i.e.,* we have blocks nested within replications. All types of lattice designs have the property of resolvability *i.e.,* a constant number of blocks can be taken at a time so as to form groups of blocks, each one of which is a complete replication. On the basis of structure of treatments number, lattice designs can be mainly classified as Square Lattice, Cubic lattice, Circular lattice and Rectangular lattice.

**Square lattice designs** (Yates, 1940): Square lattice designs have treatment structure of the form $v = s^2$. The parameters of a square lattice design is given as $v = s^2, b = is, r = i$ and $k = s$, where $i = 2,3,\dots,(s+1)$. The number of replications and thus the number of blocks are variables based on which they can be either BIB or two associate class PBIB designs. The lattices can be named as simple, triple, quadruple, $m$-ple and balanced lattice for $i = 2$, $i = 3$, $i = 4$, $i = m$ and $i = (s+1)$, respectively. The lattices leads to a PBIB design for $i = 2,3,\dots,s$, and for $i = (s+1)$ we get a BIB design. The method of construction of square lattices is based on MOLS.

For any given treatment structure, the first two replications can be generated by writing the $s^2$ treatments as a $s \times s$ array in the following fashion

$$\begin{vmatrix} 1 & 2 & \cdots & s \\ s+1 & s+2 & \cdots & 2s \\ \vdots & \vdots & & \vdots \\ s(s-1)+1 & s(s-1)+2 & \cdots & s^2 \end{vmatrix} \text{ and } \begin{vmatrix} 1 & s+1 & \cdots & s(s-1)+1 \\ 2 & s+2 & \cdots & s(s-1)+2 \\ \vdots & \vdots & & \vdots \\ s & 2s & \cdots & s^2 \end{vmatrix}.$$

The rest of maximum possible $(s-1)$ replications are generated using MOLS of order $s$, to constitute a total of $(s+1)$ replications.

**Example 3.10.1**: Here is an example for $v = s^2 = 9$. The method of construction is based on MOLS of order 3. Writing the 9 treatments in a square array row-wise and column-wise we get six blocks. Then consider the two MOLS of order 3. These two OLS are superimposed on the original $3 \times 3$ array of the symbols, one by one, and same symbol positions are taken as block contents to get six more blocks. Hence, the four replications are obtained to result in a balanced square lattice design with parameters $v = s^2 = 9, b = 12, r = 4$ and $k = 3$.

| Rep I | | | |
|---|---|---|---|
| **Blk 1** | p | q | r |
| **Blk 2** | s | t | u |
| **Blk 3** | v | w | x |
| **Using rows** | | | |

| Rep II | | | |
|---|---|---|---|
| **Blk 1** | p | s | v |
| **Blk 2** | q | t | w |
| **Blk 3** | r | u | x |
| **Using columns** | | | |

| Rep III | | | |
|---|---|---|---|
| **Blk 1** | p | u | w |
| **Blk 2** | q | s | x |
| **Blk 3** | r | t | v |
| **Using OLS I** | | | |

| Rep IV | | | |
|---|---|---|---|
| **Blk 1** | p | t | x |
| **Blk 2** | q | u | v |
| **Blk 3** | r | s | w |
| **Using OLS II** | | | |

If two replications of this design are used, then it will lead to a simple square lattice and if we consider three replications then it is known as triple lattice. The underlying association scheme can be taken as $L_2$ (Latin Square association scheme with two constraints) by considering the first $s \times s$ array, where for a given treatment, treatments occurring in either of same row or column are considered as first associates and the rest are second associates.

**Cubic lattice designs (**Das and Giri, 1986**):** Cubic lattices are formed when the treatments can be expressed in the $v = s^3$ structure. Cubic lattice designs belong to the three associate class PBIB designs. These designs have fixed 3 replications and the number of blocks is a multiple of 3. The parameters of these designs are $v = s^3, b = 3s^2, r = 3$ and $k = s$.

**Example 3.10.2:** Here is an example for $v = s^3 = 27$. In order to construct the design, first we have to consider the underlying association scheme. First we have to make the following arrangement in which the triplets are the positions of treatments column-wise, block wise and row-wise respectively. The triplet (1 2 3) means that the corresponding treatment 6 is present in the third row of second block in the first column.

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| **1** | 1 | 1 | 1 | **10** | 2 | 1 | 1 | **19** | 3 | 1 | 1 |
| **2** | 1 | 1 | 2 | **11** | 2 | 1 | 2 | **20** | 3 | 1 | 2 |
| **3** | 1 | 1 | 3 | **12** | 2 | 1 | 3 | **21** | 3 | 1 | 3 |
| **4** | 1 | 2 | 1 | **13** | 2 | 2 | 1 | **22** | 3 | 2 | 1 |
| **5** | 1 | 2 | 2 | **14** | 2 | 2 | 2 | **23** | 3 | 2 | 2 |
| **6** | 1 | 2 | 3 | **15** | 2 | 2 | 3 | **24** | 3 | 2 | 3 |
| **7** | 1 | 3 | 1 | **16** | 2 | 3 | 1 | **25** | 3 | 3 | 1 |
| **8** | 1 | 3 | 2 | **17** | 2 | 3 | 2 | **26** | 3 | 3 | 2 |
| **9** | 1 | 3 | 3 | **18** | 2 | 3 | 3 | **27** | 3 | 3 | 3 |

Now, two treatments are said to be first associates if they have two positions either of three in common, second associates if they have one position in common, otherwise they are third associates. The cubic lattice design based on the association scheme with parameters $v = 27, b = 27, r = 3$ and $k = 3$ is given below.

| Rep I | | | |
|---|---|---|---|
| **Blk 1** | 1 | 2 | 3 |
| **Blk 2** | 4 | 5 | 6 |
| **Blk 3** | 7 | 8 | 9 |
| **Blk 4** | 10 | 11 | 12 |
| **Blk 5** | 13 | 14 | 15 |
| **Blk 6** | 16 | 17 | 18 |
| **Blk 7** | 19 | 20 | 21 |
| **Blk 8** | 22 | 23 | 24 |
| **Blk 9** | 25 | 26 | 27 |

| Rep II | | | |
|---|---|---|---|
| **Blk 1** | 1 | 4 | 7 |
| **Blk 2** | 2 | 5 | 8 |
| **Blk 3** | 3 | 6 | 9 |
| **Blk 4** | 10 | 13 | 16 |
| **Blk 5** | 11 | 14 | 17 |
| **Blk 6** | 12 | 15 | 18 |
| **Blk 7** | 19 | 22 | 25 |
| **Blk 8** | 20 | 23 | 26 |
| **Blk 9** | 21 | 24 | 27 |

| Rep III | | | |
|---|---|---|---|
| **Blk 1** | 1 | 10 | 19 |
| **Blk 2** | 2 | 11 | 20 |
| **Blk 3** | 3 | 12 | 21 |
| **Blk 4** | 4 | 13 | 22 |
| **Blk 5** | 5 | 14 | 23 |
| **Blk 6** | 6 | 15 | 24 |
| **Blk 7** | 7 | 16 | 25 |
| **Blk 8** | 8 | 17 | 26 |
| **Blk 9** | 9 | 18 | 27 |

The blocks of first replication are generated by taking those treatments which are present in the same blocks or treatments having same rows and column positions. The blocks of second replication are constituted by those treatments which are having first and third positions same. The third replication includes the blocks having same second and third position.

**Circular lattice designs (**Rao, 1956**):** Circular lattices can be formed for treatments which takes the form $v = 2s^2$. Circular lattice designs are three associate class PBIB designs with parameters $v = 2s^2, b = 2s, r = 2$ and $k = 2s$. The association scheme is established by taking $s$ concentric circles and its $s$ diagonals and the lattice points formed by the intersection of circles and diameters is numbered and considered as treatments. Then, for any treatment the first associates are the treatments on same circle and same diagonal, second associates are the treatments either on same circle or same diagonal, and rest are third associates.

**Example 3.10.3**: Consider an example for $s = 2$ yielding $v = 2s^2 = 8$. Then, we have to take two concentric circles along with the two diagonals as shown in Fig. 3.10.3.

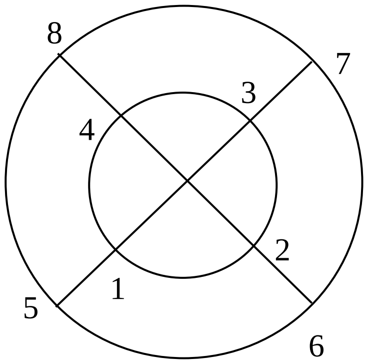


**Fig. 3.10.3**

Now, the circular lattice design can be obtained by developing the blocks of first replication by taking the treatments on one circle as one block. The blocks of second replication are obtained by taking the treatments on one diagonal as one block. Thus, for the given example the design is obtained with parameters $v = 8, b = 4, r = 2$ and $k = 4$.

| | | | | | |
|---|---|---|---|---|---|
| **Rep-1** | **Blk I** | 1 | 2 | 3 | 4 |
| | **Blk II** | 5 | 6 | 7 | 8 |
| **Rep-2** | **Blk I** | 1 | 3 | 5 | 7 |
| | **Blk II** | 2 | 4 | 6 | 8 |

**Rectangular lattice designs (**Nair, 1951**):** Rectangular lattice designs can be constructed for treatments structure expressed as $v = s(s+1)$. These designs belongs to the four associate class of PBIB designs with parameters $v = s(s+1), b = s(s+1), r = s$ and $k = s$ or $v = s(s+1)$, $b = (s+1)^2$, $r = (s+1)$ and $k = s$.

Rectangular lattice designs with parameters $v = s(s+1)$, $b = (s+1)^2$, $r = (s+1)$ and $k = s$ can be obtained using balanced lattice design for $v = (s+1)^2$. The method advocates that from the selected balanced lattice design any replication is chosen and deleted and from the rest of replications the extra treatments are discarded to give a rectangular lattice design.

**Example 3.10.4:** Here is an example for $v = s(s+1) = 6$. Consider the balanced lattice design for 9 treatments. The first replication is deleted and the treatments 7, 8 and 9 are discarded from the rest of replication to give a rectangular design with parameters $v = 6$, $b = 9$, $r = 3$ and $k = 2$ as shown below:

| | | | |
|---|---|---|---|
| **Rep I** | **Blk 1** | 1 | 4 |
| | **Blk 2** | 2 | 5 |
| | **Blk 3** | 3 | 6 |
| **Rep II** | **Blk 1** | 1 | 6 |
| | **Blk 2** | 2 | 4 |
| | **Blk 3** | 3 | 5 |
| **Rep III** | **Blk 1** | 1 | 5 |
| | **Blk 2** | 2 | 6 |
| | **Blk 3** | 3 | 4 |

## 3.11 Kronecker Product

In similar lines to Searle (1982), if there are two matrices, $\boldsymbol{K} = \{k_{ij}\}$ and $\boldsymbol{M} = \{m_{ij}\}$, of order $a \times b$ and $c \times d$ respectively, then the Kronecker product of these two matrices is given as

$$\boldsymbol{K} \otimes \boldsymbol{M} = \begin{bmatrix} k_{11}\boldsymbol{M} & \dots & k_{1a}\boldsymbol{M} \\ \vdots & \ddots & \ddots \\ k_{b1}\boldsymbol{M} & \dots & k_{ab}\boldsymbol{M} \end{bmatrix}$$

$$= \begin{bmatrix} k_{11}m_{11} & \dots & k_{11}m_{ij} & \cdots & \cdots & k_{1a}m_{ij} & \dots & k_{1a}m_{ij} \\ \vdots & \ddots & \ddots & & & \vdots & \ddots & \ddots \\ k_{11}m_{ij} & \dots & k_{11}m_{ij} & \cdots & \cdots & k_{1a}m_{ij} & \dots & k_{1a}m_{ij} \\ \vdots & & \vdots & & & \vdots & & \vdots \\ \vdots & & \vdots & & & \vdots & & \vdots \\ k_{b1}m_{ij} & \dots & k_{b1}m_{ij} & \cdots & \cdots & k_{ab}m_{ij} & \dots & k_{ab}m_{ij} \\ \vdots & \ddots & \ddots & & & \vdots & \ddots & \ddots \\ k_{b1}m_{ij} & \dots & k_{b1}m_{ij} & \cdots & \cdots & k_{ab}m_{ij} & \dots & k_{ab}m_{ij} \end{bmatrix},$$

where the order of the resultant matrices is $ac \times bd$.

This can be more easily understood through an example.

Let $\boldsymbol{K} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$ and $\boldsymbol{M} = \begin{bmatrix} 1 & 2 & 4 & 0 \\ 1 & 1 & 3 & 1 \\ 2 & 3 & 2 & 1 \end{bmatrix}$, then the Kronecker product of these two matrices a matrix of order $6 \times 8$ is given as:

$$\boldsymbol{K} \otimes \boldsymbol{M} = \begin{bmatrix} 2 & 4 & 8 & 0 & 1 & 2 & 4 & 0 \\ 2 & 2 & 6 & 2 & 1 & 1 & 3 & 1 \\ 4 & 6 & 4 & 2 & 2 & 3 & 2 & 1 \\ 1 & 2 & 4 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 3 & 1 & 0 & 0 & 0 & 0 \\ 2 & 3 & 2 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

## 3.12 Full rank factorization

Searle (1982) defined full rank factorization of a matrix, which is not of full rank. This factorization is useful in calculating Moore-Penrose inverse in this study. Let $\boldsymbol{A}$ be a matrix of order $m \times n$, such that $rank(\boldsymbol{A}) = r$. Then using full rank factorization it is possible to express $\boldsymbol{A}$ as the product of two matrices with full rank as follows:

$$\boldsymbol{A} = \boldsymbol{PG},$$

where $\boldsymbol{P}$ and $\boldsymbol{G}$ are full rank matrices of order $m \times r$ and $r \times n$, respectively.

Now, there is at least one full rank matrix $\boldsymbol{X}$ of order $r \times r$, such that we can have following representation

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{X} & \boldsymbol{Y} \\ \boldsymbol{Z} & \boldsymbol{U} \end{bmatrix},$$

where $\boldsymbol{Y}$, $\boldsymbol{Z}$ and $\boldsymbol{U}$ are matrices of order $r \times (n-r)$, $(m-r) \times r$ and $(m-r) \times (n-r)$, respectively. Since, $rank(\boldsymbol{A}) = r$, there are $r$ linearly independent rows and columns and remaining are linear combination of them.

Thus, we can write the matrices $\boldsymbol{Y}$, $\boldsymbol{Z}$ and $\boldsymbol{U}$ as:

$$[\boldsymbol{Z} \quad \boldsymbol{U}] = \boldsymbol{F}[\boldsymbol{X} \quad \boldsymbol{Y}] \text{ and } \begin{bmatrix} \boldsymbol{Y} \\ \boldsymbol{U} \end{bmatrix} = \begin{bmatrix} \boldsymbol{X} \\ \boldsymbol{Z} \end{bmatrix} \boldsymbol{H},$$

where $\boldsymbol{F}$ and $\boldsymbol{H}$ are matrices of order $(m-r) \times r$ and $r \times (n-r)$ respectively, such that $\boldsymbol{F} = \boldsymbol{Z}\boldsymbol{X}^{-1}$ and $\boldsymbol{H} = \boldsymbol{X}^{-1}\boldsymbol{Y}$. Thus we can have following representations for $\boldsymbol{A}$

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{X} & \boldsymbol{XH} \\ \boldsymbol{FX} & \boldsymbol{FXH} \end{bmatrix}$$

$$= \begin{bmatrix} \boldsymbol{I} \\ \boldsymbol{F} \end{bmatrix} \boldsymbol{X}[\boldsymbol{I} \quad \boldsymbol{H}]$$

$$= \begin{bmatrix} \boldsymbol{I} \\ \boldsymbol{Z}\boldsymbol{X}^{-1} \end{bmatrix} \boldsymbol{X}[\boldsymbol{I} \quad \boldsymbol{X}^{-1}\boldsymbol{Y}]$$

$$= \begin{bmatrix} \boldsymbol{X} \\ \boldsymbol{Z} \end{bmatrix} [\boldsymbol{I} \quad \boldsymbol{X}^{-1}\boldsymbol{Y}] = \begin{bmatrix} \boldsymbol{I} \\ \boldsymbol{Z}\boldsymbol{X}^{-1} \end{bmatrix} [\boldsymbol{X} \quad \boldsymbol{Y}].$$

**3.13 SAS Codes**

The following programs have been written in SAS [PROC IML] software to study the characterization properties of the developed designs

- SAS code for computing canonical efficiency factor of the design involving triallel crosses for estimating gca effects for half parents as well as full parents under unblocked set-up.

- SAS code for computing canonical efficiency factor of the design involving triallel crosses for estimating gca effects for half parents as well as full parents under blocked set-up.

- SAS code for computing canonical efficiency of the disturbed design involving triallel crosses for estimating gca effects for half parents as well as full parents under unblocked set-up to investigate the robustness of designs against missing observation using the connectedness and efficiency criteria.

- SAS code for computing canonical efficiency of the disturbed design involving triallel crosses for estimating gca effects for half parents as well as full parents under blocked set-up to investigate the robustness of designs against missing observation using the connectedness and efficiency criteria.

- SAS code for computing canonical efficiency of the disturbed design involving tetra-allele crosses for estimating gca effects under unblocked set-up to investigate the robustness of designs against missing observation using the connectedness and efficiency criteria.

- SAS code for computing canonical efficiency of the disturbed design involving tetra-allele crosses for estimating gca effects under blocked set-up to investigate the robustness of designs against missing observation using the connectedness and efficiency criteria.

These programs are given in Annexures.

## 4.1 Introduction

Greater genetical viability, and stability along with consistency of higher order crosses as compared to diallel crosses, is the main attraction to breeders for adopting higher order mating plans like triallel and tetra-allele crosses. Besides this, triallel and tetra-allele cross hybrids exhibit higher individual as well as population buffering mechanism because of the broad genetic base.

## 4.2 Higher order mating designs incorporating sca effects

For breeders, sca effects are of much importance in addition to gca effects. In case of diallel crosses, only a first order sca effect can be studied whereas in triallel crosses three first order sca effects alongwith a second order sca effect and in case of tetra-allele crosses first, second and even third order sca effects can be studied. Thus, these techniques provide ample amount of information on sca effects and help the breeders to improve various traits which are of economical as well as nutritional importance in crops and animals.

### 4.2.1 Triallel cross experiments with sca effects

Triallel cross hybrids find a vital role in the area of plant and animal breeding experiments due to their uniformity, stability and the relative simplicity of selecting and testing. Triallel crosses are intermediate between diallel and tetra-allele crosses with respect to number of lines used, complexity of handling the crosses and the amount of information regarding combining abilities.

**Model and experimental setup**

There are various methods of analysing the data collected through a triallel cross experiment, based on the model considered. The model may include all types of sca effects, or only lower order sca effects or may not include sca effects, along with gca effects, as per the experimental objectives.

#### 4.2.1.1 Model with first order sca effects

Let us consider that the triallel crosses $(i,j,k)$, are arranged in the order $(1,2{,}3)$, $(1,2{,}4)$,...,$(1{,}2,k)$,...,$(1{,}2,N)$,$(1{,}3{,}4)$,...,$(1,j,j+1)$,...,$(1,(N-1),N)$,$(2,3{,}4)$,..., $(i,i+1,i+2)$,..., $\{(N-2),(N-1),N\}\ \forall\ i \neq j \neq k = 1{,}2,\dots,N$. It should be noted that along with every cross of the type $(i,j,k)$ crosses of the types $(i,k,j)$ and $(j,k,i)$ are also included in the experiment simultaneously. Let $\boldsymbol{g} = (g_1,\ g_2, \dots, g_N)'$ be a $N \times 1$ vector of gca effects of full parents, $\boldsymbol{h} = (h_1,\ h_2, \dots, h_N)'$ be a $N \times 1$ vector of gca effects of half parents. $\boldsymbol{y}$ and $\boldsymbol{s}$ be $T \times 1$ vectors whose elements are $\{y_{ijk}\}$ and $\{s_{ijk}\}$, respectively. Let us define a matrix $\boldsymbol{W}$of order $2N \times T$ with rows indexed by the line numberings as $1,2,\dots,N$ repeatedly two times and, the columns by the $T$ number of crosses, in same order as described previously, of the types $(i,j,k)$, including the crosses of the types $(i,k,j)$ and $(j,k,i)$ also simultaneously. Then, the $\{t,(i,j,k)\}^{\text{th}}$ entry of $\boldsymbol{W}$ takes a value 0.5 if $t \in (i,j)$, takes a value 1 if $t \in k$ and 0, otherwise. Before arriving at the final model we have $\boldsymbol{W}\boldsymbol{1}_T = \frac{(N-1)(N-2)}{2}\boldsymbol{1}_{2N}, \boldsymbol{W}'\boldsymbol{1}_{2N} = 2\boldsymbol{1}_N$ and $\bar{y} = \frac{\boldsymbol{1}_T'\boldsymbol{y}}{T}$. Now the model can be rewritten in matrix notation as follows:

$$\boldsymbol{y} = \bar{y}\,\boldsymbol{1}_T + \boldsymbol{W}'\begin{pmatrix}\boldsymbol{g}\\ \boldsymbol{h}\end{pmatrix} + \boldsymbol{s} + \boldsymbol{e}, \tag{4.2.1.1.1}$$

where $\bar{y}$ is the average effect of the crosses, $\boldsymbol{1}_T$ is the $T \times 1$ vector of unity and $\boldsymbol{e}$ is the $T \times 1$ vector of random errors . The constraints

$$g_1 + g_2 + \cdots + g_N = 0 \text{ or } \textstyle\sum_{i=1}^{N} g_i = 0 \text{ or } \boldsymbol{1}_N'\boldsymbol{g} = 0,$$

$$h_1 + h_2 + \cdots + h_N = 0 \text{ or } \textstyle\sum_{i=1}^{N} h_i = 0 \text{ or } \boldsymbol{1}_N'\boldsymbol{h} = 0 \text{ and}$$

$$\textstyle\sum_j \sum_{k>j} s_{(jk)i} = 0\ \forall\ 1 \le i \le N \text{ or } \boldsymbol{W}\boldsymbol{s} = \boldsymbol{0}, \tag{4.2.1.1.2}$$

have been imposed to the model.

Before proceeding further it is important to derive the forms of $\boldsymbol{W}\boldsymbol{W}'$ and $(\boldsymbol{W}\boldsymbol{W}')^{+}$.

$$\boldsymbol{W}\boldsymbol{W}' = \begin{bmatrix} \frac{(N-1)(N-2)}{2}\boldsymbol{I}_N & -\frac{(N-2)}{2}(\boldsymbol{I}_N - \boldsymbol{J}_N) \\ -\frac{(N-2)}{2}(\boldsymbol{I}_N - \boldsymbol{J}_N) & \frac{(N-2)^2}{4}\boldsymbol{I}_N + \frac{(N-2)}{4}\boldsymbol{J}_N \end{bmatrix} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{B}' & \boldsymbol{D} \end{bmatrix} \text{(say)} \tag{4.2.1.1.3}$$

Now, we can see that $rank(\boldsymbol{A}) = rank\ (\boldsymbol{B}) = rank\ (\boldsymbol{D}) = N$ and $rank\ (\boldsymbol{W}\boldsymbol{W}') = 2N - 1$. Thus, it is not possible to find the true inverse of $\boldsymbol{W}\boldsymbol{W}'$. Also, we can see that the condition $rank\ (\boldsymbol{W}\boldsymbol{W}') = rank(\boldsymbol{A})$ for finding the generalized inverse of a matrix by partitioning is not satisfied. So, in order to find a unique inverse of $\boldsymbol{W}\boldsymbol{W}'$, it is better to proceed for finding the Moore Penrose inverse denoted as $(\boldsymbol{W}\boldsymbol{W}')^{+}$, through full rank factorization method.

**Moore Penrose inverse** $(\boldsymbol{WW'})^{+}$: In order to find the $(\boldsymbol{WW}')^{+}$ the first step is to write $\boldsymbol{WW'}$ as the product of two matrices which are of full column and full row rank, respectively.

Let $\boldsymbol{WW}' = \begin{bmatrix} \boldsymbol{X} & \boldsymbol{z} \\ \boldsymbol{z}' & \boldsymbol{u} \end{bmatrix}$, be written as the product of two matrices $\boldsymbol{K}$ and $\boldsymbol{L}$ which are of full column and row rank, respectively, where $\boldsymbol{K} = \begin{bmatrix} \boldsymbol{X} \\ \boldsymbol{z}' \end{bmatrix}$ and $\boldsymbol{L} = [\boldsymbol{I} \quad \boldsymbol{X}^{-1}\boldsymbol{z}]$. Since the rank of $\boldsymbol{WW'}$ is $(2N-1)$, *i.e.* one less than full rank. $\boldsymbol{X}$ can be taken as top left $(2N-1)$ rows and columns of $\boldsymbol{WW'}$. This top left $(2N-1)$ portion of $\boldsymbol{WW'}$ can be always made of full rank by pre or post multiplying $\boldsymbol{WW'}$ by permutation matrices of suitable order. Thus, $\boldsymbol{WW'}$ is partitioned and expressed as follows:

$$\boldsymbol{WW}' = \begin{bmatrix} \boldsymbol{X} & \boldsymbol{z} \\ \boldsymbol{z}' & u \end{bmatrix} = \begin{bmatrix} \boldsymbol{X}_{11} & \boldsymbol{X}_{12} & \boldsymbol{z} \\ \boldsymbol{X}_{21} & \boldsymbol{X}_{22} & \boldsymbol{z} \\ \multicolumn{2}{c}{\boldsymbol{z}'} & u \end{bmatrix}.$$

The general form of $\boldsymbol{WW'}$, in terms of number of lines is expressed as:

$$\begin{bmatrix} \left\{ \begin{matrix} \left(\frac{(N-1)(N-2)}{2}\boldsymbol{I}_N\right) & \begin{pmatrix} -\frac{(N-2)}{2}(\boldsymbol{I}_{N-1} - \boldsymbol{J}_{N-1}) \\ \frac{(N-2)}{2}\boldsymbol{1}'_{N-1} \end{pmatrix} \\ \left(-\frac{(N-2)}{2}(\boldsymbol{I}_{N-1} - \boldsymbol{J}_{N-1}) \quad \frac{(N-2)}{2}\boldsymbol{1}_{N-1}\right) & \left(\frac{(N-2)^2}{4}\boldsymbol{I}_{N-1} + \frac{(N-2)}{4}\boldsymbol{J}_{N-1}\right) \end{matrix} \right\} & \boldsymbol{z} \\ \boldsymbol{z}' & u \end{bmatrix},$$

where the vector $\boldsymbol{z} = \begin{pmatrix} \frac{(N-2)}{2}\boldsymbol{1}_{N-1} \\ 0 \\ \frac{(N-2)}{4}\boldsymbol{1}_{N-1} \end{pmatrix}$ and the scalar $u = \frac{(N-2)^2}{4} + \frac{(N-2)}{4}$. Now, to find the inverse of matrix $\boldsymbol{X}$ we have to use the method of partitioning as

$$\boldsymbol{X}^{-1} = \begin{bmatrix} \boldsymbol{X}_{11}^{-1} + \boldsymbol{FE}^{-1}\boldsymbol{F}' & -\boldsymbol{FE}^{-1} \\ -\boldsymbol{E}^{-1}\boldsymbol{F}' & \boldsymbol{E}^{-1} \end{bmatrix},$$

where $\boldsymbol{E} = \boldsymbol{X}_{22} - \boldsymbol{X}_{21}\boldsymbol{X}_{11}^{-1}\boldsymbol{X}_{12}$ and $\boldsymbol{F} = \boldsymbol{X}_{11}^{-1}\boldsymbol{X}_{12}$. The following results have been derived:

$$\boldsymbol{X}_{11}^{-1} = \left\{\frac{2}{(N-1)(N-2)}\boldsymbol{I}_N\right\},$$

$$\boldsymbol{F} = -\frac{1}{(N-1)}\begin{Bmatrix}(\boldsymbol{I}_{N-1} - \boldsymbol{J}_{N-1}) \\ -\mathbf{1}'_{N-1}\end{Bmatrix},$$

$$\boldsymbol{X}_{21}\boldsymbol{X}_{11}^{-1}\boldsymbol{X}_{12} = \frac{(N-2)}{\mathbf{2}(N-1)}\{\boldsymbol{I}_{N-1} + (N-2)\boldsymbol{J}_{N-1}\},$$

$$\boldsymbol{E} = \frac{(N-2)(N-3)}{4(N-1)}\{N\boldsymbol{I}_{N-1} - \boldsymbol{J}_{N-1}\},$$

$$\boldsymbol{E}^{-1} = \frac{4(N-1)}{N(N-2)(N-3)}\{\boldsymbol{I}_{N-1} + \boldsymbol{J}_{N-1}\},$$

$$-\boldsymbol{F}\boldsymbol{E}^{-1} = \frac{4}{N(N-2)(N-3)}\begin{Bmatrix}(\boldsymbol{I}_{N-1} - (N-1)\boldsymbol{J}_{N-1}) \\ -N\mathbf{1}'_{N-1}\end{Bmatrix},$$

$$\boldsymbol{F}\boldsymbol{E}^{-1}\boldsymbol{F}' = \frac{4}{N(N-1)(N-2)(N-3)}\begin{bmatrix}[\boldsymbol{I}_{N-1} + \{(N-1)^2 - N\}\boldsymbol{J}_{N-1}] & N(N-2)\mathbf{1}_{N-1} \\ N(N-2)\mathbf{1}'_{N-1} & N(N-1)\end{bmatrix},$$

and $\boldsymbol{X}_{11}^{-1} + \boldsymbol{F}\boldsymbol{E}^{-1}\boldsymbol{F}'$ is given as

$$\frac{4}{N(N-1)(N-2)(N-3)}\begin{bmatrix}\frac{(N-1)(N-2)}{2}[\boldsymbol{I}_{N-1} + \{(N-1)^2 - N\}\boldsymbol{J}_{N-1}] & N(N-2)\mathbf{1}_{N-1} \\ N(N-2)\mathbf{1}'_{N-1} & \frac{N(3N-5)}{2}\end{bmatrix}.$$

Hence the inverse of matrix $\boldsymbol{X}$ is derived and it can be easily seen that matrices $\boldsymbol{K} = \begin{bmatrix}\boldsymbol{X} \\ \boldsymbol{z}'\end{bmatrix}$ and $\boldsymbol{L} = [\boldsymbol{I} \quad \boldsymbol{X}^{-1}\boldsymbol{z}]$ are of full column and row rank respectively, with $\boldsymbol{X}^{-1}\boldsymbol{z} = \begin{bmatrix}\mathbf{1}_{N-1} \\ 1 \\ -\mathbf{1}_{N-1}\end{bmatrix}$. Thus the Moore Penrose inverse can be obtained by the formulae $(\boldsymbol{W}\boldsymbol{W}')^{+} = \boldsymbol{L}'(\boldsymbol{K}'\boldsymbol{W}\boldsymbol{W}'\boldsymbol{L}')^{-\mathbf{1}}\boldsymbol{K}'$**.** Now, we have to first find $(\boldsymbol{K}'\boldsymbol{W}\boldsymbol{W}'\boldsymbol{L}')^{-\mathbf{1}}$ for which the following results are needed to be obtained first $\boldsymbol{K}'\boldsymbol{W}\boldsymbol{W}' = \begin{pmatrix}\boldsymbol{A}_{11} & \boldsymbol{A}_{12} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22}\end{pmatrix}$,

where $\boldsymbol{A}_{11} = \frac{(N-2)^2}{4}\begin{bmatrix}[\{(N-1)^2+1\}\boldsymbol{I}_{N-1} + (N-2)\boldsymbol{J}_{N-1}] & (N-2)\mathbf{1}_{N-1} \\ (N-2)\mathbf{1}'_{N-1} & N(N-1)\end{bmatrix}$, $\boldsymbol{A}_{12} = \boldsymbol{A}'_{21}$,

where $\boldsymbol{A}_{21} = \frac{(N-2)^2}{8}\{-(3N-4)\boldsymbol{I}_{N-1} + (4N-5)\boldsymbol{J}_{N-1} \qquad (4N-5)\mathbf{1}'_{N-1}\}$, and

$$\boldsymbol{A}_{22} = \frac{(N-2)^2}{8}\begin{bmatrix} (4N-5)\mathbf{1}'_{N-1} & (N-1) \\ \frac{1}{2}[\{(N-2)^2+4\}\boldsymbol{I}_{N-1} + (7N{-}12)\boldsymbol{J}_{N-1}] & \frac{(7N-12)}{2}\mathbf{1}_{N-1} \end{bmatrix};$$

and $\boldsymbol{K}'\boldsymbol{W}\boldsymbol{W}'\boldsymbol{L}' = \begin{pmatrix} \boldsymbol{B}_{11} & \boldsymbol{B}_{12} & \boldsymbol{B}_{13} \\ \boldsymbol{B}_{21} & \boldsymbol{B}_{22} & \boldsymbol{B}_{23} \\ \boldsymbol{B}_{31} & \boldsymbol{B}_{32} & \boldsymbol{B}_{33} \end{pmatrix}$,

where $\boldsymbol{B}_{11} = \frac{(N-2)^2}{8}[2\{(N-1)^2+1\}\boldsymbol{I}_{N-1} + 3(2N{-}3)\boldsymbol{J}_{N-1}]$,

$$\boldsymbol{B}_{12} = \frac{3(N-2)^2(2N-3)}{8}\mathbf{1}_{N-1},$$

$$\boldsymbol{B}_{13} = -\frac{(N-2)^2(3N-4)}{8}\boldsymbol{I}_{N-1},$$

$$\boldsymbol{B}_{21} = \frac{(N-2)^2(3N-4)}{8}\mathbf{1}'_{N-1},$$

$$\boldsymbol{B}_{22} = \frac{(N-2)^2(2N+1)(N-1)}{8},$$

$$\boldsymbol{B}_{23} = \frac{(N-2)^2(3N-4)}{8}\mathbf{1}'_{N-1},$$

$$\boldsymbol{B}_{31} = -\frac{(N-2)^2}{16}[2(3N-4)\boldsymbol{I}_{N-1} - (15N{-}22)\boldsymbol{J}_{N-1}],$$

$\boldsymbol{B}_{32} = \frac{(N-2)^2(15N-22)}{16}\mathbf{1}_{N-1}$ and

$$\boldsymbol{B}_{33} = \frac{(N-2)^2\{(N-2)^2+4\}}{16}\boldsymbol{I}_{N-1}.$$

Since the matrix $\boldsymbol{K}'\boldsymbol{W}\boldsymbol{W}'\boldsymbol{L}'$ is not a symmetric matrix, hence one has to find the inverse of this matrix by the procedure available for non-symmetric matrices.

Consider that $\boldsymbol{K}'\boldsymbol{W}\boldsymbol{W}'\boldsymbol{L}'$ the matrix can be expressed in a new form as follows:

$$\boldsymbol{K}'\boldsymbol{W}\boldsymbol{W}'\boldsymbol{L}' = \begin{pmatrix} \boldsymbol{A}_1 & \boldsymbol{B}_1 \\ \boldsymbol{C}_1 & \boldsymbol{D}_1 \end{pmatrix},$$

where $\boldsymbol{A}_1 = \begin{pmatrix} \boldsymbol{B}_{11} & \boldsymbol{B}_{12} \\ \boldsymbol{B}_{21} & \boldsymbol{B}_{22} \end{pmatrix}$,

$$\boldsymbol{B}_1 = \begin{pmatrix} \boldsymbol{B}_{13} \\ \boldsymbol{B}_{23} \end{pmatrix},$$

$\boldsymbol{C}_1 = (\boldsymbol{B}_{31} \quad \boldsymbol{B}_{32})$ and

$\boldsymbol{D}_1 = \boldsymbol{B}_{23}$. Then, the inverse can be obtained by the following relation:

$$\begin{bmatrix} \boldsymbol{A}_1 & \boldsymbol{B}_1 \\ \boldsymbol{C}_1 & \boldsymbol{D}_1 \end{bmatrix}^{-1} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{D}_1^{-1} \end{bmatrix} + \begin{bmatrix} \boldsymbol{I} \\ \\ -\boldsymbol{D}_1^{-1}\boldsymbol{C}_1 \end{bmatrix} (\boldsymbol{A}_1 - \boldsymbol{B}_1\boldsymbol{D}_1^{-1}\boldsymbol{C}_1)^{-1}[\boldsymbol{I} \quad -\boldsymbol{B}_1\boldsymbol{D}_1^{-1}].$$

Now, we have following results:

$$\boldsymbol{D}_1^{-1} = \frac{16}{(N-2)^2\{(N-2)^2+4\}}\boldsymbol{I}_{N-1},$$

$$-\boldsymbol{D}_1^{-1}\boldsymbol{C}_1 = \frac{1}{\{(N-2)^2+4\}}[2(3N-4)\boldsymbol{I}_{N-1} - (15N-22)\boldsymbol{J}_{N-1} \qquad -(15N-22)\boldsymbol{1}_{N-1}],$$

$$-\boldsymbol{B}_1\boldsymbol{D}_1^{-1} = \frac{2(3N-4)}{\{(N-2)^2+4\}}\begin{bmatrix} \boldsymbol{I}_{N-1} \\ \\ -\boldsymbol{1}'_{N-1} \end{bmatrix},$$

$-\boldsymbol{B}_1\boldsymbol{D}_1^{-1}\boldsymbol{C}_1$ is given as

$$-\frac{(N-2)^2(3N-4)}{8\{(N-2)^2+4\}}\begin{bmatrix} 2(3N-4)\boldsymbol{I}_{N-1} - (15N-22)\boldsymbol{J}_{N-1} & -(15N-22)\boldsymbol{1}_{N-1} \\ \\ (15N^2-43N+30)\boldsymbol{1}'_{N-1} & (N-1)(15N-22) \end{bmatrix},$$

and $\boldsymbol{A}_1 - \boldsymbol{B}_1\boldsymbol{D}_1^{-1}\boldsymbol{C}_1$ is given as

$$\frac{(N-2)^2}{4\{(N-2)^2+4\}}\begin{bmatrix} N^2(N-3)^2\boldsymbol{I}_{N-1} + (3N^3+6N^2-21N+8)\boldsymbol{J}_{N-1} & (3N^3+6N^2-21N+8)\boldsymbol{1}_{N-1} \\ \\ -(21N^3-86N^2+109N-40)\boldsymbol{1}'_{N-1} & (N-1)(N^3-26N^2+69N-40) \end{bmatrix}.$$

Now we can see that $(\boldsymbol{A}_1 - \boldsymbol{B}_1\boldsymbol{D}_1^{-1}\boldsymbol{C}_1)$ is an asymmetric matrix, hence we have to find out the inverse as the earlier method.

Let $(\boldsymbol{A}_1 - \boldsymbol{B}_1\boldsymbol{D}_1^{-1}\boldsymbol{C}_1) = \begin{pmatrix} \boldsymbol{A}_2 & \boldsymbol{B}_2 \\ \boldsymbol{C}_2 & \boldsymbol{D}_2 \end{pmatrix}$. Then, the inverse is given as

$$\begin{bmatrix} \boldsymbol{A}_2 & \boldsymbol{B}_2 \\ \boldsymbol{C}_2 & \boldsymbol{D}_2 \end{bmatrix}^{-1} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{D}_2^{-1} \end{bmatrix} + \begin{bmatrix} \boldsymbol{I} \\ \\ -\boldsymbol{D}_2^{-1}\boldsymbol{C}_2 \end{bmatrix} (\boldsymbol{A}_2 - \boldsymbol{B}_2\boldsymbol{D}_2^{-1}\boldsymbol{C}_2)^{-1}[\boldsymbol{I} \qquad -\boldsymbol{B}_2\boldsymbol{D}_2^{-1}].$$

Now, we have to use the following results:

$$\boldsymbol{D}_2^{-1} = \frac{4\{(N-2)^2+4\}}{(N-2)^2(N-1)(N^3-26N^2+69N-40)},$$

$$-\boldsymbol{D}_2^{-1}\boldsymbol{C}_2 = \frac{(21N^3-86N^2+109N-40)}{(N-1)(N^3-26N^2+69N-40)}\mathbf{1}'_{N-1},$$

$$-\boldsymbol{B}_2\boldsymbol{D}_2^{-1} = -\frac{(3N^3+6N^2-21N+8)}{(N-1)(N^3-26N^2+69N-40)}\boldsymbol{J}_{N-1},$$

$$-\boldsymbol{B}_2\boldsymbol{D}_2^{-1}\boldsymbol{C}_2 = \frac{(N-2)^2(21N^3-86N^2+109N-40)(3N^3+6N^2-21N+8)}{4\{(N-2)^2+4\}(N-1)(N^3-26N^2+69N-40)},$$

$$\boldsymbol{A}_2 - \boldsymbol{B}_2\boldsymbol{D}_2^{-1}\boldsymbol{C}_2 = k\left\{\boldsymbol{I}_{N-1} + \frac{(3N^3+6N^2-21N+8)}{(N-1)(N^3-26N^2+69N-40)}\right\}\boldsymbol{J}_{N-1},$$

and thus, finally we get $(\boldsymbol{A}_2 - \boldsymbol{B}_2\boldsymbol{D}_2^{-1}\boldsymbol{C}_2)^{-1}$ as

$$\frac{1}{k}\left\{\boldsymbol{I}_{N-1} - \frac{(3N^3+6N^2-21N+8)}{4(N-1)(N^3-5N^2+12N-8)}\boldsymbol{J}_{N-1}\right\}, \text{ where } k = \frac{\{N(N-2)(N-3)\}^2}{4\{(N-2)^2+4\}}.$$

Now, $\begin{bmatrix} \boldsymbol{I} \\ -\boldsymbol{D}_2^{-1}\boldsymbol{C}_2 \end{bmatrix}(\boldsymbol{A}_2 - \boldsymbol{B}_2\boldsymbol{D}_2^{-1}\boldsymbol{C}_2)^{-1}$ is given as follows:

$$\frac{1}{k}\begin{bmatrix} \boldsymbol{I}_{N-1} - \frac{(3N^3+6N^2-21N+8)}{4(N-1)(N^3-5N^2+12N-8)}\boldsymbol{J}_{N-1} \\ \frac{(21N^3-86N^2+109N-40)}{(N-1)(N^3-26N^2+69N-40)}\left\{1 - \frac{(3N^3+6N^2-21N+8)}{4(N-1)(N^3-5N^2+12N-8)}\right\}\mathbf{1}'_{N-1} \end{bmatrix},$$

and $\begin{bmatrix} \boldsymbol{I} \\ -\boldsymbol{D}_2^{-1}\boldsymbol{C}_2 \end{bmatrix}(\boldsymbol{A}_2 - \boldsymbol{B}_2\boldsymbol{D}_2^{-1}\boldsymbol{C}_2)^{-1}[\boldsymbol{I} \quad -\boldsymbol{B}_2\boldsymbol{D}_2^{-1}]$ is given as follows:

$$\frac{1}{k}\begin{bmatrix} \boldsymbol{I}_{N-1} - \frac{(3N^3+6N^2-21N+8)}{4(N-1)(N^3-5N^2+12N-8)}\boldsymbol{J}_{N-1} & -\frac{(3N^3+6N^2-21N+8)}{4(N-1)(N^3-5N^2+12N-8)}\mathbf{1}_{N-1} \\ \frac{(21N^3-86N^2+109N-40)}{4(N-1)(N^3-5N^2+12N-8)}\mathbf{1}'_{N-1} & -\frac{(21N^3-86N^2+109N-40)(3N^3+6N^2-21N+8)}{4(N-1)(N^3-5N^2+12N-8)(N^3-26N^2+69N-40)} \end{bmatrix}.$$

Also, $(\boldsymbol{A}_1 - \boldsymbol{B}_1\boldsymbol{D}_1^{-1}\boldsymbol{C}_1)^{-1} = \begin{bmatrix} \boldsymbol{A}_2 & \boldsymbol{B}_2 \\ \boldsymbol{C}_2 & \boldsymbol{D}_2 \end{bmatrix}^{-1}$ is given as:

$$\frac{1}{k}\begin{bmatrix} \boldsymbol{I}_{N-1} - \frac{(3N^3+6N^2-21N+8)}{4(N-1)(N^3-5N^2+12N-8)}\boldsymbol{J}_{N-1} & -\frac{(3N^3+6N^2-21N+8)}{4(N-1)(N^3-5N^2+12N-8)}\mathbf{1}_{N-1} \\ \frac{(21N^3-86N^2+109N-40)}{4(N-1)(N^3-5N^2+12N-8)}\mathbf{1}'_{N-1} & \left\{\frac{N^2(N-3)^2}{(N-1)(N^3-26N^2+69N-40)} - \frac{(21N^3-86N^2+109N-40)(3N^3+6N^2-21N+8)}{4(N^3-5N^2+12N-8)(N-1)(N^3-26N^2+69N-40)}\right\} \end{bmatrix}$$

Thus, finally we get, $(\boldsymbol{K}'\boldsymbol{W}\boldsymbol{W}'\boldsymbol{L}')^{-1} = \begin{bmatrix} \boldsymbol{A}_1 & \boldsymbol{B}_1 \\ \boldsymbol{C}_1 & \boldsymbol{D}_1 \end{bmatrix}^{-1}$ as follows:

$$\frac{1}{k}\begin{bmatrix} \boldsymbol{I}_{N-1}-\frac{(3N^3+6N^2-21N+8)}{4(N-1)(N^3-5N^2+12N-8)}\boldsymbol{J}_{N-1} & -\frac{(3N^3+6N^2-21N+8)}{4(N-1)(N^3-5N^2+12N-8)}\boldsymbol{1}_{N-1} & \frac{2(3N-4)}{(N^2-4N+8)}\boldsymbol{I}_{N-1} \\ \frac{(21N^3-86N^2+109N-40)}{4(N-1)(N^3-5N^2+12N-8)}\boldsymbol{1}'_{N-1} & \left\{\frac{N^2(N-3)^2}{(N-1)\begin{pmatrix}N^3-26N^2\\+69N-40\end{pmatrix}}-\frac{\begin{pmatrix}21N^3-86N^2\\+109N-40\end{pmatrix}\begin{pmatrix}3N^3+6N^2\\-21N+8\end{pmatrix}}{4\begin{pmatrix}N^3-5N^2\\+12N-8\end{pmatrix}(N-1)\begin{pmatrix}N^3-26N^2\\+69N-40\end{pmatrix}}\right\} & -\frac{2(3N-4)}{(N^2-4N+8)}\boldsymbol{1}'_{N-1} \\ \frac{2(3N-4)}{(N^2-4N+8)}\boldsymbol{I}_{N-1}+\frac{\begin{matrix}(-15N^5+94N^4-279N^3\\+372N^2-216N+64)\end{matrix}}{4(N-1)\begin{pmatrix}N^2\\-4N+8\end{pmatrix}\begin{pmatrix}N^3-5N^2\\+12N-8\end{pmatrix}}\boldsymbol{J}_{N-1} & \frac{(-15N^5+94N^4-279N^3+372N^2-216N+64)}{4(N-1)(N^2-4N+8)(N^3-5N^2+12N-8)}\boldsymbol{1}_{N-1} & \frac{\begin{matrix}4(N^4-6N^3\\+18N^2-24N+16)\end{matrix}}{(N^2-4N+8)^2}\boldsymbol{I}_{N-1} \end{bmatrix}$$

and $\boldsymbol{L}'(\boldsymbol{K}'\boldsymbol{W}\boldsymbol{W}'\boldsymbol{L}')^{-\mathbf{1}}$ is given as:

$$\frac{1}{k}\begin{bmatrix} \boldsymbol{I}_{N-1}-\frac{(3N^3+6N^2-21N+8)}{4(N-1)(N^3-5N^2+12N-8)}\boldsymbol{J}_{N-1} & -\frac{(3N^3+6N^2-21N+8)}{4(N-1)(N^3-5N^2+12N-8)}\boldsymbol{1}_{N-1} & \frac{2(3N-4)}{(N^2-4N+8)}\boldsymbol{I}_{N-1} \\ \frac{(21N^3-86N^2+109N-40)}{4(N-1)(N^3-5N^2+12N-8)}\boldsymbol{1}'_{N-1} & \left\{\frac{N^2(N-3)^2}{(N-1)\begin{pmatrix}N^3-26N^2\\+69N-40\end{pmatrix}}-\frac{\begin{pmatrix}21N^3-86N^2\\+109N-40\end{pmatrix}\begin{pmatrix}3N^3+6N^2\\-21N+8\end{pmatrix}}{4\begin{pmatrix}N^3-5N^2\\+12N-8\end{pmatrix}(N-1)\begin{pmatrix}N^3-26N^2\\+69N-40\end{pmatrix}}\right\} & -\frac{2(3N-4)}{(N^2-4N+8)}\boldsymbol{1}'_{N-1} \\ \frac{2(3N-4)}{(N^2-4N+8)}\boldsymbol{I}_{N-1}+\frac{\begin{matrix}(-15N^5+94N^4-279N^3\\+372N^2-216N+64)\end{matrix}}{4(N-1)\begin{pmatrix}N^2\\-4N+8\end{pmatrix}\begin{pmatrix}N^3-5N^2\\+12N-8\end{pmatrix}}\boldsymbol{J}_{N-1} & \frac{(-15N^5+94N^4-279N^3+372N^2-216N+64)}{4(N-1)(N^2-4N+8)(N^3-5N^2+12N-8)}\boldsymbol{1}_{N-1} & \frac{\begin{matrix}4(N^4-6N^3\\+18N^2-24N+16)\end{matrix}}{(N^2-4N+8)^2}\boldsymbol{I}_{N-1} \\ \frac{\begin{matrix}(16N^6-143N^5+590N^4-135N^3\\+1684N^2-1112N+320)\end{matrix}}{4(N-1)\begin{pmatrix}N^2\\-4N+8\end{pmatrix}\begin{pmatrix}N^3-5N^2\\+12N-8\end{pmatrix}}\boldsymbol{1}'_{N-1} & \frac{\begin{matrix}(16N^9-535N^8+4612N^7-20350N^6+53096N^5\\-84715N^4+8228N^3-4738N^2+15616N-2560)\end{matrix}}{4(N-1)\begin{pmatrix}N^3-5N^2\\+12N-8\end{pmatrix}\begin{pmatrix}N^3-26N^2\\+69N-40\end{pmatrix}(N^2-4N+8)}\boldsymbol{1}_{N-1} & -\frac{\begin{matrix}4(N^4-6N^3\\+18N^2-24N+16)\end{matrix}}{(N^2-4N+8)^2}\boldsymbol{1}'_{N-1} \end{bmatrix}$$

After substituting all the intermediate forms involved in the calculation, the final form of the Moore Penrose inverse $(\boldsymbol{W}\boldsymbol{W}')^{+}=\boldsymbol{L}'(\boldsymbol{K}'\boldsymbol{W}\boldsymbol{W}'\boldsymbol{L}')^{-\mathbf{1}}\boldsymbol{K}'$ is

$$\frac{2}{N(N-2)}\begin{bmatrix} \frac{(N-2)}{(N-3)}\left(\boldsymbol{I}_N-\frac{(3N-21N^3+68N^2-104N+64)}{4N(N-1)(N-2)(N^2-4N+8)}\boldsymbol{J}_{N-1}\right) & \frac{2}{(N-3)}\left(\boldsymbol{I}_N-\frac{\begin{matrix}(N^6-19N^5+128N^4-480N^3\\+1024N^2-1216N+512)\end{matrix}}{4N(N-1)(N-2)(N^2-4N+8)^2}\right)\boldsymbol{J}_N) \\ \frac{2}{(N-3)}\left(\boldsymbol{I}_N-\frac{\begin{matrix}(N^6-19N^5+128N^4-480N^3\\+1024N^2-1216N+512)\end{matrix}}{4N(N-1)(N^2-4N+8)^2}\right)\boldsymbol{J}_N) & \frac{2(N-1)}{(N-3)}\left(\boldsymbol{I}_N-\frac{\begin{matrix}(7N^6-69N^5+336N^4-928N^3\\+1536N^2-1344N+512)\end{matrix}}{8N(N-1)^2(N^2-4N+8)^2}\right)\boldsymbol{J}_N) \end{bmatrix}$$

.

**Estimates of gca and sca effects**

Now, pre-multiplying (4.2.1.1.1) with $\boldsymbol{W}$ we get

$\boldsymbol{W}\boldsymbol{y}=\bar{y}\,\boldsymbol{W}\boldsymbol{1}_T+\boldsymbol{W}\boldsymbol{W}'\begin{pmatrix}\boldsymbol{g}\\\boldsymbol{h}\end{pmatrix}+\boldsymbol{W}\boldsymbol{s}+\boldsymbol{W}\boldsymbol{e}$, and thus we get the following:

$$\begin{pmatrix}\widehat{\boldsymbol{g}}\\\widehat{\boldsymbol{h}}\end{pmatrix}=(\boldsymbol{W}\boldsymbol{W}')^{+}(\boldsymbol{W}\boldsymbol{y}-\ \bar{y}\,\boldsymbol{W}\boldsymbol{1}_T)$$

$$=(\boldsymbol{W}\boldsymbol{W}')^{+}\boldsymbol{W}\boldsymbol{y}-\ (\boldsymbol{W}\boldsymbol{W}')^{+}\boldsymbol{W}\boldsymbol{1}_T\bar{y}$$

$$= (\boldsymbol{WW'})^{+}\boldsymbol{Wy} - (\boldsymbol{WW'})^{+}\frac{(N-1)(N-2)}{2}\boldsymbol{1}_{2N}\frac{\boldsymbol{1}_T'\boldsymbol{y}}{T}.$$

Thus, the estimates of joint gca effects can be simplified and expressed as:

$$\begin{pmatrix}\widehat{\boldsymbol{g}}\\ \widehat{\boldsymbol{h}}\end{pmatrix} = \left\{(\boldsymbol{WW'})^{+}\boldsymbol{W} - \frac{\boldsymbol{1}}{N}(\boldsymbol{WW'})^{+}\boldsymbol{J}_{2N\times T}\right\}\boldsymbol{y}$$

$= \boldsymbol{H}_1\boldsymbol{y}$, where

$$\boldsymbol{H}_1 = (\boldsymbol{WW'})^{+}\boldsymbol{W} - \frac{\boldsymbol{1}}{N}(\boldsymbol{WW'})^{+}\boldsymbol{J}_{2N\times T},$$

$$= (\boldsymbol{WW'})^{+}\boldsymbol{W} - \frac{\boldsymbol{1}}{2T}\boldsymbol{J}_{2N\times T} \qquad (4.2.1.1.4)$$

Now, $\boldsymbol{y} = \bar{y}\,\boldsymbol{1}_T + \boldsymbol{W}'\begin{pmatrix}\boldsymbol{g}\\ \boldsymbol{h}\end{pmatrix} + \boldsymbol{s} + \boldsymbol{e}$, and using (4.2.1.1.4) we get

$$\hat{\boldsymbol{s}} = \boldsymbol{y} - \bar{y}\,\boldsymbol{1}_T - \boldsymbol{W}'\begin{pmatrix}\widehat{\boldsymbol{g}}\\ \widehat{\boldsymbol{h}}\end{pmatrix}$$

$$= \boldsymbol{y} - \bar{y}\,\boldsymbol{1}_T - \boldsymbol{W}'\left\{(\boldsymbol{WW'})^{+}\boldsymbol{W} - \frac{\boldsymbol{1}}{N}(\boldsymbol{WW'})^{+}\boldsymbol{J}_{2N\times T}\right\}$$

$$= \boldsymbol{y} - \frac{\boldsymbol{1}_T\boldsymbol{1}_T'}{T}\boldsymbol{y} - \boldsymbol{W}'(\boldsymbol{WW'})^{+}\boldsymbol{Wy} - \frac{\boldsymbol{1}}{N}\boldsymbol{W}'(\boldsymbol{WW'})^{+}\boldsymbol{J}_{2N\times T}\boldsymbol{y}.$$

Thus, the final estimate of sca effects after simplification is given as

$$\hat{\boldsymbol{s}} = (\boldsymbol{I} - \boldsymbol{W}'(\boldsymbol{WW'})^{+}\boldsymbol{W})\boldsymbol{y}$$

$= \boldsymbol{H}_2\boldsymbol{y}$, where

$$\boldsymbol{H}_2 = (\boldsymbol{I} - \boldsymbol{W}'(\boldsymbol{WW'})^{+}\boldsymbol{W}). \qquad (4.2.1.1.5)$$

Also, it can be seen that $rank(\boldsymbol{H}_1) = 2(N-1)$ and $rank(\boldsymbol{H}_2) = T - 2N + 1$.

Now, it is important to check the orthogonality of the model and verify following three conditions:

(i) $\boldsymbol{H}_1\boldsymbol{1} = \boldsymbol{0}$.

Proof: We have, $\boldsymbol{H}_1\boldsymbol{1} = \left\{(\boldsymbol{WW'})^{+}\boldsymbol{W} - \frac{\boldsymbol{1}}{N}(\boldsymbol{WW'})^{+}\boldsymbol{J}_{2N\times T}\right\}\boldsymbol{1}$

$$= (\boldsymbol{WW'})^{+}\boldsymbol{W1} - \frac{\boldsymbol{1}}{N}(\boldsymbol{WW'})^{+}\boldsymbol{J}_{2N\times T}\boldsymbol{1}$$

$= (\boldsymbol{WW'})^{+}\left(\frac{(N-1)(N-2)}{2}\mathbf{1} - \frac{\boldsymbol{T}}{N}\mathbf{1}\right) = \mathbf{0}.$

(ii) $\boldsymbol{H}_2\mathbf{1} = \mathbf{0}$.

Proof: As $\boldsymbol{H}_2\mathbf{1} = (\boldsymbol{I} - \boldsymbol{W}'(\boldsymbol{WW}')^{+}\boldsymbol{W})\mathbf{1}$

$= \mathbf{1} - \frac{(N-1)(N-2)}{2}\ \boldsymbol{W}'(\boldsymbol{WW}')^{+}\mathbf{1}$

$= \mathbf{1} - \frac{(N-1)(N-2)}{2(N-1)(N-2)}\ \boldsymbol{W}'\mathbf{1}$

$= \mathbf{1} - \frac{\mathbf{2}(N-1)(N-2)}{2(N-1)(N-2)}\mathbf{1} = \mathbf{0}.$

(iii) $\boldsymbol{H}_1\boldsymbol{H}_2 = \boldsymbol{H}_2\boldsymbol{H}_1 = \mathbf{0}$.

Proof: $\boldsymbol{H}_1\boldsymbol{H}_2 = \left\{(\boldsymbol{WW}')^{+}\boldsymbol{W} - \frac{\mathbf{1}}{2T}\boldsymbol{J}_{2N\times T}\right\}\{\boldsymbol{I} - \boldsymbol{W}'(\boldsymbol{WW}')^{+}\boldsymbol{W}\}$

$= (\boldsymbol{WW}')^{+}\boldsymbol{W} - \frac{\mathbf{1}}{2T}\boldsymbol{J}_{2N\times T} - (\boldsymbol{WW}')^{+}(\boldsymbol{WW}')(\boldsymbol{WW}')^{+}\boldsymbol{W} + \frac{\mathbf{1}}{2T}\boldsymbol{J}_{2N\times T}\boldsymbol{W}'(\boldsymbol{WW}')^{+}\boldsymbol{W}$

$=(\boldsymbol{WW}')^{+}\boldsymbol{W} - \frac{\mathbf{1}}{2T}\boldsymbol{J}_{2N\times T} - (\boldsymbol{WW}')^{+}\boldsymbol{W} + \frac{\mathbf{1}}{2T}\boldsymbol{J}_{2N\times T} = \mathbf{0}$. **(**using property of Moore Penrose inverse**).** Similarly we can also prove that $\boldsymbol{H}_2\boldsymbol{H}_1 = \mathbf{0}$.

It is clear that $\begin{pmatrix}\boldsymbol{g}\\ \boldsymbol{h}\end{pmatrix}$ and $\boldsymbol{s}$ represent orthogonal treatment contrasts having $2(N-1)$ and $(T-2N+1)$ degrees of freedom, respectively and can be used for obtaining orthogonal estimates of functions of gca and sca effects.

**Information matrices**

Now, under the usual set up of a block design $d$, the joint information matrix regarding $\begin{pmatrix}\boldsymbol{H}_1\\ \boldsymbol{H}_2\end{pmatrix}\boldsymbol{y}$ is given by the expression:

$$\boldsymbol{C}_{d\_ca} = \begin{bmatrix}\boldsymbol{H}_1\boldsymbol{C}_d\boldsymbol{H}_1' & \boldsymbol{H}_1\boldsymbol{C}_d\boldsymbol{H}_2' \\ \boldsymbol{H}_2\boldsymbol{C}_d\boldsymbol{H}_1' & \boldsymbol{H}_2\boldsymbol{C}_d\boldsymbol{H}_2'\end{bmatrix},$$

where $\boldsymbol{C}_d = \boldsymbol{R}_d - \frac{1}{k}\boldsymbol{N}_d\boldsymbol{N}_d'$, $\boldsymbol{R}_d = diag(r_1, r_2, \dots, r_T)$ is the diagonal matrix of replications of the crosses under the design $d$ and $\boldsymbol{N}_d$ is the incidence matrix of crosses versus blocks.

Here, $\boldsymbol{C}_d$ is the information matrix of the general block design $d$, where treatments are the $T$ crosses with $\boldsymbol{C}_d\mathbf{1}_T = \mathbf{0}$. As discussed earlier regarding orthogonality, in order to estimate $\boldsymbol{H}_1\boldsymbol{C}_d\boldsymbol{H}_1'$ and $\boldsymbol{H}_2\boldsymbol{C}_d\boldsymbol{H}_2'$ orthogonally the off diagonal components must vanish and we must have $\boldsymbol{H}_2\boldsymbol{C}_d\boldsymbol{H}_1' = \boldsymbol{H}_1\boldsymbol{C}_d\boldsymbol{H}_2' = \mathbf{0}$. Thus, we have $\boldsymbol{C}_{\text{gca}} = \boldsymbol{H}_1\boldsymbol{C}_d\boldsymbol{H}_1'$ and $\boldsymbol{C}_{\text{sca}} = \boldsymbol{H}_2\boldsymbol{C}_d\boldsymbol{H}_2'$.

**4.2.1.2 Model excluding sca effects**

In this approach, gca effects of first and second kinds corresponding to half and full parents will be estimated for which it is assumed that the sca effects are contributing less to the total combining ability effects as compared to gca effects. The model can be written as

$$y_{ijk} = \bar{y} + h_i + h_j + g_k + e_{ijk}, \qquad (4.2.1.2.1)$$

where $\bar{y}$ is the average effect of the crosses, $\{h_\alpha\}, \alpha = i, j$, represents the gca effects of first kind corresponding to the lines occurring as half parents, $\{g_k\}$ represents the gca effects of second kind corresponding to the lines occurring as full parents, $e_{ijk}$ is random error and

$$g_1 + g_2 + \cdots + g_N = 0 \text{ or } \textstyle\sum_{i=1}^{N} g_i = 0, \qquad (4.2.1.2.2)$$

$$h_1 + h_2 + \cdots + h_N = 0 \text{ or } \textstyle\sum_{i=1}^{N} h_i = 0. \qquad (4.2.1.2.3)$$

The model in matrix notation is expressed as:

$$\boldsymbol{y} = \bar{y}\,\mathbf{1}_T + \boldsymbol{W}_1'\boldsymbol{h} + \boldsymbol{W}_2'\boldsymbol{g} + \boldsymbol{e} \qquad (4.2.1.2.4)$$

where $\boldsymbol{y}$ is the $T \times 1$ vector of responses due to crosses, $\bar{y}$ is the mean effect of crosses, $\boldsymbol{h}$ is the $N \times 1$ vector of gca effects due to half parents, $\boldsymbol{g}$ is the $N \times 1$ vector of gca effects due to full parents and $\boldsymbol{e}$ is the $T \times 1$ vector of random errors. $\boldsymbol{W}_1$ and $\boldsymbol{W}_2$ are $N \times T$ matrices with rows indexed by the line numbers $1,2, \ldots N$ and columns by the three-way crosses arranged in the manner described earlier, such that the $\{t, (i, j, k)\}^{th}$entry of $\boldsymbol{W_1}$ is 0.5 if $t \in (ij)$ and zero, otherwise and the $\{t, (i, j, k)\}^{th}$entry of $\boldsymbol{W_2}$ is 1 if $t \in k$ and zero, otherwise. The normal equations are:

$$E(\boldsymbol{y}) = \bar{y}\,\mathbf{1}_T + \boldsymbol{W}_1'\boldsymbol{h} + \boldsymbol{W}_2'\boldsymbol{g}$$

$$\boldsymbol{W}_1 E(\boldsymbol{y}) = \bar{y}\,\boldsymbol{W}_1 \mathbf{1}_T + \boldsymbol{W}_1 \boldsymbol{W}_1' \boldsymbol{h} + \boldsymbol{W}_1 \boldsymbol{W}_2' \boldsymbol{g}, \text{ and}$$

$$\boldsymbol{W}_2 E(\boldsymbol{y}) = \bar{y}\,\boldsymbol{W}_2 \mathbf{1}_T + \boldsymbol{W}_2 \boldsymbol{W}_1' \boldsymbol{h} + \boldsymbol{W}_2 \boldsymbol{W}_2' \boldsymbol{g}.$$

On solving these three normal equations, the estimate of gca effects of half parent is given as:

$$\begin{aligned}\widehat{\boldsymbol{h}} &= \left(\boldsymbol{W}_1 \boldsymbol{W}_1'\right)^{-} (\boldsymbol{W}_1 \boldsymbol{y} - \boldsymbol{W}_1 \bar{y}\,\mathbf{1}_T)\\ &= [\left(\boldsymbol{W}_1 \mathbf{W}_1'\right)^{-} \boldsymbol{W}_1 - \left(\boldsymbol{W}_1 \mathbf{W}_1'\right)^{-} \boldsymbol{W}_1 \mathbf{J}_T/T)]\boldsymbol{y}\\ &= \boldsymbol{G}_1 \boldsymbol{y},\end{aligned} \tag{4.2.1.2.5}$$

and the estimate of gca effects of full parent is given as:

$$\begin{aligned}\widehat{\boldsymbol{g}} &= \left(\boldsymbol{W}_2 \boldsymbol{W}_2'\right)^{-} (\boldsymbol{W}_2 \boldsymbol{y} - \boldsymbol{W}_2 \bar{y}\,\mathbf{1}_T),\\ &= [\left(\boldsymbol{W}_2 \boldsymbol{W}_2'\right)^{-} \boldsymbol{W}_2 - \left(\boldsymbol{W}_2 \boldsymbol{W}_2'\right)^{-} \boldsymbol{W}_2 \boldsymbol{J}_T/N)]\boldsymbol{y}\\ &= \boldsymbol{G}_2 \boldsymbol{y}.\end{aligned} \tag{4.2.1.2.6}$$

It is clear that $\boldsymbol{h}$ and $\boldsymbol{g}$ represents orthogonal treatment contrasts, both having $(N-1)$ degrees of freedom and can be used for obtaining orthogonal estimates of function of gca effects of half and full parents.

The restrictions being imposed in order to estimate the gca effect of half parents free from gca effect of full parents are as:

$$\mathbf{1}'\widehat{\boldsymbol{h}} = \mathbf{1}'\widehat{\boldsymbol{g}} = \boldsymbol{G}_1 \mathbf{1} = \boldsymbol{G}_2 \mathbf{1} = \boldsymbol{G}_1' \boldsymbol{G}_2 = \mathbf{0},\ rank(\boldsymbol{G}_1) = rank(\boldsymbol{G}_2) = N-1.$$

Now, under the usual set up of a block design $d$, the joint information matrix regarding $\begin{pmatrix}\boldsymbol{G}_1\\ \boldsymbol{G}_2\end{pmatrix}\boldsymbol{y}$ is given by:

$$\boldsymbol{C}_{d\_gca} = \begin{bmatrix}\boldsymbol{G}_1 \boldsymbol{C}_d \boldsymbol{G}_1' & \boldsymbol{G}_1 \boldsymbol{C}_d \boldsymbol{G}_2' \\ \boldsymbol{G}_2 \boldsymbol{C}_d \boldsymbol{G}_1' & \boldsymbol{G}_2 \boldsymbol{C}_d \boldsymbol{G}_2'\end{bmatrix},$$

where $\boldsymbol{C}_d = \boldsymbol{R}_d - \frac{1}{k}\boldsymbol{N}_d \boldsymbol{N}_d'$, $\boldsymbol{R}_d = diag(r_1, r_2, \ldots, r_T)$ is the diagonal matrix of replications of the crosses under the design $d$ and $\boldsymbol{N}_d$ is the incidence matrix of crosses versus blocks. Here, $\boldsymbol{C}_d$ is the information matrix of the general block design $d$ where treatments are nothing but the $T$ number of triallel crosses, hence we have $\boldsymbol{C}_d \mathbf{1}_T = \mathbf{0}$. In order to estimate $\boldsymbol{G}_1 \boldsymbol{C}_d \boldsymbol{G}_1'$ and

$\boldsymbol{G}_2\boldsymbol{C}_d\boldsymbol{G}_2'$ orthogonally the off diagonal components have to vanish and hence $\boldsymbol{G}_2\boldsymbol{C}_d\boldsymbol{G}_1' = \boldsymbol{G}_1\boldsymbol{C}_d\boldsymbol{G}_2' = \boldsymbol{0}$. Thus, we have $\boldsymbol{C}_{\text{gca\_half}} = \boldsymbol{G}_1\boldsymbol{C}_d\boldsymbol{G}_1'$ and $\boldsymbol{C}_{\text{gca\_full}} = \boldsymbol{G}_2\boldsymbol{C}_d\boldsymbol{G}_2'$.

### 4.2.1.3 Methods of construction

General methods of constructing $\text{PT}_\text{r}\text{C}$ plans are described in this section using various types of designs and association schemes.

**Method 1: $\text{PT}_\text{r}\text{C}$ plans using triangular association scheme**

Let there be $N = \frac{n(n-1)}{2}$ lines, where $n > 4$. Arrange these $N$ lines in a two-associate triangular association scheme, *i.e.*, allot $N$ lines to the off diagonal positions above the principal diagonal in a natural order and repeat the same below the diagonal such that the final arrangement is symmetrical about the diagonal. Diagonal positions are left empty. Consider all possible pair of lines that can be made from each row of the array. Add a third line to each of these pairs to form triplets. Line that appears at the intersection of the second row containing the first line in the pair and column containing the second line in the pair is considered, and added to each pair to form triplets. Make three-way crosses from these triplets considering lines in the pairs as half parents and third added line in the triplet as full parent. This will result in a partial three-way cross design with parameters $N = \frac{n(n-1)}{2}$, $T = \frac{n(n-1)(n-2)}{2}$, $b = n$, $k = \frac{(n-1)(n-2)}{2}$, $r_h = 2(n-2)$ and $r_f = (n-2)$.

**Remark:** It can be seen that the above method of construction gives a layout in which the crosses are arranged in six groups. This excludes the necessity for an environmental design for laying out the crosses as these groups can be treated as blocks of a design. Hence, through this method one can get a combination of mating as well as environmental design, at one go.

**Example 4.2.1.3.1:** The method can be well understood by an example for $n = 6$ giving rise to $N$ = 15.

| | | | | | |
|---|---|---|---|---|---|
| * | A | B | C | D | E |
| A | * | F | G | H | I |
| B | F | * | J | K | L |
| C | G | J | * | M | N |
| D | H | K | M | * | O |
| E | I | L | N | O | * |

The first cross of first block is obtained by considering the first pair of lines (*i.e.*, A & B) as half parents and then crossing it with the line present at the row-column intersection of lines A and B (*i.e.*, F), treating F as the full parent in the cross. Proceeding in the same manner, for all possible pairs of first row one can obtain the three-way crosses to be placed in first block. In a similar manner, from other rows, remaining blocks can be obtained. The design so obtained is:

| **Block 1** | **Block 2** | **Block 3** | **Block 4** | **Block 5** | **Block 6** |
|---|---|---|---|---|---|
| (A×B)×F | (A×F)×B | (B×F)×A | (C×G)×A | (D×H)×A | (E×I)×A |
| (A×C)×G | (A×G)×C | (B×J)×C | (C×J)×B | (D×K)×B | (E×L)×B |
| (A×D)×H | (A×H)×D | (B×K)×D | (C×M)×D | (D×M)×C | (E×)N×C |
| (A×E)×I | (A×I)×E | (B×L)×E | (C×N)×E | (D×O)×E | (E×O)×D |
| (B×C)×J | (F×G)×J | (F×J)×G | (G×J)×F | (H×K)×F | (I×L)×F |
| (B×D)×K | (F×H)×K | (F×K)×H | (G×M)×H | (H×M)×G | (I×N)×G |
| (B×E)×L | (F×I)×L | (F×L)×I | (G×N)×I | (H×O)×I | (I×O)×H |
| (C×D)×M | (G×H)×M | (J×K)×M | (J×M)×K | (K×M)×J | (L×N)×J |
| (C×E)×N | (G×I)×N | (J×L)×N | (J×N)×L | (K×O)×L | (L×O)×K |
| (D×E)×O | (H×I)×O | (K×L)×O | (M×N)×O | (M×O)×N | (N×O)×M |

The parameters of this design are $n = 6,\ N = 15, T = 60, b = 6, k = 10, r_h = 8$ and $r_f = 4$.

**Information matrices**

Let $\mathbf{I}_N$ is an identity matrix of order $N$, $\mathbf{A}_N$ is a matrix of order $N$ whose elements, $\{a_{ij}\}$ takes value 1 if $i$ and $j$ are first associates otherwise 0, and $\mathbf{B}_N$ is a matrix of order $N$ whose elements, $\{b_{ij}\}$ takes value 1 if $i$ and $j$ are second associates, otherwise 0.

The general form of information matrix related to half parents ($\mathbf{C}_{\text{half}}$) is $a_0\,\mathbf{I}_N + a_1\,\mathbf{A}_N + a_2\,\mathbf{B}_N$, where $a_0 = \frac{2(n-3)(n-4)}{(n-2)}$, $a_1 = -\frac{2(n-3)(n-4)}{(n-2)^2}$ and $a_2 = \frac{4(n-4)}{(n-2)^2}$.

The general form of information matrix related to full parents ($\mathbf{C}_{\text{full}}$) is $b_0\,\mathbf{I}_N + b_1\,\mathbf{A}_v + b_2\,\mathbf{B}_N$, where $b_0 = (n-4)$, $b_1 = -\frac{(n-4)}{(n-2)}$ and $b_2 = \frac{2(n-4)}{(n-2)(n-3)}$.

For Example 4.2.1.3.1, $\mathbf{C}_{\text{half}} = 3\mathbf{I}_{15} - 0.75\mathbf{A}_{15} + 0.5\mathbf{B}_{15}$ and $\mathbf{C}_{full} = 2\mathbf{I}_{15} - 0.5\mathbf{A}_{15} + 0.33\mathbf{B}_{15}$.

**Inverted information matrices**

The general form of inverse of information matrix related to half parents ($\mathbf{C}_{\text{half}}^{-}$) is $c_0\,\mathbf{I}_N + c_1\,\mathbf{A}_N + c_2\,\mathbf{B}_N$, where $c_0 = \frac{(n-2)(n-3)}{2(n-1)^2(n-4)}$, $c_1 = -\frac{(n-3)}{2(n-1)^2(n-4)}$ and $c_2 = \frac{1}{(n-1)^2(n-4)}$.

The general form of inverted information matrix related to full parents ($\mathbf{C}_{\text{full}}^{-}$) is $d_0\,\mathbf{I}_N + d_1\mathbf{A}_N + d_2\mathbf{B}_N$, where $d_0 = \frac{(n-3)^2}{(n-1)^2(n-4)}$, $d_1 = -\frac{(n-3)^2}{(n-1)^2(n-2)(n-4)}$ and $d_2 = \frac{2(n-3)}{(n-1)^2(n-2)(n-4)}$.

For Example 4.2.1.3.1, $\mathbf{C}_{\text{full}}^{-} = 0.18\mathbf{I}_{15} - 0.045\mathbf{A}_{15} + 0.03\mathbf{B}_{15}$ and $\mathbf{C}_{\text{half}}^{-} = 0.12\mathbf{I}_{15} - 0.03\mathbf{A}_{15} + 0.02\mathbf{B}_{15}$.

**Eigenvalues**

The eigenvalues of $\mathbf{C}_{\text{half}}$ are $a_0 + (n-2)a_2$ and 0, whereas the eigenvalues of the $\mathbf{C}_{\text{full}}$ are $b_0 + (n-2)b_2$ and 0.

**Variance factors**

The general expression for variance factor of estimated contrasts for half parents ($\mathrm{V}_{\text{half}}(\widehat{h_i - h_j})$) is $2(c_0 - c_1) = \frac{(n-3)}{(n-1)(n-4)}$, when $i$ and $j (i \neq j)$ are first associates to each other, and $2(c_0 - c_2) = \frac{1}{(n-1)}$, when $i$ and $j (i \neq j)$are second associates to each other. The general expression for average variance factor of estimated contrasts for half parents ($\overline{\mathrm{V}}_{\text{half}}(\widehat{h_i - h_j})$) is $\frac{n(n-3)}{(n^2-1)(n-4)}$.

The general expressions for variance factor of estimated contrasts for full parents ($\mathrm{V}_{\text{full}}(\widehat{g_i - g_j})$) is $2(d_0 - d_1) = \frac{2(n-3)^2}{(n-1)(n-2)(n-4)}$, when $i$ and $j (i \neq j)$ are first associates to each other, and $2(d_0 - d_2) = \frac{2(n-3)}{(n-1)(n-2)}$ when $i$ and $j (i \neq j)$ are second associates to each other. The general expressions for average variance factor of estimated contrasts for full parents ($\overline{\mathrm{V}}_{\text{full}}(\widehat{g_i - g_j})$) is $\frac{2n(n-3)^2}{(n^2-1)(n-2)(n-4)}$.

**Degree of fractionation and efficiency factor**

The degree of fractionation ($f$) for this series of designs involving triallel crosses is $\frac{8}{(n+1)(n^2-n-4)}$.

The canonical efficiency factor for the developed class of designs pertaining to gca effects of half parents ($E_h$) is $\frac{(n-1)(n-4)}{(n-2)^2}$ and of full parents ($E_f$) is $\frac{(n-1)(n-4)}{(n-2)(n-3)}$.

**List of designs**

Considering the model under blocked setup, the canonical efficiency factor of the designs as compared to an orthogonal design with same number of replications has been calculated and listed along with other parameters in Table 4.2.1.3.1.

**Table 4.2.1.3.1. List of designs using triangular association scheme for triallel crosses under blocked setup**

| $n$ | $N$ | $b$ | $k$ | $T$ | $f$ | $r_h$ | $r_f$ | $\bar{V}_{\text{half}}(\widehat{h_i - h_j})$ | $\bar{V}_{\text{full}}(\widehat{g_i - g_j})$ | $E_h$ | $E_f$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 5 | 10 | 5 | 6 | 30 | 0.08 | 6 | 3 | 0.42 | 0.56 | 0.44 | 0.67 |
| 6 | 15 | 6 | 10 | 60 | 0.04 | 8 | 4 | 0.26 | 0.39 | 0.63 | 0.83 |
| 7 | 21 | 7 | 15 | 105 | 0.03 | 10 | 5 | 0.20 | 0.31 | 0.72 | 0.90 |
| 8 | 28 | 8 | 21 | 168 | 0.02 | 12 | 6 | 0.16 | 0.27 | 0.78 | 0.93 |
| 9 | 36 | 9 | 28 | 252 | 0.01 | 14 | 7 | 0.14 | 0.23 | 0.82 | 0.95 |
| 10 | 45 | 10 | 36 | 360 | 0.01 | 16 | 8 | 0.12 | 0.21 | 0.84 | 0.96 |
| 11 | 55 | 11 | 45 | 495 | 0.01 | 18 | 9 | 0.11 | 0.19 | 0.86 | 0.97 |
| 12 | 66 | 12 | 55 | 660 | 0.01 | 20 | 10 | 0.10 | 0.17 | 0.88 | 0.98 |
| 13 | 78 | 13 | 66 | 858 | < 0.01 | 22 | 11 | 0.09 | 0.16 | 0.89 | 0.98 |
| 14 | 91 | 14 | 78 | 1092 | < 0.01 | 24 | 12 | 0.08 | 0.15 | 0.90 | 0.98 |

**Method 2: PT$_r$C designs based on Lattice designs**

This method can be used to obtain block designs for PT$_r$C for a wide range of parameters. In this method, any lattice designs with standard parameters $(v, b^*, r^*, k^*, s)$ is considered and the block contents are taken as lines. Now, from each block all possible three-way crosses are made such that the set of crosses made from all the blocks of a replication of lattice design constitute the block of the new design. The parameters of the resultant class of PT$_r$C designs are $N = v, b = r^*, k = \frac{sk^*(k^*-1)(k^*-2)}{2}$ and $T = \frac{bsk^*(k^*-1)(k^*-2)}{2}$. Different classes of PT$_r$C designs constructed using different types of lattice designs, along with example is given. (It may be noted that along with every cross of the type $(i, j, k)$, crosses of the types $(i, k, j)$ and $(j, k, i)$ are to be considered in the same block, but are not shown here in the design layouts)

**Class I (Square lattices based PT$_r$C designs):** Any square lattice with parameters, $v = s^2$, $b^* = s(s+1)$, $r^* = (s+1)$ and $k^* = s$, can be used to obtain PT$_r$C designs with parameters, $N = s^2, b = (s+1)$, $k = \frac{sk^*(k^*-1)(k^*-2)}{2}$ and $T = \frac{bsk^*(k^*-1)(k^*-2)}{2}$. An example is illustrated here for $s = 3$ to construct a PT$_r$C design for 9 lines. Consider a square lattice design with parameters, $v = 9,\ b^* = 12,\ r^* = 4,\ k^* = 3$ and $s = 3$. The four replications of the lattice designs forms the four blocks of PT$_r$C designs with block contents as the three-way crosses formed by taking all the possible triplets from each block of a replication. Finally we get a PT$_r$C designs with parameters, $N = 9,\ b = 4,\ \ k = 9, T = 36$ and $f = 0.142$. (Only 3 crosses within a block are shown here. However, to maintain SSP, remaining 6 crosses are also to be taken)

| | | | |
|---|---|---|---|
| **Rep 1** | **Blk 1** | **Blk 2** | **Blk 3** |
| | 1,2,3 | 4,5,6 | 7,8,9 |
| **Rep 2** | **Blk 1** | **Blk 2** | **Blk 3** |
| | 1,4,7 | 2,5,8 | 3,6,9 |
| **Rep 3** | **Blk 1** | **Blk 2** | **Blk 3** |
| | 1,5,9 | 3,4,8 | 2,6,7 |
| **Rep 4** | **Blk 1** | **Blk 2** | **Blk 3** |
| | 1,6,8 | 2,4,9 | 3,5,7 |
| **Square Lattice design** | | | |

| | | | |
|---|---|---|---|
| **Blk 1** | (1×2)×3 | (4×5)×6 | (7×8)×9 |
| **Blk 2** | (1×4)×7 | (2×5)×8 | (3×6)×9 |
| **Blk 3** | (1×5)×9 | (3×4)×8 | (2×6)×7 |
| **Blk 4** | (1×6)×8 | (2×4)×9 | (3×5)×7 |
| **PT$_r$C design** | | | |

**Class II (Rectangular lattices based PT$_r$C designs):** Any rectangular lattice design with parameters, $= s(s-1),\ b^* = s^2,\ r^* = s$ and $k^* = (s-1)$, can be used to obtain PT$_r$C designs with parameters, $N = v = s(s-1), b = s,\ \ k = \frac{sk^*(k^*-1)(k^*-2)}{2}$ and $T = \frac{bsk^*(k^*-1)(k^*-2)}{2}$. An example is illustrated here for $s = 4$, which can be used to construct PT$_r$C design for 12 lines. Consider a rectangular lattice design with parameters, $v = 12$, $b^* = 16,\ r^* = 4,\ k^* = 3$ and $s = 4$.

| | | | | | |
|---|---|---|---|---|---|
| **Rectangular Lattice design** | **Rep 1** | **Blk 1** | **Blk 2** | **Blk 3** | **Blk 4** |
| | | 1,5,9 | 2,6,10 | 3,7,11 | 4,8,12 |
| | **Rep 2** | **Blk 1** | **Blk 2** | **Blk 3** | **Blk 4** |
| | | 1,6,11 | 2,5,12 | 3,8,9 | 4,7,10 |
| | **Rep 3** | **Blk 1** | **Blk 2** | **Blk 3** | **Blk 4** |
| | | 1,8,10 | 4,5,11 | 2,7,9 | 3,6,12 |
| | **Rep 4** | **Blk 1** | **Blk 2** | **Blk 3** | **Blk 4** |
| | | 1,7,12 | 3,5,10 | 4,6,9 | 2,8,11 |

Three-way crosses are made within blocks of each replication to construct a PT$_r$C design with parameters, $N = 12,\ b = 4,\ \ k = 12,\ T = 36$ and $f = 0.018$.

| PT$_r$C Designs | | | | | |
|---|---|---|---|---|---|
| | **Blk 1** | (1×5)×9 | (2×6)×10 | (3×7)×11 | (4×8)×12 |
| | **Blk 2** | (1×6)×11 | (2×5)×12 | (3×8)×9 | (4×7)×10 |
| | **Blk 3** | (1×8)×10 | (4×5)×11 | (2×7)×9 | (3×6)×12 |
| | **Blk 4** | (1×7)×12 | (3×5)×10 | (4×6)×9 | (2×8)×11 |

**Class III (Circular Lattices based PT$_r$C designs):** Any circular lattice design with parameters, $v = 2s^2,\ b^* = 2s,\ r^* = 2,$ and $k^* = 2s$, can be used to obtain PT$_r$C designs with parameters, $N = v = 2s^2, b = r^*,\ k = \frac{sk^*(k^*-1)(k^*-2)}{2}$ and $T = \frac{bsk^*(k^*-1)(k^*-2)}{2}$. An example is illustrated here for $s = 2$, which can be used to construct PT$_r$C design for 8 lines. Consider a circular lattice design with parameters, $v = 8,\ b^* = 4,\ r^* = 2,\ k^* = 4$ and $s = 2$. All possible three-way crosses are made within each block of a replication of this lattice design to yield a PT$_r$C designs with parameters, $N = 8,\ b = 2,\ \ k = 24$ and $T = 16$.

| Circular Lattice design | | |
|---|---|---|
| **Rep 1** | **Blk 1** | **Blk 2** |
| | 1,2,3,4 | 5,6,7,8 |
| **Rep 2** | **Blk 1** | **Blk 2** |
| | 1,3,5,7 | 2,4,6,8 |

| PT$_r$C design | | | | |
|---|---|---|---|---|
| **Blk 1** | (1×2)×3 | (1×2)×4 | (1×3)×4 | (2×3)×4 |
| | (5×6)×7 | (5×6)×8 | (5×7)×8 | (6×7)×8 |
| **Blk 2** | (1×3)×5 | (1×3)×7 | (1×5)×7 | (3×5)×7 |
| | (2×4)×6 | (2×4)×8 | (2×6)×8 | (4×6)×8 |

**Class IV (Cubic lattices based PT$_r$C designs):** Any cubic lattice design with parameters, $v = s^3,\ b^* = 3s^2,\ r^* = 3,$ and $k^* = s$, can be used to obtain PT$_r$C designs with parameters, $N = s^3, b = 3,\ k = \frac{sk^*(k^*-1)(k^*-2)}{2}$ and $T = \frac{bsk^*(k^*-1)(k^*-2)}{2}$. An example is illustrated here for $s = 3$, which can be used to construct PT$_r$C design for 27 lines. Consider a circular lattice design with parameters, $v = 27,\ b^* = 27,\ r^* = 3,$ and $k^* = 3$ .

| Cubic Lattice design | | | | | | |
|---|---|---|---|---|---|---|
| | **Rep 1** | **Blk 1** | **Blk 2** | **Blk 3** | **Blk 4** | **Blk 5** |
| | | 1,2,3 | 4,5,6 | 7,8,9 | 10,11,12 | 13,14,15 |
| | | **Blk 6** | **Blk 7** | **Blk 8** | **Blk 9** | |
| | | 16,17,18 | 19,20,21 | 22,23,24 | 25,26,27 | |
| | **Rep 2** | **Blk 1** | **Blk 2** | **Blk 3** | **Blk 4** | **Blk 5** |
| | | 1,4,7 | 2,5,6 | 3,6,9 | 10,13,16 | 11,14,17 |

| | | | | | | |
|---|---|---|---|---|---|---|
| | | **Blk 6** | **Blk 7** | **Blk 8** | **Blk 9** | |
| | | 12,15,18 | 19,22,25 | 20,23,26 | 21,24,27 | |
| | **Rep 3** | **Blk 1** | **Blk 2** | **Blk 3** | **Blk 4** | **Blk 5** |
| | | 1,10,19 | 2,11,20 | 3,12,21 | 4,13,22 | 5,14,23 |
| | | **Blk 6** | **Blk 7** | **Blk 8** | **Blk 9** | |
| | | 6,15,24 | 7,16,25 | 8,17,26 | 9,18,27 | |

Treating the treatment numbers as line numbers, triallel crosses are made within block of each replication of this lattice design to give a PT$_r$C designs with parameters, $N = 27,\ b = 3,\ \ k = 27,\ T = 81$ and $f = 0.003$ as shown below:

| | | | | | | |
|---|---|---|---|---|---|---|
| **PT$_r$C Designs** | **Blk 1** | (1×2)×3 | (4×5) ×6 | (7×8)×9 | (10×11)×12 | (13×14)×15 |
| | | (16×17)×18 | (19×20)×21 | (22×23)×24 | (25×26)×27 | |
| | **Blk 2** | (1×4)×7 | (2×5)×6 | (3×6)×9 | (10×13)×16 | (11×14)×17 |
| | | (12×15)×18 | (19×22) ×25 | (20×23) ×26 | (21×24) ×27 | |
| | **Blk 3** | (1×10) ×19 | (2×11) ×20 | (3×12) ×21 | (4×13) ×22 | (5×14) ×23 |
| | | (6×15) ×24 | (7×16) ×25 | (8×17) ×26 | (9×18) ×27 | |

**List of designs**

A list of parameters of PT$_r$C designs constructed using different types of lattice design alongwith degree of fractionation and efficiency factor has been given in the Table 4.2.1.3.2.

**Table 4.2.1.3.2 List of designs using lattice designs for triallel crosses under blocked setup**

| $N$ | $b$ | $k$ | $T$ | $f$ | $\bar{V}_{\mathbf{half}}(\widehat{h_i - h_j})$ | $\bar{V}_{\mathbf{full}}(\widehat{g_i - g_j})$ | $E_h$ | $E_f$ | Type of lattices |
|---|---|---|---|---|---|---|---|---|---|
| 8 | 2 | 24 | 48 | 0.29 | 0.31 | 0.47 | 0.54 | 0.71 | Circular |
| 9 | 4 | 9 | 36 | 0.14 | 0.30 | 0.52 | 0.84 | 0.96 | Square |
| 12 | 4 | 9 | 36 | 0.02 | 0.20 | 0.41 | 0.99 | 0.99 | Rectangular |
| 27 | 3 | 27 | 81 | < 0.01 | 0.16 | 0.27 | 0.99 | 0.86 | Cubic |

**Method 3: PT$_r$C plans using Kronecker product**

This method can be used to obtain PT$_r$C plans for composite number of lines. In this method we have to consider the incidence matrices of any two BIB designs. The Kronecker product of these two matrices is obtained and is considered as an incidence matrix of a block design. Now, from each block of this design all possible triplet combinations are considered to make three-way crosses. The process is carried out for all the blocks and hence a $PT_rC$ plan is obtained. The method can be well understood through the example given below:

**Example 4.2.1.3.2**: A $PT_rC$ plan for number of lines, $N = 12$ and number of crosses $T = 216$ can be obtained using two BIB designs, viz., Design 1 (3,3,2,2,1) and Design 2 (4,6,3,2,1) as explained below:

| | | |
|---|---|---|
| **Blk 1** | 1 | 2 |
| **Blk 2** | 1 | 3 |
| **Blk 3** | 2 | 3 |

**Design 1: BIBD (3, 3, 2, 2, 1)**

| | **Trt 1** | **Trt 2** | **Trt 3** |
|---|---|---|---|
| **Blk 1** | 1 | 1 | 0 |
| **Blk 2** | 1 | 0 | 1 |
| **Blk 3** | 0 | 1 | 1 |

**Incidence matrix**

| | | |
|---|---|---|
| **Blk 1** | 1 | 2 |
| **Blk 2** | 1 | 3 |
| **Blk 3** | 1 | 4 |
| **Blk 4** | 2 | 3 |
| **Blk 5** | 2 | 4 |
| **Blk 6** | 3 | 4 |

**Design 2: BIBD (4, 6, 3, 2, 1)**

| | **Trt 1** | **Trt 2** | **Trt 3** | **Trt 4** |
|---|---|---|---|---|
| **Blk 1** | 1 | 1 | 0 | 0 |
| **Blk 2** | 1 | 0 | 1 | 0 |
| **Blk 3** | 1 | 0 | 0 | 1 |
| **Blk 4** | 0 | 1 | 1 | 0 |
| **Blk 5** | 0 | 1 | 0 | 1 |
| **Blk 6** | 0 | 0 | 1 | 1 |

**Incidence matrix**

Now, write down the block versus treatment incidence matrices of order $3 \times 3$ and $6 \times 4$ respectively. The Kronecker product of these two matrices results in matrix of order $18 \times 12$, yielding the incidence matrix of a new incomplete design *i.e.*, a rectangular design with parameters, $v = 12,\ b = 18,\ r = 6,\ k = 4\ ,\ \lambda_1 = 3, \lambda_2 = 2, \lambda_3 = 1,\ n_1 = 2,\ n_2 = 3$ and $n_3 = 6$. This is shown ahead:

| | TREATMENTS | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| BLOCKS | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 |
| | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 |
| | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 |
| | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 |
| | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 |
| | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 |
| | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 |
| | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 |
| | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 |
| | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 1 |
| | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 |
| | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 |

**Incidence matrix**

| | TREATMENTS | | | |
|---|---|---|---|---|
| **B-1** | 1 | 2 | 5 | 6 |
| **B-2** | 1 | 3 | 5 | 7 |
| **B-3** | 1 | 4 | 5 | 8 |
| **B-4** | 2 | 3 | 6 | 7 |
| **B-5** | 2 | 4 | 6 | 8 |
| **B-6** | 3 | 4 | 7 | 8 |
| **B-7** | 1 | 2 | 9 | 10 |
| **B-8** | 1 | 3 | 9 | 11 |
| **B-9** | 1 | 4 | 9 | 12 |
| **B-10** | 2 | 3 | 10 | 11 |
| **B-11** | 2 | 4 | 10 | 12 |
| **B-12** | 3 | 4 | 11 | 12 |
| **B-13** | 5 | 6 | 9 | 10 |
| **B-14** | 5 | 7 | 9 | 11 |
| **B-15** | 5 | 8 | 9 | 12 |
| **B-16** | 5 | 7 | 10 | 11 |
| **B-17** | 6 | 8 | 10 | 12 |
| **B-18** | 7 | 8 | 11 | 12 |

**Design 3: PBIB Rectangular Design (12,18,6,4,3,2,1)**

Now making all the possible three-way crosses from each block the following PT$_r$C plan can be obtained for 12 lines involving 216 crosses with a degree of fractionation 0.27:

| | | | | |
|---|---|---|---|---|
| (1×2)×5 | (2×3)×10 | (1×2)×6 | (2×3)×11 | (1×5)×6 |
| (1×3)×5 | (2×4)×10 | (1×3)×7 | (2×4)×12 | (1×5)×7 |
| (1×4)×5 | (3×4)×11 | (1×4)×8 | (3×4)×12 | (1×5)×8 |
| (2×3)×6 | (5×6)×9 | (2×3)×7 | (5×6)×10 | (2×6)×7 |
| (2×4)×6 | (5×7)×9 | (2×4)×8 | (5×7)×11 | (2×6)×8 |
| (3×4)×7 | (5×8)×9 | (3×4)×8 | (5×8)×12 | (3×7)×8 |
| (1×2)×9 | (6×7)×10 | (1×2)×10 | (6×7)×11 | (1×9)×10 |
| (1×3)×9 | (6×8)×10 | (1×3)×11 | (6×8)×12 | (1×9)×11 |

| | | | | |
|---|---|---|---|---|
| (1×4)×9 | (7×8)×11 | (1×4)×12 | (7×8)×12 | (1×9)×12 |
| (2×10)×11 | (2×5)×6 | (6×9)×10 | (3×6)×7 | (7×10)×11 |
| (4×10)×12 | (3×5)×7 | (7×9)×11 | (4×6)×8 | (8×10)×12 |
| (4×11)×12 | (4×5)×8 | (8×9)×12 | (4×7)×8 | (8×11)×12 |

Here, it should be noted that along with every cross of the type $(i, j, k)$ crosses of the types $(i, k, j)$ and $(j, k, i)$ are not shown in the plan and hence there are only 72 crosses.

**List of design**

A list of parameters PT$_r$C plans constructed using Kronecker product method alongwith the efficiency factors have been given in the Table 4.2.1.3.3.

**Table 4.2.1.3.3 List of triallel cross plans using kronecker product**

| $N$ | $T$ | $f$ | $r_h$ | $r_f$ | $\bar{V}_{half}(\widehat{h_i - h_j})$ | $\bar{V}_{full}(\widehat{g_i - g_j})$ | $E_h$ | $E_f$ |
|---|---|---|---|---|---|---|---|---|
| 9 | 108 | 0.43 | 24 | 12 | 0.1083 | 0.1833 | 0.9116 | 0.9427 |
| 12 | 180 | 0.27 | 36 | 18 | 0.0686 | 0.1206 | 0.9063 | 0.9382 |
| 15 | 360 | 0.26 | 48 | 24 | 0.0504 | 0.0903 | 0.8993 | 0.9321 |
| 18 | 540 | 0.22 | 60 | 30 | 0.0421 | 0.0795 | 0.8753 | 0.9124 |
| 20 | 720 | 0.21 | 72 | 36 | 0.0385 | 0.0654 | 0.8612 | 0.9009 |

**4.2.2 Tetra-allele crosses with sca effects**

Tetra-allele crosses are commonly used in animal and plant breeding experiments to study the various genetic properties especially on the gca effects of the lines involved or the sca effects of the various crosses. Tetra-allele cross experiment provides us more information regarding the combining abilities and the hybrids developed based on them are found to be more stable and consistent in performance due to broad genetic base.

**Model and experimental setup**

The model for tetra-allele cross experiments differs due to sca effects. A full model with sca involves estimation of sca effects upto third order alongwith gca effects. A restricted model may involve sca effects or may drop sca effects totally.

**4.2.2.1 Restricted model including lower order sca effects**

Since the second order and third order sca effects are negligible as compared to the first order sca effects these can be dropped from the model. Then for $y_{ijkl}$, we can have the following representation with the conditions on $g_i$'s as stated before:

$$y_{ijkl} = \bar{y} + g_i + g_j + g_k + g_l + s_{ij} s_{ik} + s_{il} + s_{jk} + s_{jl} + s_{kl} + e_{ijkl}, \qquad (4.2.2.1.1)$$

where $\bar{y}$ is the average effect of the treatments, $\{g_\alpha\}, \alpha = i, j, k, l$, represents the gca effects, $\{s_{\alpha\beta}\}, (\alpha, \beta) \in (i, j, k, l)$ represents the first order sca effects, $e_{ijkl}$ is the random errors, and

$$g_1 + g_2 + \cdots + g_N = 0 \text{ or } \sum_{i=1}^{N} g_i = 0,$$

$$s_{1i} + \cdots + s_{(i-1)i} + s_{i(i+1)} + \cdots + s_{iN} = 0, \text{and } \sum_{\alpha\beta} s_{\alpha\beta} = 0, \qquad (4.2.2.1.2)$$

for every $(\alpha, \beta) \in (i, j, k, l),\ i \neq j \neq k \neq l, i < j, k < l\,, i, j, k, l\ = 1,2, \ldots, N$.

**4.2.2.2 Estimates of combining abilities**

Consider that the tetra-allele crosses of the type, are arranged in the order $(1, 2, 3, 4)$, $\ldots, (1, 2, 3, l), \ldots, (1, 2, k, l), \ldots, (1, j, k, l), \ldots, (i, j, k, l),\ \ldots\ , \{(N-3), (N-2), (N-1),\ N\}$, $(i, j, k, l);\ i \neq j \neq k \neq l;\ \ i < j;\ k < l\,;\ i, j, k, l\ = 1,2, \ldots, N$. It should be noted that along with every cross of the type $(i, j, k, l)$ crosses of the types $(i, k, j, l)$ and $(i, l, j, k)$ are also included in the experiment, simultaneously. Let $\boldsymbol{g} = (g_1,\ g_2, \ldots, g_N)'$ be an $N \times 1$ vector of gca effects, $\boldsymbol{y}$ and $\boldsymbol{s}$ be $N \times 1$ vectors whose elements are $\{y_{ijkl}\}$ and $\{s_{\alpha\beta}\}$, respectively. Let us define a matrix $\boldsymbol{W}$ of order $N \times T$ with rows indexed by the line numberings as $1, 2, \ldots m, \ldots N$ and, the columns by the crosses, in same order as described earlier, of the types $(i, j, k, l)$, including the crosses of the types $(i, k, j, l)$ and $(i, l, j, k)$ also, simultaneously. Then, the $\{m, (i, j, k, l)\}^{\text{th}}$ entry of $\boldsymbol{W}$ takes a value 1 if $m \in (i, j, k, l)$ and 0, otherwise. The followings results are obtained:

$$\boldsymbol{W}\boldsymbol{W}' = \frac{(N-2)(N-3)}{2}\{(N-4)\boldsymbol{I}_N + 3\boldsymbol{J}_N\},$$

$$(\boldsymbol{W}\boldsymbol{W}')^{-1} = \frac{2}{(N-2)(N-3)(N-4)}\{\boldsymbol{I}_N - \frac{3}{4(N-1)}\boldsymbol{J}_N\},$$

$$\boldsymbol{W}\boldsymbol{1}_T = \frac{(N-1)(N-2)(N-3)}{2}\boldsymbol{1}_N \text{ and } \boldsymbol{W}'\boldsymbol{1}_N = 4\boldsymbol{1}_T. \qquad (4.2.2.2.1)$$

where $\boldsymbol{I}_N$ is an identity matrix of order $N$, $\boldsymbol{J}_N = \boldsymbol{1}_N\boldsymbol{1}_N'$ and $\boldsymbol{1}_N$ is an $N \times 1$ column vector of unities. Now, these results can be used to represent the cross $(i, j, k, l)$ effect as

$$\boldsymbol{y} = \bar{y}\,\boldsymbol{1}_T + \boldsymbol{W}'\boldsymbol{g} + \boldsymbol{s} + \boldsymbol{e}, \text{ with} \qquad (4.2.2.2.2)$$

$$\mathbf{1}_N^{'}\boldsymbol{g} = 0, \boldsymbol{W's} = \mathbf{0}, \quad (4.2.2.2.3)$$

and random error vector $\boldsymbol{e}$.

Premultiplying (4.2.2.2.2) by $\boldsymbol{W}$ and using (4.2.2.2.1) and (4.2.2.2.3) we get

$\hat{\boldsymbol{g}} = \boldsymbol{Ay}$, where

$$\boldsymbol{A} = (\boldsymbol{WW'})^{-1}\boldsymbol{W} - \frac{1}{4T}\boldsymbol{J}_{N\times T},$$

$$= \frac{2}{(N-2)(N-3)(N-4)}\{\boldsymbol{W} - \frac{4}{N}\boldsymbol{J}_{N\times T}\}, \text{ and} \quad (4.2.2.2.4)$$

$\hat{\boldsymbol{s}} = \boldsymbol{By}$, where

$$\boldsymbol{B} = \boldsymbol{I}_T - \frac{2}{(N-2)(N-3)(N-4)}\{\boldsymbol{W'W} - \frac{12}{(N-1)}\boldsymbol{J}_T\}. \quad (4.2.2.2.5)$$

Also, we have

$$\boldsymbol{A}\mathbf{1}_T = \boldsymbol{B}\mathbf{1}_T = \mathbf{0}, \boldsymbol{AB}^{'} = \boldsymbol{BA}^{'} = \mathbf{0}, rank(\boldsymbol{A}) = N-1 \text{ and}$$

$$rank(\boldsymbol{B}) = T - N. \quad (4.2.2.2.6)$$

Hence, $\boldsymbol{g}$ represents contrasts pertaining to gca effect with $(N-1)$degrees of freedom and, $\boldsymbol{s}$ for sca effects with $(T-N)$ degrees of freedom. From (4.2.2.2.6), it can be verified that the contrasts representing $\boldsymbol{g}$ are orthogonal to those representing $\boldsymbol{s}$. It is important to note here that for number of lines $N = 5$, $\boldsymbol{s} = \mathbf{0}$. Hence, it is considered that $N > 5$ throughout.

### 4.2.2.3 Orthogonal tetra-allele cross designs

Consider an arrangement of $T$ crosses, each replicated $r_i$ times such that $r_1 + r_2 + \cdots + r_T = v$, in a block design setup with $b$ blocks of size $k$ $(\geq 2)$. The fixed effect model incorporating both gca and sca effects is

$$y_{ijk} = \bar{y} + \tau_i + \beta_j + e_{ijk}, \quad (4.2.2.3.1)$$

where $y_{ijk}$ is the response from $k^{th}(k = 1, 2, \dots, r_i)$ replication of $i^{th}$ $(1, 2, \dots, T)$ cross from $j^{th}$ $(1,2, \dots, b)$ block, $\bar{y}$ is the general mean effect, $\tau_i$ is the $i^{th}$ cross effect, $\beta_j$ is the $j^{th}$ block effect and $e_{ijk}$ is the random error. The model can also be represented in matrix notation as follows:

$$\boldsymbol{y} = \bar{y}\mathbf{1} + \Delta_{\mathbf{1}}'\boldsymbol{\tau} + \Delta_2'\boldsymbol{\beta} + \boldsymbol{e}, \quad (4.2.2.3.2)$$

where, $\boldsymbol{y}$ is a $v \times 1$ vector of observations, $\bar{y}$ is the general mean effect, $\mathbf{1}$ is the $v \times 1$ vector of unity, $\Delta_{\mathbf{1}}'$ is the observation-cross incidence matrix of order $v \times T$, $\boldsymbol{\tau}$ is $T \times 1$ vector of cross effects, $\Delta_2'$ is the observation-block incidence matrix of order $v \times b$, $\boldsymbol{\beta}$ is $b \times 1$ vector of block effects and $\boldsymbol{e}$ is $v \times 1$ vector of random error. Also, $\boldsymbol{N} = \Delta_2\Delta_1'$ is the $b \times T$ incidence matrix of blocks versus crosses and $\boldsymbol{M} = \boldsymbol{W}\boldsymbol{N} = \boldsymbol{W}\Delta_2\Delta_1'$ is the $N \times b$ lines versus blocks incidence matrix. Here, our main interest will be in the estimation of contrasts pertaining to the gca effects of various lines while considering sca effects along with the block effects as nuisance factors in the aforesaid model. Under this model set up, we will be deriving conditions for block designs such that gca effects can be estimated independently from the sca effects, after eliminating the blocks effects.

Now, we can have the following definition:

**Definition 1:** Under a general block design set up for a usual fixed effects model incorporating both gca and sca effects, an *orthogonal tetra-allele cross design* can be defined as a block design which allows the estimation of contrasts pertaining to gca effects free from contrasts pertaining to sca effects, after eliminating the block effects.

The problem for finding orthogonal designs for tetra-allele crosses can be simplified if one starts by taking a complete set of orthonormal contrasts. Let us take two complete sets of orthonormal contrasts *viz.* $\boldsymbol{L}_1\boldsymbol{y}$ and $\boldsymbol{L}_2\boldsymbol{y}$ corresponding to $\boldsymbol{g}$ and $\boldsymbol{s}$, respectively. Now, using (4.2.2.2.3) and (4.2.2.2.6) we have the following results:

$$\boldsymbol{L}_1\boldsymbol{L}_1' = \boldsymbol{I}_{N-1}, \boldsymbol{L}_2\boldsymbol{L}_2' = \boldsymbol{I}_{T-N}, \boldsymbol{L}_1\boldsymbol{L}_2' = \mathbf{0}, \quad (4.2.2.3.3)$$

$$\boldsymbol{\mathcal{R}}(\boldsymbol{L}_1) = \boldsymbol{\mathcal{R}}(\boldsymbol{A}), \boldsymbol{\mathcal{R}}(\boldsymbol{L}_2) = \boldsymbol{\mathcal{R}}(\boldsymbol{B}), \quad (4.2.2.3.4)$$

where for a matrix $\boldsymbol{X}$, $\boldsymbol{\mathcal{R}}(\boldsymbol{X})$ denotes the row span of $\boldsymbol{X}$. Moreover, the results further obtained are not restricted for a specific choice of set of orthonormal contrasts corresponding to $\boldsymbol{g}$ and

$\boldsymbol{s}$, respectively. Let $\boldsymbol{X}_1$ be a $(N-1)\times N$ matrix such that $\begin{pmatrix} \frac{1}{\sqrt{N}}\boldsymbol{1}'_N \\ \boldsymbol{X}_1 \end{pmatrix}$ is an orthogonal matrix and $\boldsymbol{X}_2$ be a $(T-N)\times T$ matrix satisfying

$$\boldsymbol{X}_2\boldsymbol{X}'_2 = \boldsymbol{I}_{T-N} \text{ and } \boldsymbol{X}_2\boldsymbol{W}' = \boldsymbol{0}. \tag{4.2.2.3.5}$$

Hence, using (4.2.2.3.1) and (4.2.2.3.5) following results can be obtained:

$$\boldsymbol{X}_1\boldsymbol{1}_N = \boldsymbol{0},\ \boldsymbol{X}_1\boldsymbol{X}'_1 = \boldsymbol{I}_{N-1},\ \boldsymbol{X}_2\boldsymbol{1}_T = \boldsymbol{0}. \tag{4.2.2.3.6}$$

Now, consider any two matrices $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ satisfying (4.2.2.3.5) and (4.2.2.3.6), then corresponding to these we can express $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$ in the following form:

$$\boldsymbol{L}_1 = \frac{\sqrt{2}}{\sqrt{(N-2)(N-3)(N-4)}}\boldsymbol{X}_1\boldsymbol{W}, \boldsymbol{L}_2 = \boldsymbol{X}_2, \tag{4.2.2.3.7}$$

where $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$ are satisfying (4.2.2.3.3) and (4.2.2.3.4) with

$$\boldsymbol{L}'_1\boldsymbol{L}_1 = \frac{2}{(N-2)(N-3)(N-4)}\left(\boldsymbol{W}'\boldsymbol{W} - \frac{16}{N}\boldsymbol{J}_T\right). \tag{4.2.2.3.8}$$

Now, under the usual setup of a block design $d$, the joint information matrix for $\begin{pmatrix} \boldsymbol{L}_1 \\ \boldsymbol{L}_2 \end{pmatrix}\boldsymbol{y}$ is obtained as

$$\boldsymbol{C}_{joint} = \begin{bmatrix} \boldsymbol{L}_1\boldsymbol{C}_d\boldsymbol{L}'_1 & \boldsymbol{L}_1\boldsymbol{C}_d\boldsymbol{L}'_2 \\ \boldsymbol{L}_2\boldsymbol{C}_d\boldsymbol{L}'_1 & \boldsymbol{L}_2\boldsymbol{C}_d\boldsymbol{L}'_2 \end{bmatrix} \tag{4.2.2.3.9}$$

where $\boldsymbol{C}_d = \boldsymbol{R}_d - \frac{1}{k}\boldsymbol{N}_d\boldsymbol{N}'_d$, $\boldsymbol{R}_d = diag(r_1, r_2, \ldots, r_T)$ is the diagonal matrix of replications of the tetra-allele crosses under the design $d$ and $\boldsymbol{N}_d$ is the incidence matrix of crosses versus blocks. Here, $\boldsymbol{C}_d$ is the information matrix of the general block design $d$ where treatments are nothing but the $T$ number of tetra-allele crosses, hence we have $\boldsymbol{C}_d\boldsymbol{1}_T = \boldsymbol{0}$. As discussed earlier regarding orthogonality in Definition 1, and shown later in Lemma 1, to estimate $\boldsymbol{L}_1\boldsymbol{C}_d\boldsymbol{L}'_1$ and $\boldsymbol{L}_2\boldsymbol{C}_d\boldsymbol{L}'_2$ orthogonally the off diagonal components must vanish and we must have

$$\boldsymbol{L}_1\boldsymbol{C}_d\boldsymbol{L}'_2 = \boldsymbol{0}. \tag{4.2.2.3.10}$$

If we use the supposition from (4.2.2.3.7), then this is equivalent to

$$\boldsymbol{X}_1\boldsymbol{W}\boldsymbol{C}_d\boldsymbol{X}_2' = \boldsymbol{0}. \tag{4.2.2.3.11}$$

Let the information matrix related to $\boldsymbol{L}_1\boldsymbol{y}$ be $\boldsymbol{C}_{gd}$. Then, we can prove the following results:

**Lemma 1.** Under a design $d$, and $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$satisfying (4.2.2.3.3) and (4.2.2.3.4) we get that

(a) $\boldsymbol{L}_1\boldsymbol{C}_d\boldsymbol{L}_1' - \boldsymbol{C}_{gd}$ is a non-negative definite (n.n.d) matrix, and

(b) $\boldsymbol{C}_{gd} = \boldsymbol{L}_1\boldsymbol{C}_d\boldsymbol{L}_1'$, if and only if $\boldsymbol{L}_1\boldsymbol{C}_d\boldsymbol{L}_2' = \boldsymbol{0}$.

**Proof:** Consider the matrices $\boldsymbol{L}_1$and $\boldsymbol{L}_2$, then it can be easily established that the rows of $[\boldsymbol{L}_1' \quad \boldsymbol{L}_2']'$ form an orthonormal basis of the orthogonal complement of $\boldsymbol{\mathcal{R}}(\boldsymbol{1}_T')$ in the $T$-dimensional Euclidean space. Using (4.2.2.3.9) we can easily obtain $\boldsymbol{C}_{gd} = \boldsymbol{L}_1\boldsymbol{C}_d\boldsymbol{L}_1' - \boldsymbol{L}_1\boldsymbol{C}_d\boldsymbol{L}_2'(\boldsymbol{L}_2\boldsymbol{C}_d\boldsymbol{L}_2')^{-}\boldsymbol{L}_2\boldsymbol{C}_d\boldsymbol{L}_1'$, where $(\boldsymbol{L}_2\boldsymbol{C}_d\boldsymbol{L}_2')^{-}$ is the generalized inverse of $\boldsymbol{L}_2\boldsymbol{C}_d\boldsymbol{L}_2'$. Hence,

$$\boldsymbol{L}_1\boldsymbol{C}_d\boldsymbol{L}_1' - \boldsymbol{C}_{gd} = \boldsymbol{L}_1\boldsymbol{C}_d\boldsymbol{L}_2'(\boldsymbol{L}_2\boldsymbol{C}_d\boldsymbol{L}_2')^{-}\boldsymbol{L}_2\boldsymbol{C}_d\boldsymbol{L}_1'. \tag{4.2.2.3.12}$$

If $rank(\boldsymbol{C}_d) = t$, and as we know that $\boldsymbol{C}_d$ is a n.n.d. matrix, there exist a $N \times t$ matrix $\boldsymbol{G}$ of full column rank, such that $\boldsymbol{C}_d = \boldsymbol{G}'\boldsymbol{G}$. Then, (4.2.2.3.12) can be expressed in the form

$$\boldsymbol{L}_1\boldsymbol{C}_d\boldsymbol{L}_1' - \boldsymbol{C}_{gd} = \boldsymbol{L}_1\boldsymbol{G}'pr(\boldsymbol{G}\boldsymbol{L}_2')\boldsymbol{G}\boldsymbol{L}_1', \tag{4.2.2.3.13}$$

where $pr(\boldsymbol{G}\boldsymbol{L}_2') = \boldsymbol{G}\boldsymbol{L}_2'(\boldsymbol{L}_2\boldsymbol{G}'\boldsymbol{G}\boldsymbol{L}_2')^{-}\boldsymbol{L}_2\boldsymbol{G}'$ refers to the projection on to the column span of $\boldsymbol{G}\boldsymbol{L}_2'$. Since $pr(\boldsymbol{G}\boldsymbol{L}_2')$ is n.n.d., it can be easily established that $\boldsymbol{L}_1\boldsymbol{C}_d\boldsymbol{L}_1' - \boldsymbol{C}_{gd}$ is n.n.d. Hence we proved Lemma 1(a).

Now, using (4.2.2.3.13) it can be established that for $\boldsymbol{L}_1\boldsymbol{C}_d\boldsymbol{L}_1' = \boldsymbol{C}_{gd}$, we must have $\boldsymbol{L}_1\boldsymbol{G}'\boldsymbol{G}\boldsymbol{L}_2'(\boldsymbol{L}_2\boldsymbol{G}'\boldsymbol{G}\boldsymbol{L}_2')^{-}\boldsymbol{L}_2\boldsymbol{G}'\boldsymbol{G}\boldsymbol{L}_1' = \boldsymbol{0},$ which is equivalent to $\boldsymbol{L}_1\boldsymbol{G}'\boldsymbol{G}\boldsymbol{L}_2'(\boldsymbol{L}_2\boldsymbol{G}'\boldsymbol{G}\boldsymbol{L}_2')^{-} = \boldsymbol{0}$ or $\boldsymbol{L}_1\boldsymbol{G}'pr(\boldsymbol{G}\boldsymbol{L}_2')\boldsymbol{G}\boldsymbol{L}_1' = \boldsymbol{0}$ which is equivalent to $\boldsymbol{L}_1\boldsymbol{G}'pr(\boldsymbol{G}\boldsymbol{L}_2') = \boldsymbol{0}.$ Hence we get $\boldsymbol{L}_1\boldsymbol{G}'\boldsymbol{G}\boldsymbol{L}_2' = \boldsymbol{0}$ or $\boldsymbol{L}_1\boldsymbol{C}_d\boldsymbol{L}_2' = \boldsymbol{0},$ which proves Lemma 1(b).

Before proceeding further, let us prove the following results which will be needed further to prove Lemma 3.

**Lemma 2:** Let $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$ be defined as above. Then we have the following results:

(a) $\boldsymbol{L}_1'\boldsymbol{L}_1 = \frac{2}{(N-2)(N-3)(N-4)}\left(\boldsymbol{W}'\boldsymbol{W} - \frac{16}{N}\boldsymbol{J}_T\right).$

(b) $\boldsymbol{L}_2'\boldsymbol{L}_2 = \boldsymbol{I}_T - \frac{2}{(N-2)(N-3)(N-4)}\left(\boldsymbol{W'W} - \frac{12}{(N-1)}\boldsymbol{J}_T\right)$.

(c) $\boldsymbol{L}_1'\boldsymbol{L}_1 + \boldsymbol{L}_2'\boldsymbol{L}_2 = \boldsymbol{I}_T - \frac{1}{T}\boldsymbol{J}_T$.

**Proof:** Let us consider a matrix $\boldsymbol{L} = \begin{bmatrix} \boldsymbol{l}_1' \\ \boldsymbol{l}_2' \\ \vdots \\ \boldsymbol{l}_r' \end{bmatrix}$, rows of which forms an orthonormal basis of $\boldsymbol{\mathcal{R}}(\boldsymbol{G})$, where $\boldsymbol{G}$ is symmetric, idempotent and $rank(\boldsymbol{G}) = r$. Let $\boldsymbol{\Omega}$ is a diagonal matrix with diagonal elements being the $r$ distinct non-zero eigenvalues of $\boldsymbol{G}$**.** Thus, a spectral decomposition of $\boldsymbol{G}$ will lead us to the following:

$$\boldsymbol{L'}\Omega\boldsymbol{L} = \boldsymbol{G}.$$

Since $\boldsymbol{G}$ is an idempotent and symmetric matrix, the eigenvalues can take values either zero or one. Thus, we get that the diagonal elements of $\boldsymbol{\Omega}$ are only unities, which gives us

$$\boldsymbol{L'L} = \boldsymbol{G}. \tag{4.2.2.3.14}$$

Let $\boldsymbol{A}^* = \boldsymbol{W'A}$. As we know that $\boldsymbol{W'}$ is having full column rank, thus $\boldsymbol{\mathcal{R}}(\boldsymbol{A}^*) = \boldsymbol{\mathcal{R}}(\boldsymbol{A})$. Thus using (4.2.2.3.3) and (4.2.2.3.4) we find that the rows of $\boldsymbol{L}_1$ forms an orthonormal basis of $\boldsymbol{\mathcal{R}}(\boldsymbol{A}^*)$ where $\boldsymbol{A}^*$ is symmetric and idempotent. Hence, using (4.2.2.3.14) it can be easily established that $\boldsymbol{L}_1'\boldsymbol{L}_1 = \boldsymbol{A}^* = \boldsymbol{W'A}$, which can be further simplified using (4.2.2.2.1) and (4.2.2.2.4) to yield $\boldsymbol{L}_1'\boldsymbol{L}_1 = \boldsymbol{A}^* = \boldsymbol{W'A} = \frac{2}{(N-2)(N-3)(N-4)}\left(\boldsymbol{W'W} - \frac{16}{N}\boldsymbol{J}_T\right)$, which proves Lemma 2(a). Similarly, we find that the rows of $\boldsymbol{L}_2$ forms an orthonormal basis of $\boldsymbol{\mathcal{R}}(\boldsymbol{B})$where $\boldsymbol{B}$ is symmetric and idempotent. Hence, using (4.2.2.3.14) it can be easily established that $\boldsymbol{L}_2'\boldsymbol{L}_2 = \boldsymbol{B}$, which can be further simplified using (4.2.2.2.1) and (4.2.2.2.4) to yield $\boldsymbol{L}_2'\boldsymbol{L}_2 = \boldsymbol{B} = \boldsymbol{I}_T - \frac{2}{(N-2)(N-3)(N-4)}\left(\boldsymbol{W'W} - \frac{12}{(N-1)}\boldsymbol{J}_T\right)$, which proves Lemma 2(b). Adding the results of Lemma 2(a) & Lemma 2(b) we prove Lemma 2(c).

Now using these results following lemma will be proved.

**Lemma 3:** The following conditions are equivalent to each other:

(a) $\boldsymbol{L}_1\boldsymbol{C}_d\boldsymbol{L}_2' = \boldsymbol{0}$,

(b) $\boldsymbol{L}_1'\boldsymbol{L}_1\boldsymbol{C}_d = \boldsymbol{C}_d\boldsymbol{L}_1'\boldsymbol{L}_1$and

(c) $\boldsymbol{W'W}\boldsymbol{C}_d = \boldsymbol{C}_d\boldsymbol{W'W}$.

**Proof:** Let $\boldsymbol{L}_1'\boldsymbol{L}_1\boldsymbol{C}_d = \boldsymbol{C}_d\boldsymbol{L}_1'\boldsymbol{L}_1$.

Then, $\boldsymbol{L}_1\boldsymbol{C}_d\boldsymbol{L}_2' = \boldsymbol{L}_1\boldsymbol{L}_1'\boldsymbol{L}_1\boldsymbol{C}_d\boldsymbol{L}_2'$

$= \boldsymbol{L}_1\boldsymbol{C}_d\boldsymbol{L}_1'\boldsymbol{L}_1\boldsymbol{L}_2' = \boldsymbol{0}$, which proves Lemma 3(a).

Now, suppose that $\boldsymbol{L}_1\boldsymbol{C}_d\boldsymbol{L}_2' = \boldsymbol{0}$.

Then, $\boldsymbol{L}_1\boldsymbol{C}_d\boldsymbol{L}_2'\boldsymbol{L}_2 = \boldsymbol{0}$, and using (c) of Lemma 2 we get

$$\boldsymbol{L}_1\boldsymbol{C}_d\left(\boldsymbol{I}_T - \frac{1}{T}\boldsymbol{J}_T - \boldsymbol{L}_1'\boldsymbol{L}_1\right) = 0,$$

$$\Rightarrow \boldsymbol{L}_1\boldsymbol{C}_d = \boldsymbol{L}_1\boldsymbol{C}_d\boldsymbol{L}_1'\boldsymbol{L}_1$$

$$\Rightarrow \boldsymbol{L}_1'\boldsymbol{L}_1\boldsymbol{C}_d = \boldsymbol{L}_1'\boldsymbol{L}_1\boldsymbol{C}_d\boldsymbol{L}_1'\boldsymbol{L}_1.$$

Now, since $\boldsymbol{L}_1'\boldsymbol{L}_1\boldsymbol{C}_d\boldsymbol{L}_1'\boldsymbol{L}_1$ is symmetric, we get

$\boldsymbol{L}_1'\boldsymbol{L}_1\boldsymbol{C}_d = (\boldsymbol{L}_1'\boldsymbol{L}_1\boldsymbol{C}_d)' = \boldsymbol{C}_d\boldsymbol{L}_1'\boldsymbol{L}_1$**,** which proves Lemma 3(b).

Now, as we have $\boldsymbol{L}_1'\boldsymbol{L}_1\boldsymbol{C}_d = \boldsymbol{C}_d\boldsymbol{L}_1'\boldsymbol{L}_1$

$$\Rightarrow \frac{2}{(N-2)(N-3)(N-4)}\left(\boldsymbol{W}'\boldsymbol{W} - \frac{16}{N}\boldsymbol{J}_T\right)\boldsymbol{C}_d = \boldsymbol{C}_d\frac{2}{(N-2)(N-3)(N-4)}\left(\boldsymbol{W}'\boldsymbol{W} - \frac{16}{N}\boldsymbol{J}_T\right)$$

$\Rightarrow \ \boldsymbol{W}'\boldsymbol{W}\boldsymbol{C}_d = \boldsymbol{C}_d\boldsymbol{W}'\boldsymbol{W}$, which proves Lemma 3(c).

#### 4.2.2.4 Optimal tetra-allele cross designs

Our first step in the search for optimal designs for tetra-allele cross experiment will be finding an upper bound to the trace of information matrix $\boldsymbol{C}_{gd}$, represented by $tr(\boldsymbol{C}_{gd})$. Let $\mathcal{D}$ be the class of designs under which contrasts pertaining to gca effects, $\boldsymbol{L}_1\boldsymbol{y}$ is estimable. Now, using the result of Lemma 1(a), for any design, $d \in \mathcal{D}$, $\boldsymbol{L}_1\boldsymbol{C}_d\boldsymbol{L}_1' - \boldsymbol{C}_{gd}$ is n.n.d., thus we have

Trace $(\boldsymbol{C}_{gd}) = \ tr(\boldsymbol{C}_{gd}) \leq tr(\boldsymbol{L}_1\boldsymbol{C}_d\boldsymbol{L}_1') = tr(\boldsymbol{C}_d\boldsymbol{L}_1'\boldsymbol{L}_1)$, [using Lemma 3(b)] and

$= tr(\boldsymbol{C}_d\{\frac{2}{(N-2)(N-3)(N-4)}\left(\boldsymbol{W}'\boldsymbol{W} - \frac{16}{N}\boldsymbol{J}_T\right)\})$ [using Lemma 2(a)]

$$= \frac{2}{(N-2)(N-3)(N-4)} tr(\boldsymbol{C}_d\boldsymbol{W}'\boldsymbol{W})$$

$$= \frac{2}{(N-2)(N-3)(N-4)} tr\left(\boldsymbol{R}_d\boldsymbol{W}'\boldsymbol{W} - \frac{1}{k}\boldsymbol{N}_d\boldsymbol{N}_d'\boldsymbol{W}'\boldsymbol{W}\right).$$

Now, $tr(\boldsymbol{N}_d\boldsymbol{N}_d'\boldsymbol{W}'\boldsymbol{W})$

$$= tr(\boldsymbol{W}\boldsymbol{N}_d\boldsymbol{N}_d'\boldsymbol{W}')$$

$$= tr(\boldsymbol{M}_d\boldsymbol{M}_d')$$

$$= \sum_{i=1}^{N}\sum_{j=1}^{b} m_{dij}^2,$$

where $\boldsymbol{M}_d = \boldsymbol{W}\boldsymbol{N}_d = (m_{dij})$ is the $N \times b$ lines versus blocks incidence matrix. We can find that

$$\sum_{i=1}^{N}\sum_{j=1}^{b} m_{dij} = 4bk,$$

and since $\{m_{dij}\}$ can take only integral values,

$$\sum_{i=1}^{N}\sum_{j=1}^{b} m_{dij}^2 \geq b\{4k(2x+1) - Nx(x+1)\},$$

where $x = \left\lfloor \frac{4k}{N} \right\rfloor$ and $\lfloor \quad \rfloor$ denotes the greatest integer function. Also,

$$tr(\boldsymbol{R}_d\boldsymbol{W}'\boldsymbol{W}) = 4tr(\boldsymbol{R}_d) = 4bk,$$

since each diagonal element of $\boldsymbol{W}'\boldsymbol{W}$equals 4. Thus,

$$tr(\boldsymbol{C}_{gd}) \leq \frac{2}{(N-2)(N-3)(N-4)}\{4bk - \frac{b}{k}(4k(2x+1) - Nx(x+1))\},$$

or this can be simplified as

$$tr(\boldsymbol{C}_{gd}) \leq \frac{2b}{k(N-2)(N-3)(N-4)}\{4k(k-2x-1) + Nx(x+1))\} = \theta. \qquad (4.2.2.4.1)$$

The expression in (4.2.2.4.1) is proved in similar lines to Theorem 2.1 (which gives a lower bound to the trace of information matrix so that a block design is universal optimal) from Das *et al.* (1998). Now, it can be seen that $tr(\boldsymbol{C}_d\boldsymbol{W}'\boldsymbol{W}) = tr(\boldsymbol{W}\boldsymbol{C}_d\boldsymbol{W}')$, and $\boldsymbol{W}\boldsymbol{C}_d\boldsymbol{W}'$ is the information matrix of the block design where lines are regarded as treatments rather than crosses. Now, another important result is given below.

**Theorem 4.2.2.4:** Suppose there exists a design $d_0 \in \mathcal{D}$ such that

(a) $\boldsymbol{W}'\boldsymbol{W}\boldsymbol{C}_{d_0} = \boldsymbol{C}_{d_0}\boldsymbol{W}'\boldsymbol{W}$, and

(b) $\boldsymbol{W}\boldsymbol{C}_{d_0}\boldsymbol{W}' = \frac{(N-2)(N-3)(N-4)}{2(N-1)}\theta\left(\boldsymbol{I}_N - \frac{1}{N}\boldsymbol{J}_N\right)$, where $\theta > 0$.

Then, $d_0$ is universally optimal in the competing class of designs $\mathcal{D}$ for making inference regarding $\boldsymbol{L}_1\boldsymbol{y}$.

**Proof:** Let $\boldsymbol{C}_{gd_0}$ be the information matrix related to $\boldsymbol{L}_1\boldsymbol{y}$ under the design $d_0$. Hence, using (a) of Theorem 4.2.2.4, Lemma 1(b) and Lemma 3, we arrive at

$$\boldsymbol{C}_{gd_0} = \boldsymbol{L}_1\boldsymbol{C}_{d_0}\boldsymbol{L}_1'.$$

Since, we have

$$\boldsymbol{L}_1\boldsymbol{C}_{d_0}\boldsymbol{L}_1' = \frac{2}{(N-2)(N-3)(N-4)}\boldsymbol{X}_1\boldsymbol{W}\boldsymbol{C}_{d_0}\boldsymbol{W}'\boldsymbol{X}_1',$$

thus using (b) it is easy to show that

$$\boldsymbol{L}_1\boldsymbol{C}_{d_0}\boldsymbol{L}_1' = \frac{1}{(N-1)}\theta\boldsymbol{X}_1\left(\boldsymbol{I}_N - \frac{1}{N}\boldsymbol{J}_N\right)\boldsymbol{X}_1'$$

$= \frac{1}{(N-1)}\theta\boldsymbol{X}_1\boldsymbol{X}_1' = \frac{1}{(N-1)}\theta\boldsymbol{I}_{N-1}$. Thus, we get

$$\boldsymbol{C}_{gd_0} = \frac{1}{(N-1)}\theta\boldsymbol{I}_{N-1}. \tag{4.2.2.4.2}$$

From (4.2.2.4.1) it is obvious that for every design $d \in \mathcal{D}$,

$$tr(\boldsymbol{C}_{gd}) \leq \theta, \tag{4.2.2.4.3}$$

and using (4.2.2.4.2), we have

$$tr(\boldsymbol{C}_{gd_0}) = tr\left(\frac{1}{(N-1)}\theta\boldsymbol{I}_{N-1}\right) = \theta. \tag{4.2.2.4.4}$$

As we can see from (4.2.2.4.4) that the information matrix, $\boldsymbol{C}_{gd_0}$ is nothing but an identity matrix multiplied by a constant and from (4.2.2.4.3) and (4.2.2.4.4) we can easily say that $tr(\boldsymbol{C}_{gd_0})$ is maximized here to achieve the upper bound. Hence, our claim for universal optimality of the design $d_0$ is in agreement with results from Kiefer (1975), Sinha and Mukerjee (1982) and Das and Dey (2004).

Now, we can derive a more general condition, which is equivalent to the conditions in Theorem 4.2.2.4.

**Lemma 4:** The conditions described in Theorem 4.2.2.4 are equivalent to the following generalized result:

$$\boldsymbol{W}\boldsymbol{C}_{d_0} = \frac{1}{(N-1)}\theta\left(\boldsymbol{W} - \frac{4}{N}\boldsymbol{J}_{N\times T}\right). \qquad (4.2.2.4.5)$$

**Proof:** Suppose $\boldsymbol{W}\boldsymbol{C}_{d_0} = \frac{1}{(N-1)}\theta\left(\boldsymbol{W} - \frac{4}{N}\boldsymbol{J}_{N\times T}\right)$ holds. Then

$$\boldsymbol{W}\boldsymbol{C}_{d_0}\boldsymbol{W}' = \frac{1}{(N-1)}\theta\left(\boldsymbol{W} - \frac{4}{N}\boldsymbol{J}_{N\times T}\right)\boldsymbol{W}'$$

$$= \frac{(N-2)(N-3)(N-4)}{2(N-1)}\theta\left(\boldsymbol{I}_N - \frac{1}{N}\boldsymbol{J}_N\right), \text{ and}$$

$$\boldsymbol{W}'\boldsymbol{W}\boldsymbol{C}_{d_0} = \frac{1}{(N-1)}\theta\left(\boldsymbol{W}'\boldsymbol{W} - \frac{16}{N}\boldsymbol{J}_T\right),$$

which is symmetric, *i.e.* $\boldsymbol{W}'\boldsymbol{W}\boldsymbol{C}_{d_0} = \boldsymbol{C}_{d_0}\boldsymbol{W}'\boldsymbol{W}$.

Thus,

$$\boldsymbol{W}'\boldsymbol{W}\boldsymbol{C}_{d_0} = \frac{1}{(N-1)}\theta\left(\boldsymbol{W}'\boldsymbol{W} - \frac{16}{N}\boldsymbol{J}_T\right) \Rightarrow \text{Theorem 4.2.2.4.}$$

Conversely, suppose Theorem 4.2.2.4 holds true. Then, using (b) part of Theorem 4.2.2.4, we can have

$$\boldsymbol{W}\boldsymbol{C}_{d_0}\boldsymbol{W}'\boldsymbol{W} = \frac{(N-2)(N-3)(N-4)}{2(N-1)}\theta\left(\boldsymbol{I}_N - \frac{1}{N}\boldsymbol{J}_N\right)\boldsymbol{W}.$$

Now using (a) part of Theorem 4.2.2.4, we have

$$\boldsymbol{W}\boldsymbol{W}'\boldsymbol{W}\boldsymbol{C}_{d_0} = \frac{(N-2)(N-3)(N-4)}{2(N-1)}\theta\left(\boldsymbol{W} - \frac{4}{N}\boldsymbol{J}_{N\times T}\right)$$

$$\Rightarrow \boldsymbol{W}\boldsymbol{C}_{d_0} = \frac{(N-2)(N-3)(N-4)}{2(N-1)}\theta(\boldsymbol{W}\boldsymbol{W}')^{-\mathbf{1}}\left(\boldsymbol{W} - \frac{4}{N}\boldsymbol{J}_{N\times T}\right)$$

$$= \frac{(N-2)(N-3)(N-4)}{2(N-1)} \theta \frac{2}{(N-2)(N-3)(N-4)} \left\{ \boldsymbol{I}_N - \frac{3}{4(N-1)} \boldsymbol{J}_N \right\} \left( \boldsymbol{W} - \frac{4}{N} \boldsymbol{J}_{N\times T} \right)$$

$$= \frac{1}{(N-1)} \theta \left( \boldsymbol{W} - \frac{4}{N} \boldsymbol{J}_{N\times T} \right).$$

So, we have, $\boldsymbol{W}\boldsymbol{C}_{d_0} = \frac{1}{(N-1)} \theta \left( \boldsymbol{W} - \frac{4}{N} \boldsymbol{J}_{N\times T} \right)$ and thus

Theorem 4.2.2.4 ⇒ $\boldsymbol{W}'\boldsymbol{W}\boldsymbol{C}_{d_0} = \frac{1}{(N-1)} \theta \left( \boldsymbol{W}'\boldsymbol{W} - \frac{16}{N} \boldsymbol{J}_T \right)$.

#### 4.2.2.5 Designs satisfying the conditions

After establishing the orthogonality and optimality conditions for general block designs involving tetra-allele cross experiments, the problem remains is to search a class of designs that satisfy the conditions of Theorem 4.2.2.4. for this purpose, we state and prove the following lemmas and theorems:

**Lemma 5:** To ensure the estimability of $\boldsymbol{L}_1\boldsymbol{y}$ under a design $d \in \mathcal{D}$, whether for a blocked or unblocked set up, it is necessary that every cross (treatment) must appear at least once in the design.

**Proof:** We will try to prove this by method of contradiction. If $\boldsymbol{L}_1\boldsymbol{y}$ is estimable under a design $d \in \mathcal{D}$, and if possible let us consider that a cross, say $(1,2,3,4)$ is not present in the design $d$. Now, it can be easily seen that $\boldsymbol{\mathcal{R}}(\boldsymbol{L}_1) \subset \boldsymbol{\mathcal{R}}(\boldsymbol{C}_d)$, and the first column of $\boldsymbol{C}_d$ consists of zeros and so the first column of $\boldsymbol{L}_1$ are also zeros. Here, arises a contradiction.

To prove it, let us consider that the first column of $\boldsymbol{L}_1$ is a null vector. Then, the first column of $\boldsymbol{L}_1'\boldsymbol{L}_1$is a null vector, which means that by using (a) of Lemma 2, the first column of $\left( \boldsymbol{W}'\boldsymbol{W} - \frac{16}{N} \boldsymbol{J}_T \right)$is a null vector. This means that the first element of the first column of $\boldsymbol{W}'\boldsymbol{W}$ is $\frac{16}{N}$. But, by the definition of $\boldsymbol{W}$, the first element of the first column of $\boldsymbol{W}'\boldsymbol{W}$ equals to 4, which leads to contradiction, since $n > 4$.

The result from Lemma 5 leads us to the statement that, *under our stated model the smallest design under which $\boldsymbol{L}_1\boldsymbol{y}$ is estimable is one in which each cross is exactly replicated once.*

Now, for a single replicate design $d_1$,

$$\boldsymbol{C}_{d_1} = \boldsymbol{I}_T - \frac{1}{k}\boldsymbol{N}_{d_1}\boldsymbol{N}'_{d_1}.$$

Thus, using $\boldsymbol{W}'\boldsymbol{W}\boldsymbol{C}_{d_0} = \frac{1}{(N-1)}\theta\left(\boldsymbol{W}'\boldsymbol{W} - \frac{16}{N}\boldsymbol{J}_T\right)$ we can easily get the result

$$\boldsymbol{W}\boldsymbol{C}_{d_1} = \boldsymbol{W}(\boldsymbol{I}_T - \frac{1}{k}\boldsymbol{N}_{d_1}\boldsymbol{N}'_{d_1}) = \frac{1}{(N-1)}\theta\left(\boldsymbol{W} - \frac{4}{N}\boldsymbol{J}_{N\times T}\right),$$

which is equivalent to

$$(N-1-\theta)\boldsymbol{W} + \frac{4\theta}{N}\boldsymbol{J}_{N\times T} = \frac{(N-1)}{k}\theta\boldsymbol{W}\boldsymbol{N}_{d_1}\boldsymbol{N}'_{d_1}. \qquad (4.2.2.5.1)$$

Now, for $d_1$, $\boldsymbol{N}'_{d_1}\boldsymbol{N}_{d_1} = k\boldsymbol{I}_b$. Hence, post multiplying (4.2.2.5.1) by $\boldsymbol{N}_{d_1}$ and then simplifying it we get

$$\boldsymbol{W}\boldsymbol{N}_{d_1} = \boldsymbol{M}_{d_1} = \frac{4k}{N}\boldsymbol{J}_{N\times b}. \qquad (4.2.2.5.2)$$

Thus we get that (4.2.2.5.1) $\Rightarrow$ (4.2.2.5.2), where $\boldsymbol{M}_{d_1}$ is the $N \times b$ lines versus blocks incidence matrix. From (4.2.2.5.2) we can interpret that each lines occurs $\frac{4k}{N}$ times in each block. Now we will prove the converse. Suppose (4.2.2.5.2) holds, then

$$\boldsymbol{W}\boldsymbol{N}_{d_1}\boldsymbol{N}'_{d_1} = \frac{4k}{N}\boldsymbol{J}_{N\times b}\boldsymbol{N}'_{d_1} = \frac{4k}{N}\boldsymbol{1}_N(\boldsymbol{N}_{d_1}\boldsymbol{1}_b)' = \frac{4k}{N}\boldsymbol{J}_{N\times T}. \qquad (4.2.2.5.3)$$

It should be noted that if (4.2.2.5.2) holds, then $\frac{4k}{N}$ must be an integer i.e., $x = \frac{4k}{N}$ is an integer. In continuation we can show that $\theta = N-1$. Hence, using (4.2.2.5.2) the right hand side of (4.2.2.5.1) equals $\frac{4(N-1)}{N}\boldsymbol{J}_{N\times T}$ and thus by (4.2.2.5.3) it can be seen that (4.2.2.5.1) $\Rightarrow$ (4.2.2.5.2). Thus these results can be summarized in Theorem 4.2.2.5.

**Theorem 4.2.2.5:** For a single replicate design $d_1$, $\boldsymbol{W}\boldsymbol{C}_{d_0} = \frac{1}{(N-1)}\theta\left(\boldsymbol{W} - \frac{4}{N}\boldsymbol{J}_{N\times T}\right)$ holds if and only if

$$\boldsymbol{W}\boldsymbol{N}_{d_1} = \boldsymbol{M}_{d_1} = \frac{4k}{N}\boldsymbol{J}_{N\times b}.$$

Now, let us consider a general equireplicate design, $d_2$. The information matrix for such a design will be $\boldsymbol{C}_{d_2} = r\boldsymbol{I}_T - \frac{1}{k}\boldsymbol{N}_{d_2}\boldsymbol{N}'_{d_2}$, where $r$ denotes the number of replications of crosses under the design $d_2$ and $\boldsymbol{N}_{d_2}$denotes the $T \times b$ crosses versus blocks incidence matrix of the design $d_2$.

Thus, in this case

$$\boldsymbol{W}\boldsymbol{C}_{d_0} = \frac{1}{(N-1)}\theta\left(\boldsymbol{W} - \frac{4}{N}\boldsymbol{J}_{N\times T}\right) \Leftrightarrow (r(N-1)-\theta)\boldsymbol{W} + \frac{4\theta}{N}\boldsymbol{J}_{N\times T}$$

$$= \frac{(N-1)}{k}\boldsymbol{W}\boldsymbol{N}_{d_2}\boldsymbol{N}'_{d_2}. \qquad (4.2.2.5.4)$$

In order to proceed further, first the following lemma needs to be proved.

**Lemma 6:** The $T \times T$ matrix $\boldsymbol{Q} = \begin{bmatrix}\boldsymbol{W}\\ \boldsymbol{X}_2\end{bmatrix}$ is nonsingular.

**Proof:** The result can be easily established as $\boldsymbol{Q} = \begin{bmatrix}\boldsymbol{W}\\ \boldsymbol{X}_2\end{bmatrix}$, hence

$$\boldsymbol{Q}\boldsymbol{Q}' = \begin{bmatrix}\boldsymbol{W}\boldsymbol{W}' & \boldsymbol{W}\boldsymbol{X}'_2\\ \boldsymbol{X}_2\boldsymbol{W}' & \boldsymbol{X}_2\boldsymbol{X}'_2\end{bmatrix}$$

$$= \begin{bmatrix}\frac{(N-2)(N-3)}{2}\{(N-4)\boldsymbol{I}_N + 3\boldsymbol{J}_N\} & \boldsymbol{0}\\ \boldsymbol{0}' & \boldsymbol{I}_{T-N}\end{bmatrix},$$

So $\boldsymbol{Q}$ is a full rank square matrix *i.e.*, nonsingular. Hence, the result is proved.

Now, since $\boldsymbol{Q}$ is a non-singular matrix of order $T \times T$, and we have $T \times b$ matrix $\boldsymbol{N}_{d_2}$, which means that the column span of $\boldsymbol{N}_{d_2}$ is a subspace of the column span of $\boldsymbol{Q}'$, which means that there exist matrices $\boldsymbol{Y}_1$ and $\boldsymbol{Y}_2$ of order $N \times b$ and $(T-N) \times b$ respectively, such that

$$\boldsymbol{N}_{d_2} = \boldsymbol{W}'\boldsymbol{Y}_1 + \boldsymbol{X}'_2\boldsymbol{Y}_2. \qquad (4.2.2.5.5)$$

Now, pre-multiplication with $\boldsymbol{1}'_N$ in (4.2.2.5.5) and further simplification gives

$$k\boldsymbol{1}'_b = \boldsymbol{1}'_T\boldsymbol{N}_{d_2}$$

$$= \boldsymbol{1}'_T\boldsymbol{W}'\boldsymbol{Y}_1 + \boldsymbol{1}'_T\boldsymbol{X}'_2\boldsymbol{Y}_2$$

$$= \boldsymbol{1}'_T\boldsymbol{W}'\boldsymbol{Y}_1 = \frac{(N-1)(N-2)(N-3)}{2}\boldsymbol{1}'_N\boldsymbol{Y}_1,$$ hence we have

$$\mathbf{1}_N'\boldsymbol{Y}_1 = \frac{2k}{(N-1)(N-2)(N-3)}\mathbf{1}_b'.$$

Pre-multiplication with $\boldsymbol{W}$ and post multiplication $\mathbf{1}_b$ in (4.2.2.5.5) gives us

$$r\boldsymbol{W}\mathbf{1}_T = \boldsymbol{W}\boldsymbol{N}_{d_2}\mathbf{1}_b$$

$$= \boldsymbol{W}\boldsymbol{W}'\boldsymbol{Y}_1\mathbf{1}_b + \boldsymbol{W}\boldsymbol{X}_2'\boldsymbol{Y}_2\mathbf{1}_b$$

$= \boldsymbol{W}\boldsymbol{W}'\boldsymbol{Y}_1\mathbf{1}_b$, which gives more simple form as

$\boldsymbol{Y}_1\mathbf{1}_b = r(\boldsymbol{W}\boldsymbol{W}')^{-\mathbf{1}}\boldsymbol{W}\mathbf{1}_N$, which is further simplified using (4.2.2.2.1) as

$$\boldsymbol{Y}_1\mathbf{1}_b = \frac{r}{4}\mathbf{1}_N.$$

At last, we have,

$$r\mathbf{1}_T = \boldsymbol{N}_{d_2}\mathbf{1}_b$$

$$= \boldsymbol{W}'\boldsymbol{Y}_1\mathbf{1}_b + \boldsymbol{X}_2'\boldsymbol{Y}_2\mathbf{1}_b$$

$$= \boldsymbol{W}'\left(\frac{r}{4}\mathbf{1}_N\right) + \boldsymbol{X}_2'\boldsymbol{Y}_2\mathbf{1}_b$$

$= r\mathbf{1}_T + \boldsymbol{X}_2'\boldsymbol{Y}_2\mathbf{1}_b$, which gives

$r\mathbf{1}_T = r\mathbf{1}_T + \boldsymbol{X}_2'\boldsymbol{Y}_2\mathbf{1}_b$, which on pre-multiplication with $\boldsymbol{X}_2$ gives

$r\boldsymbol{X}_2\mathbf{1}_T = r\boldsymbol{X}_2\mathbf{1}_T + \boldsymbol{X}_2\boldsymbol{X}_2'\boldsymbol{Y}_2\mathbf{1}_b$, implying that

$$\boldsymbol{Y}_2\mathbf{1}_b = \mathbf{0}.$$

The results discussed here lead to a new Lemma.

**Lemma 7:** For a single replicate design, the result discussed in (4.2.2.5.4) is equivalent to

$\boldsymbol{Y}_1\boldsymbol{Y}_2' = \mathbf{0}$, and (4.2.2.5.6)

$$\boldsymbol{Y}_1\boldsymbol{Y}_{\mathbf{1}}' = \frac{2k}{(N-1)(N-2)(N-3)(N-4)}\{(r(N-1)-\theta)\boldsymbol{I}_N - \frac{(3rN-4\theta)}{4N}\boldsymbol{J}_N\}. \quad (4.2.2.5.7)$$

**Proof:** If we pre-multiply (4.2.2.5.5) by $\boldsymbol{W}$ we get

$\boldsymbol{W}\boldsymbol{N}_{d_2} = \boldsymbol{W}\boldsymbol{W}'\boldsymbol{Y}_1 + \boldsymbol{W}\boldsymbol{X}_2'\boldsymbol{Y}_2 = \boldsymbol{W}\boldsymbol{W}'\boldsymbol{Y}_1$, which further gives

$$\boldsymbol{W}\boldsymbol{N}_{d_2}\boldsymbol{N}_{d_2}' = \boldsymbol{W}\boldsymbol{W}'\boldsymbol{Y}_1(\boldsymbol{Y}_1'\boldsymbol{W} + \boldsymbol{Y}_2'\boldsymbol{X}_2). \quad (4.2.2.5.8)$$

Let us consider that (4.2.2.5.4) hold true. Then we have

$$(r(N-1)-\theta)\boldsymbol{W} + \frac{4\theta}{N}\boldsymbol{J}_{N\times T} = \frac{(N-1)}{k}\boldsymbol{W}\boldsymbol{N}_{d_2}\boldsymbol{N}_{d_2}'$$

$$= \frac{(N-1)}{k}\boldsymbol{W}\boldsymbol{W}'\boldsymbol{Y}_1(\boldsymbol{Y}_1'\boldsymbol{W} + \boldsymbol{Y}_2'\boldsymbol{X}_2). \quad (4.2.2.5.9)$$

Now, if we post-multiply (4.2.2.5.9) with $\boldsymbol{X}_2'$ on both the sides and then using $\boldsymbol{W}\boldsymbol{X}_2' = \boldsymbol{0}$, $\boldsymbol{J}_{N\times T}\boldsymbol{X}_2' = \boldsymbol{0}$, $\boldsymbol{X}_2\boldsymbol{X}_2' = \boldsymbol{I}_{T-N}$, we get

$\boldsymbol{0} = \frac{(N-1)}{k}\boldsymbol{W}\boldsymbol{W}'\boldsymbol{Y}_1\boldsymbol{Y}_2'$, which implies that

$$\boldsymbol{Y}_1\boldsymbol{Y}_2' = \boldsymbol{0},$$

because $\boldsymbol{W}\boldsymbol{W}'$ is a non-singular matrix. Hence proved (4.2.2.5.6).

Similarly, if we post-multiply (4.2.2.5.9) with $\boldsymbol{W}'$on both the sides, then

$$\boldsymbol{Y}_1\boldsymbol{Y}_1' = \frac{k(r(N-1)-\theta)}{(N-1)}(\boldsymbol{W}\boldsymbol{W}')^{-1} + \frac{4k\theta}{N(N-1)}(\boldsymbol{W}\boldsymbol{W}')^{-1}\boldsymbol{J}_{N\times T}\boldsymbol{W}'(\boldsymbol{W}\boldsymbol{W}')^{-1},$$

further simplification of which proves (4.2.2.5.7).

In order to prove the converse part, let us assume that (4.2.2.5.5) and (4.2.2.5.6) holds true. Then, using (4.2.2.5.8) and (4.2.2.5.6), we get

$$\frac{(N-1)}{k}\boldsymbol{W}\boldsymbol{N}_{d_2}\boldsymbol{N}_{d_2}' = \frac{(N-1)}{k}\boldsymbol{W}\boldsymbol{W}'\boldsymbol{Y}_1\left(\boldsymbol{Y}_1'\boldsymbol{W} + \boldsymbol{Y}_2'\boldsymbol{X}_2\right)$$

$$= \frac{(N-1)}{k}\boldsymbol{W}\boldsymbol{W}'\boldsymbol{Y}_1\boldsymbol{Y}_1'\boldsymbol{W},$$

which can be further simplified using (4.2.2.2.1) and (4.2.2.5.7) and the we get

$$\frac{(N-1)}{k}\boldsymbol{W}\boldsymbol{W}'\boldsymbol{Y}_1\boldsymbol{Y}_1'\boldsymbol{W} = [r(N-1)-\theta]\boldsymbol{W} + \frac{4\theta}{N}\boldsymbol{J}_{N\times T},$$

which is same as (4.2.2.5.4), hence proved the result.

All the results which follows (4.2.2.5.4) are derived to establish the following result in Lemma 8.

**Lemma 8:** In order that the result established in Theorem 4.2.2.4 can be extended for the case of an equireplicate design $d_2$, the following conditions must hold true:

(a) The matrix must be of the form $\boldsymbol{N}_{d_2} = \boldsymbol{W}'\boldsymbol{Y}_1 + \boldsymbol{X}_2'\boldsymbol{Y}_2$

(b) $\boldsymbol{Y}_1\boldsymbol{Y}_2' = \boldsymbol{0}$

(c) $\boldsymbol{Y}_1\boldsymbol{Y}_1' = \frac{2k}{(N-1)(N-2)(N-3)(N-4)}\left[\{r(N-1)-\theta\}\boldsymbol{I}_N - \frac{(3rN-4\theta)}{4N}\boldsymbol{J}_N\right]$

(d) $\boldsymbol{Y}_1\boldsymbol{1}_b = \frac{r}{4}\boldsymbol{1}_N$, $\boldsymbol{1}_N'\boldsymbol{Y}_1 = \frac{2k}{(N-1)(N-2)(N-3)}\boldsymbol{1}_b'$ and $\boldsymbol{Y}_2\boldsymbol{1}_b = \boldsymbol{0}$.

Now, from (4.2.2.5.4) it can be easily established that the design $d_2$ is orthogonal and balanced for estimation of gca effects if

$$[r(N-1)-\varphi]\boldsymbol{W} + \frac{4\varphi}{N}\boldsymbol{J}_{N\times T} = \frac{(N-1)}{k}\boldsymbol{W}\boldsymbol{N}_{d_2}\boldsymbol{N}_{d_2}', \tag{4.2.2.5.8}$$

where $\varphi$ is any positive scalar. Also, when $\varphi = \theta$, then we can also claim for the optimality of the class of designs. Here, we can also say that if

$\boldsymbol{N}_{d_2}\boldsymbol{N}_{d_2}' = \mu\boldsymbol{W}'\boldsymbol{W} + \sigma\boldsymbol{J}_{N\times T}$, for some scalars $\mu$ and $\sigma$ such that

$(N-1)(16\mu + N\sigma) = 4rk$, then (4.2.2.5.8) holds true.

Now, if we consider the case of an equireplicate design $d_2$, such that

$$\boldsymbol{M}_{d_2} = \boldsymbol{W}\boldsymbol{N}_{d_2} = \frac{4k}{N}\boldsymbol{J}_{N\times b}.$$

Then we can easily establish using (4.2.2.4.1) that $\theta = r(N-1)$ and thus, (4.2.2.5.4) holds. Moreover, such an equireplicated design is universally optimal for estimating gca effects while including sca effect in the underlying model.

### 4.2.2.6 Class of optimal tetra-allele cross designs

A series of universally optimal family of designs, which satisfies the above mentioned conditions, can be obtained using MOLS. Consider $N$, the number of lines as a prime or prime power. Out of total $(N-1)$ possible MOLS, consider any of the $\frac{(N-1)}{2}$ MOLS. Retaining the first four rows of each Latin square, $N$ number of crosses corresponding to each column can be made from each MOLS. Thus, the parameters of the developed class of design is $T = \frac{N(N-1)}{2}$, $b = \frac{(N-1)}{2}$, $r$, $k = N$. It can be easily seen that for the given class of designs $\boldsymbol{M}_{d_2} = \boldsymbol{W}\boldsymbol{N}_{d_2} = \frac{4k}{N}\boldsymbol{J}_{N\times b}$, hence this class of designs are universally optimal in the class of competing designs.

**Example 4.2.2.6:** The method can be well understood by an example for $N = 6$, $b = 3$, $k = 7$ and $T = 21$. Considering 3 MOLS of order 7 chosen at random out of the total 6 possible MOLS of order 7, and retaining only first 4 rows of each, based on the symbols A, B, C, D, E, F and G as given below.

| **MOLS I** | | | | | | |
|---|---|---|---|---|---|---|
| A | B | C | D | E | F | G |
| B | C | D | E | F | G | A |
| C | D | E | F | G | A | B |
| D | E | F | G | A | B | C |

| **MOLS II** | | | | | | |
|---|---|---|---|---|---|---|
| A | B | C | D | E | F | G |
| C | D | E | F | G | A | B |
| E | F | G | A | B | C | D |
| G | A | B | C | D | E | F |

| **MOLS III** | | | | | | |
|---|---|---|---|---|---|---|
| A | B | C | D | E | F | G |
| D | E | F | G | A | B | C |
| G | A | B | C | D | E | F |
| C | D | E | F | G | A | B |

Now, considering the seven symbols as lines, from each LS a tetra-allele crosses can be made by taking the four lines of each column. The crosses made from each LS will be constituting a block. The final layout of the design so obtained is given below.

| **Block 1** | **Block 2** | **Block 3** |
|---|---|---|
| (A×B)×(C×D) | (A×C)×(E×G) | (A×D)×(G×C) |
| (B×C)×(D×E) | (B×D)×(F×A) | (B×E)×(A×D) |
| (C×D)×(E×F) | (C×E)×(G×B) | (C×F)×(B×E) |
| (D×E)×(F×G) | (D×F)×(A×C) | (D×G)×(C×F) |
| (E×F)×(G×A) | (E×G)×(B×D) | (E×A)×(D×G) |
| (F×G)×(A×B) | (F×A)×(C×E) | (F×B)×(E×A) |
| (G×A)×(B×C) | (G×B)×(D×F) | (G×C)×(F×B) |

## 4.3 Prediction of combining ability effects

There are many situations where one wants to quantify the realization of an unobservable random variable. An example from breeding sector is that of predicting the genetic merit of a dairy bull from the milk yield capacity of his daughters. In a similar manner, predicting the yielding capacity of the cross from the sample of inbred lines is important for the breeders. Observations on some random variables are used to predict the value of some other related random variables that cannot be observed. The concept of BLUP is used for unbiased prediction of the yielding capacity of the crosses from the sample of inbred lines under mixed effects model.

As the number of lines increases, it becomes impossible to conduct a complete tetra-allele cross due to limitations of experimental units and so one may consider partial tetra-allele cross experiment with some unobserved crosses for predicting the yielding capacities of all the possible tetra-allele crosses among $N$ inbred lines. Ignoring the sca effects, the yielding capacity of the cross $(i, j, k, l)$ is estimated by $\hat{y} + \widehat{f_i} + \widehat{f_j} + \widehat{f_k} + \widehat{f_l}$, where $\{\widehat{f_\alpha}\}, \alpha = i, j, k, l,$ are some predicted values of gca.

### 4.3.1 Model and experimental setup

Consider a tetra-allele cross experiment being performed under unblocked design set up with $N$ number of lines and $T$ crosses, such that each line occurs exactly $p$ times in the design $d$. Reconsidering the mixed effects model $\boldsymbol{y} = \boldsymbol{X\theta} + \boldsymbol{Zb} + \boldsymbol{e}$, explained earlier in Section 3.3.2 of Chapter 3, to the situation of tetra-allele cross experiments, the model can be expressed as

$$\boldsymbol{y} = \bar{y}\mathbf{1}_N + \boldsymbol{W}_4'\boldsymbol{f} + \boldsymbol{e}, \qquad (4.3.1.1)$$

where $\boldsymbol{y}$ is an $T \times 1$ vector of observations, $\boldsymbol{f}$ is a $N \times 1$ vector of gca effects with $E(\boldsymbol{f}) = \mathbf{0}$ and $D(\boldsymbol{f}) = \boldsymbol{V_f} = \sigma_f^2\boldsymbol{I}_N$, $\boldsymbol{e}$ is a $T \times 1$ vector of random errors with $E(\boldsymbol{e}) = \mathbf{0}$ and $D(\boldsymbol{e}) = \sigma_e^2\boldsymbol{I}_T$, and $\boldsymbol{W_4}$ is an $N \times T$ incidence matrix with rows indexed by the line numbers $1,2, \ldots N$ and columns by the $T$ crosses such that the $\{m, (i, j, k, l)\}^{\text{th}}$ entry of $\boldsymbol{W_4}$ takes a value 1 if $m \in (i, j, k, l)$ and 0, otherwise. Thus we have

$$\boldsymbol{W}_4'\boldsymbol{W}_4\mathbf{1}_T = p\boldsymbol{W}_4'\mathbf{1}_T = 4p\mathbf{1}_N.$$

Now, borrowing the results from the mixed effect model regarding BLUP, we have

$$E(\boldsymbol{y}) = \bar{y}\mathbf{1}_N,$$

$$D\left(\boldsymbol{y}/\sigma_f^2, \sigma_e^2\right) = \sigma_f^2 \boldsymbol{W}_4' \boldsymbol{W}_4 + \sigma_e^2 \boldsymbol{I}_T = \boldsymbol{V_y}, \text{ and} \quad (4.3.1.2)$$

$$Cov(\boldsymbol{f}, \boldsymbol{y}) = Cov\left(\boldsymbol{f}, \bar{y}\boldsymbol{1}_N + \boldsymbol{W}_4'\boldsymbol{f} + \boldsymbol{e}\right) = \sigma_f^2 \boldsymbol{W}_4 = \boldsymbol{C_{fy}} \quad (4.3.1.3)$$

### 4.3.2 Prediction of yielding capacity of crosses

For the purpose of predicting the yielding capacity of tetra-allele crosses, following predictor has been considered

$$\boldsymbol{w} = \boldsymbol{L}'\boldsymbol{\theta} + \boldsymbol{b} \equiv \bar{y}\boldsymbol{1}_N + \boldsymbol{f}. \quad (4.3.2.1)$$

Now, as discussed earlier the best linear unbiased predictor (BLUP) of $\boldsymbol{w}$ is given as

$$\widetilde{\boldsymbol{w}} = \boldsymbol{1}_N \bar{y}^0 + \boldsymbol{C_{fy}} \boldsymbol{V_y^{-1}} (\boldsymbol{y} - \boldsymbol{1}_T \bar{y}^0)$$

$$= \boldsymbol{1}_N \bar{y}^0 + \boldsymbol{f}^0 \quad (4.3.2.2)$$

where $\boldsymbol{f}^0 = \boldsymbol{C_{fy}} \boldsymbol{V_y^{-1}} (\boldsymbol{y} - \boldsymbol{1}_T \bar{y}^0)$ and

$\bar{y}^0 = (\boldsymbol{1}_T' \boldsymbol{V_y^{-1}} \boldsymbol{1}_T)^{-} \boldsymbol{1}_T' \boldsymbol{V_y^{-1}} \boldsymbol{y}$ is the Generalised Least Square Estimate (GLSE) of $\bar{y}$.

### 4.3.3 Optimality of designs

Consider the class of designs $\mathcal{D}(N, T)$ under the usual unblocked setup involving $T$ tetra-allele crosses based on $N$ lines. Then for a given design $d \in \mathcal{D}(N, T)$, the mean square prediction error,

$$MSE(BLUP(\boldsymbol{w}), d) = E(\boldsymbol{w} - \widetilde{\boldsymbol{w}})'(\boldsymbol{w} - \widetilde{\boldsymbol{w}})$$

$$= E\{tr(\boldsymbol{w} - \widetilde{\boldsymbol{w}})'(\boldsymbol{w} - \widetilde{\boldsymbol{w}})\}$$

$$= E\{tr(\boldsymbol{w} - \widetilde{\boldsymbol{w}})(\boldsymbol{w} - \widetilde{\boldsymbol{w}})'\}$$

$$= tr\{E(\boldsymbol{w} - \widetilde{\boldsymbol{w}})(\boldsymbol{w} - \widetilde{\boldsymbol{w}})'\}$$

$$= tr\{D(\boldsymbol{w} - \widetilde{\boldsymbol{w}})\}.$$

We can see that the $MSE(BLUP(\boldsymbol{w}), d)$ depends on the design $d$. Thus, a design $d_{opt} \in \mathcal{D}(N, T)$ is said to be $A$-optimal over the competing class of designs $\mathcal{D}(N, T)$ if

$$MSE\left(BLUP(\boldsymbol{w}), d_{opt}\right) = \min_{d \in \mathcal{D}(N,T)} \{MSE(BLUP(\boldsymbol{w}), d)\} \quad (4.3.3.1)$$

Now, using the properties of BLUP, and

$Cov(\boldsymbol{X},\boldsymbol{Y}) = Cov(\boldsymbol{Y},\boldsymbol{X})$,

$Cov(\bar{y}^0\mathbf{1}_N,\boldsymbol{f}^0) = \mathbf{0}$,

$D(\boldsymbol{f}^0-\boldsymbol{f}) = \boldsymbol{V_f} - D(\boldsymbol{f}^0)$, and

$Cov(\bar{y}^0\mathbf{1}_N,\boldsymbol{f}) = \mathbf{1}_N(\mathbf{1}_T'\boldsymbol{V_y^{-1}}\mathbf{1}_T)^{-}\mathbf{1}_T'\boldsymbol{V_y^{-1}C_{fy}'}$, thus we can proceed as

$$\begin{aligned}
tr\{D(\boldsymbol{w}-\widetilde{\boldsymbol{w}})\} &= tr\{D(\bar{y}\mathbf{1}_N+\boldsymbol{f}-\bar{y}^0\mathbf{1}_N-\boldsymbol{f}^0)\}\\
&= tr[D\{(\bar{y}^0\mathbf{1}_N-\bar{y}\mathbf{1}_N)+(\boldsymbol{f}^0-\boldsymbol{f})\}]\\
&= tr[E(\bar{y}^0\mathbf{1}_N-\bar{y}\mathbf{1}_N)(\bar{y}^0\mathbf{1}_N-\bar{y}\mathbf{1}_N)'+E(\boldsymbol{f}^0-\boldsymbol{f})(\boldsymbol{f}^0-\boldsymbol{f})'\}\\
&\quad + E(\bar{y}^0\mathbf{1}_N-\bar{y}\mathbf{1}_N)(\boldsymbol{f}^0-\boldsymbol{f})'+E(\boldsymbol{f}^0-\boldsymbol{f})(\bar{y}^0\mathbf{1}_N-\bar{y}\mathbf{1}_N)']\\
&= tr[D(\bar{y}^0\mathbf{1}_N)+D(\boldsymbol{f}^0-\boldsymbol{f})\}+2Cov(\bar{y}^0\mathbf{1}_N,\boldsymbol{f}^0-\boldsymbol{f})]\\
&= tr[D(\bar{y}^0\mathbf{1}_N)+D(\boldsymbol{f}^0-\boldsymbol{f})\}+2Cov(\bar{y}^0\mathbf{1}_N,\boldsymbol{f}^0-\boldsymbol{f})]\\
&= tr\{D(\bar{y}^0\mathbf{1}_N)\}+tr\{D(\boldsymbol{f}^0-\boldsymbol{f})\}-2tr\{Cov(\bar{y}^0\mathbf{1}_N,\boldsymbol{f})\}\\
&= tr\{\left(\mathbf{1}_N(\mathbf{1}_T'\boldsymbol{V_y^{-1}}\mathbf{1}_T)^{-}\mathbf{1}_N'\right)\}+tr\{\boldsymbol{V_f}-D(\boldsymbol{f}^0)\}\\
&\quad -2tr\{\mathbf{1}_N(\mathbf{1}_T'\boldsymbol{V_y^{-1}}\mathbf{1}_T)^{-}\mathbf{1}_T'\boldsymbol{V_y^{-1}C_{fy}'}\}\\
&= tr\{\left(\mathbf{1}_N(\mathbf{1}_T'\boldsymbol{V_y^{-1}}\mathbf{1}_T)^{-}\mathbf{1}_N'\right)\}+tr(\boldsymbol{V_f})-tr(\boldsymbol{C_{fy}V_y^{-1}C_{fy}'})\\
&\quad +tr(\boldsymbol{C_{fy}V_y^{-1}}\mathbf{1}_T(\mathbf{1}_T'\boldsymbol{V_y^{-1}}\mathbf{1}_T)^{-}\mathbf{1}_T'\boldsymbol{V_y^{-1}C_{fy}'})\\
&\quad -2tr\{\mathbf{1}_N(\mathbf{1}_T'\boldsymbol{V_y^{-1}}\mathbf{1}_T)^{-}\mathbf{1}_T'\boldsymbol{V_y^{-1}C_{fy}'}\}. \qquad (4.3.3.2)
\end{aligned}$$

Now, the first term of (4.3.3.2) can be further simplified and expressed as:

$$tr\{\left(\mathbf{1}_N(\mathbf{1}_T'\boldsymbol{V_y^{-1}}\mathbf{1}_T)^{-}\mathbf{1}_N'\right)\} = \frac{N}{\mathbf{1}_T'\boldsymbol{V_y^{-1}}\mathbf{1}_T} \qquad (4.3.3.3)$$

As the matrix $\boldsymbol{V_y}$ is dependent on $\boldsymbol{W}_4'\boldsymbol{W}_4$, and we have to find the eigen vectors of $\boldsymbol{W}_4'\boldsymbol{W}_4$ orthogonal to the vector of unity $\mathbf{1}_T$, the following result will be helpful.

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two positive definite matrices of order $N$, such that $\boldsymbol{AB} = \boldsymbol{BA}$. Then there exists an orthogonal matrix $\boldsymbol{P} = (\boldsymbol{p_1}, \boldsymbol{p_2}, \dots, \boldsymbol{p_N})$ such that $\boldsymbol{P'AP} = \boldsymbol{D}_1$, $\boldsymbol{P'BP} = \boldsymbol{D}_2$ where $\boldsymbol{D}_1 = diag(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \dots, \boldsymbol{\alpha}_N)$ and $\boldsymbol{D}_2 = diag(\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \dots, \boldsymbol{\gamma}_N)$, where $\boldsymbol{\alpha}_i$ and $\boldsymbol{\gamma}_i$ are the eigenvalues of $\boldsymbol{A}$ and $\boldsymbol{B}$, respectively, $i = 1,2, \dots, N$. Then according to the definition of eigen vector $\boldsymbol{Ap}_i = \boldsymbol{\alpha}_i \boldsymbol{p}_i$ and $\boldsymbol{Bp}_i = \boldsymbol{\gamma}_i \boldsymbol{p}_i$, $i = 1,2, \dots, N$. Then we have, $(\boldsymbol{A} + \boldsymbol{B})\boldsymbol{p}_i = (\boldsymbol{\alpha}_i + \boldsymbol{\gamma}_i)\boldsymbol{p}_i$, $i = 1,2, \dots, N$.

Since, we have

$$\boldsymbol{W}_4'\boldsymbol{W}_4\boldsymbol{1}_T = 4p\boldsymbol{1}_N, \text{ and}$$

$$\sigma_f^2\boldsymbol{W}_4'\boldsymbol{W}_4\frac{\boldsymbol{1}_T}{\sqrt{\boldsymbol{T}}} = 4p\sigma_f^2\frac{\boldsymbol{1}_N}{\sqrt{\boldsymbol{T}}}, \text{ we get}$$

$$\left(\sigma_f^2\boldsymbol{W}_4'\boldsymbol{W}_4 + \sigma_e^2\boldsymbol{I}_T\right)\frac{\boldsymbol{1}_T}{\sqrt{\boldsymbol{T}}} = \left(4p\sigma_f^2 + \sigma_e^2\right)\frac{\boldsymbol{1}_N}{\sqrt{T}},$$

which means $\boldsymbol{p}_1 = \frac{\boldsymbol{1}_T}{\sqrt{T}}$ is an eigen vector corresponding to the eigenvalue $\boldsymbol{\lambda}_1 = \left(4p\sigma_f^2 + \sigma_e^2\right)$. In the same way, if $\boldsymbol{p}_2, \boldsymbol{p}_3, \dots, \boldsymbol{p}_T$ be the rest of $(T-1)$ orthonormal eigen vectors of $\boldsymbol{V_y}$ corresponding to eigenvalues $\boldsymbol{\lambda}_2, \boldsymbol{\lambda}_3, \dots, \boldsymbol{\lambda}_T$.

Hence we can have,

$$\boldsymbol{1}_T'\boldsymbol{V_y^{-1}}\boldsymbol{1}_T = \boldsymbol{1}_T'[\textstyle\sum_{i=1}^{T}\frac{\boldsymbol{p}_i\boldsymbol{p}_i'}{\boldsymbol{\lambda}_i}]\boldsymbol{1}_T$$

$$= \boldsymbol{1}_T'[(4p\sigma_f^2 + \sigma_e^2)^{-1}\frac{\boldsymbol{1}_T}{\sqrt{T}}\frac{\boldsymbol{1}_T'}{\sqrt{T}} + \textstyle\sum_{i=2}^{T}\frac{\boldsymbol{p}_i\boldsymbol{p}_i'}{\boldsymbol{\lambda}_i}]\boldsymbol{1}_T$$

$$= \frac{T}{(4p\sigma_f^2+\sigma_e^2)}. \tag{4.3.3.4}$$

Thus, $tr\{\left(\boldsymbol{1}_N(\boldsymbol{1}_T'\boldsymbol{V_y^{-1}}\boldsymbol{1}_T)^{-}\boldsymbol{1}_N'\right)\} = \frac{\boldsymbol{N}(4p\sigma_f^2+\sigma_e^2)}{T}$ (4.3.3.5)

The second term of (4.3.3.2) can be simplified as

$$tr(\boldsymbol{V_f}) = tr(\sigma_f^2\boldsymbol{I}_N) = N\sigma_f^2. \tag{4.3.3.6}$$

Here, we consider $\boldsymbol{W}_4\boldsymbol{W}_4' = \boldsymbol{G}$ and as the non zero eigenvalues are same for $\boldsymbol{W}_4\boldsymbol{W}_4'$ and $\boldsymbol{W}_4'\boldsymbol{W}_4$ and are given as $\lambda_{G1} = 4p$, $\lambda_{G2}$, ... , $\lambda_{GN}$ and thus $\lambda_i = \sigma_f^2\lambda_{Gi} + \sigma_e^2$, for $i = 1,2, \dots, N$ and $\lambda_i = \sigma_e^2$, for $i = N+1,2, \dots, T$.

Now, consider the third term of (4.3.3.2), which can be expressed as

$$tr(\boldsymbol{C_{fy}}\boldsymbol{V_y^{-1}}\boldsymbol{C'_{fy}}) = tr(\sigma_f^2\boldsymbol{W}_4\boldsymbol{V_y^{-1}}\sigma_f^2\ \boldsymbol{W}'_4)$$

$$= \sigma_f^4 tr(\boldsymbol{W}_4\boldsymbol{V_y^{-1}}\boldsymbol{W}'_4)$$

$$= \sigma_f^4 tr(\boldsymbol{W}'_4\boldsymbol{W}_4\boldsymbol{V_y^{-1}})$$

$$= \sigma_f^2 tr(\boldsymbol{V_y} - \sigma_e^2\boldsymbol{I}_T)\boldsymbol{V_y^{-1}}$$

$$= \sigma_f^2 tr(\boldsymbol{I}_T - \sigma_e^2\boldsymbol{V}_y^{-1})$$

$$= \sigma_f^2[T - \frac{\sigma_e^2}{\left(4p\sigma_f^2+\sigma_e^2\right)} - \sigma_e^2\sum_{i=2}^{T}\frac{1}{\boldsymbol{\lambda}_i}]$$

$$= \sigma_f^2[T - \frac{\sigma_e^2}{\left(4p\sigma_f^2+\sigma_e^2\right)} - \sigma_e^2\sum_{i=2}^{N}\frac{1}{\sigma_f^2\boldsymbol{\lambda}_{Gi}+\sigma_e^2} - \sigma_e^2\sum_{i=N+1}^{T}\frac{1}{\sigma_e^2}]$$

$$= \sigma_f^2[T - \frac{\sigma_e^2}{\left(4p\sigma_f^2+\sigma_e^2\right)} - \sum_{i=2}^{N}\frac{\sigma_e^2}{\sigma_f^2\boldsymbol{\lambda}_{Gi}+\sigma_e^2} - (T-N)]$$

$$= \sigma_f^2[N - \frac{\sigma_e^2}{\left(4p\sigma_f^2+\sigma_e^2\right)} - \sum_{i=2}^{N}\frac{\sigma_e^2}{\sigma_f^2\boldsymbol{\lambda}_{Gi}+\sigma_e^2}]. \quad \textbf{(}4.3.3.7\textbf{)}$$

Now, the fourth term in (4.3.3.2) can be simplified as

$$tr(\boldsymbol{C_{fy}}\boldsymbol{V_y^{-1}}\boldsymbol{1}_T(\boldsymbol{1}'_T\boldsymbol{V_y^{-1}}\boldsymbol{1}_T)^{-}\boldsymbol{1}'_T\boldsymbol{V_y^{-1}}\boldsymbol{C'_{fy}})$$

$$= tr(\sigma_f^2\boldsymbol{W}_4\boldsymbol{V_y^{-1}}\boldsymbol{1}_T(\boldsymbol{1}'_T\boldsymbol{V_y^{-1}}\boldsymbol{1}_T)^{-}\boldsymbol{1}'_T\boldsymbol{V_y^{-1}}\sigma_f^2\ \boldsymbol{W}'_4)$$

$$= \sigma_f^4 tr(\boldsymbol{W}'_4\boldsymbol{W}_4\boldsymbol{V_y^{-1}}\boldsymbol{1}_T\boldsymbol{1}'_T\frac{\boldsymbol{1}_T\boldsymbol{1}'_T}{\left(4p\sigma_f^2+\sigma_e^2\right)T}) \ / \ (\boldsymbol{1}'_T\boldsymbol{V_y^{-1}}\boldsymbol{1}_T)$$

$$= \frac{\sigma_f^4\left(4p\sigma_f^2+\sigma_e^2\right)}{T\left(4p\sigma_f^2+\sigma_e^2\right)} tr(\boldsymbol{W}'_4\boldsymbol{W}_4\boldsymbol{V_y^{-1}}\boldsymbol{1}_T\boldsymbol{1}'_T)$$

$$= \frac{\sigma_f^2}{T} tr(\boldsymbol{V_y} - \sigma_e^2\boldsymbol{I}_T)\boldsymbol{V_y^{-1}}\boldsymbol{1}_T\boldsymbol{1}'_T)$$

$$= \frac{\sigma_f^2}{T} tr[\left(\boldsymbol{I}_T - \sigma_e^2\boldsymbol{V}_y^{-1}\right)\boldsymbol{1}_T\boldsymbol{1}'_T]$$

$$= \frac{\sigma_f^2}{T}\left(T - \sigma_e^2 tr(\boldsymbol{1}'_T\boldsymbol{V}_y^{-1}\boldsymbol{1}_T)\right)$$

$$= \frac{\sigma_f^2}{T}\left(T - \sigma_e^2 \frac{T}{\left(4p\sigma_f^2+\sigma_e^2\right)}\right)$$

$$= \frac{4p\sigma_f^4}{4p\sigma_f^2+\sigma_e^2}. \quad (4.3.3.8)$$

Finally the last term in (4.3.3.2) can be simplified as

$$2tr\{\mathbf{1}_N(\mathbf{1}_T'\boldsymbol{V}_{\boldsymbol{y}}^{-\mathbf{1}}\mathbf{1}_T)^{-}\mathbf{1}_T'\boldsymbol{V}_{\boldsymbol{y}}^{-\mathbf{1}}\boldsymbol{C}_{\boldsymbol{fy}}'\}$$

$$= 2tr\{\mathbf{1}_N(\mathbf{1}_T'\boldsymbol{V}_{\boldsymbol{y}}^{-\mathbf{1}}\mathbf{1}_T)^{-}\mathbf{1}_T'\boldsymbol{V}_{\boldsymbol{y}}^{-\mathbf{1}}\sigma_f^2\ \boldsymbol{W}_4'\}$$

$$= 2\sigma_f^2 tr\left(\mathbf{1}_T'\boldsymbol{V}_{\boldsymbol{y}}^{-\mathbf{1}}\ \boldsymbol{W}_4'\mathbf{1}_N\right)/\mathbf{1}_T'\boldsymbol{V}_{\boldsymbol{y}}^{-\mathbf{1}}\mathbf{1}_T$$

$$= 2\sigma_f^2 tr(\mathbf{1}_T'\boldsymbol{V}_{\boldsymbol{y}}^{-\mathbf{1}}4\mathbf{1}_T)\ /\ (\mathbf{1}_T'\boldsymbol{V}_{\boldsymbol{y}}^{-\mathbf{1}}\mathbf{1}_T)$$

$$= 8\sigma_f^2. \quad (4.3.3.9)$$

Now, using the simplified results from (4.3.3.3) to (4.3.3.9) we can get the final expression as

$$MSE\left(BLUP(\boldsymbol{w}), d_{opt}\right)$$

$$= \frac{\boldsymbol{N}(4p\sigma_f^2+\sigma_e^2)}{T} + N\sigma_f^2 - \sigma_f^2[N - \frac{\sigma_e^2}{\left(4p\sigma_f^2+\sigma_e^2\right)} - \sum_{i=2}^{N}\frac{\sigma_e^2}{\sigma_f^2\boldsymbol{\lambda}_{Gi}+\sigma_e^2}] + \frac{4p\sigma_f^4}{4p\sigma_f^2+\sigma_e^2} - 8\sigma_f^2$$

$$= \frac{(4Np-7T)\sigma_f^2+N\sigma_e^2)}{T} + \sigma_e^2\sum_{i=2}^{N}\frac{1}{\boldsymbol{\lambda}_{Gi}+\frac{\sigma_e^2}{\sigma_f^2}}. \quad (4.3.3.10)$$

Minimizing $MSE(BLUP(\boldsymbol{w}), d)$ means minimizing the following

$$\sum_{i=2}^{N}\frac{1}{\boldsymbol{\lambda}_{G_d i}+\frac{\sigma_e^2}{\sigma_f^2}} = \sum_{i=2}^{N}\frac{1}{\boldsymbol{\lambda}_{G_d i}^*} \text{ (say).}$$

Now, we have

$$\sum_{i=1}^{N}\boldsymbol{\lambda}_{G_d i}^* = \sum_{i=1}^{N}(\lambda_{G_d i} + \frac{\sigma_e^2}{\sigma_f^2})$$

$$= N\frac{\sigma_e^2}{\sigma_f^2} + \sum_{i=1}^{N}(\lambda_{G_d i})$$

$$= N\frac{\sigma_e^2}{\sigma_f^2} + tr(\boldsymbol{G}_d)$$

$$= N\frac{\sigma_e^2}{\sigma_f^2} + tr(\boldsymbol{W}_4\boldsymbol{W}_4')$$

$$= N\frac{\sigma_e^2}{\sigma_f^2} + 4T.$$

Since $\lambda_{G_d1} = 4p$, we have

$$\sum_{i=2}^{N} \boldsymbol{\lambda}_{G_di}^{*} = (N-1)\frac{\sigma_e^2}{\sigma_f^2} + 4(T-p).$$

In order to attain a lower bound, we consider the AM-HM inequality for $(N-1)$ positive real numbers *i.e.* the eigenvalues $\lambda_{G_d2}^{*}, \lambda_{G_d3}^{*}, \dots, \lambda_{G_di}^{*}, \dots, \lambda_{G_dN}^{*}$ to attain the result

$$\sum_{i=2}^{N}\frac{1}{\lambda_{G_di}^{*}} \geq \frac{(N-1)^2}{\sum_{i=2}^{N}\lambda_{G_di}^{*}}, \text{ or}$$

$$\sum_{i=2}^{N}\frac{1}{\lambda_{G_di}^{*}} \geq \frac{(N-1)^2}{(N-1)\frac{\sigma_e^2}{\sigma_f^2}+4(T-p)},$$

equality being attained iff all $\lambda_{G_di}^{*},\ i = 2, \dots, N$ are equal which is same condition as all $\lambda_{G_di},\ i = 2, \dots, N$ being equal.

Thus, a design $d_{opt} \in \mathcal{D}(N,T)$ is $A$-optimal for BLUP of $\boldsymbol{w} = \boldsymbol{L}'\boldsymbol{\theta} + \boldsymbol{b} = \bar{y}\boldsymbol{1}_N + \boldsymbol{f}$, where $\boldsymbol{f}$ is a $N \times 1$ vector of gca effects with eigenvalues $\lambda_{G_{d_{opt}}i} = \lambda$ for $i = 2, \dots, N$.

### 4.4 Variance component estimates

The variance components are of much interest as they are needed for the estimation of genetic parameters like heritability, genotypic and phenotypic correlations and repeatability. All these important parameters are some functions of variance components. The parameters are also needed for the prediction of breeding value as well as response due to various selection procedures. Animal breeders are interested in using these breeding techniques to increase the production of economically important products from farm animals (e.g., eggs, milk, butter, wool, tallow and bacon).

#### 4.4.1 Unbiased estimates of variance components

The identity related to quadratic forms which can be used to obtain the unbiased estimates of variance components is

$$\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-}\boldsymbol{X}' = \boldsymbol{X}_1(\boldsymbol{X}_1'\boldsymbol{X}_1)^{-}\boldsymbol{X}_1' + \boldsymbol{T}_2\boldsymbol{X}_2(\boldsymbol{X}_2'\boldsymbol{T}_2\boldsymbol{X}_2)^{-}\boldsymbol{X}_2'\boldsymbol{T}_2,$$

where $\boldsymbol{T}_2 = \boldsymbol{I} - \boldsymbol{X}(\boldsymbol{X'X})^{-}\boldsymbol{X'}$ is an idempotent matrix.

Reconsider the model for tetra-allele cross experiment

$$\boldsymbol{y} = \boldsymbol{X\beta} + \boldsymbol{e} = \boldsymbol{y} = \bar{y}\boldsymbol{1}_N + \boldsymbol{W}_4'\boldsymbol{f} + \boldsymbol{e}, \quad (4.4.1.1)$$

such that $\boldsymbol{X} = [\boldsymbol{X}_1 \quad \boldsymbol{X}_2] = [\boldsymbol{1}_N \quad \boldsymbol{W}_4']$ and $\boldsymbol{\beta} = \begin{bmatrix}\boldsymbol{\beta}_1\\ \boldsymbol{\beta}_2\end{bmatrix} = \begin{bmatrix}\bar{y}\\ \boldsymbol{f}\end{bmatrix}$.

For this model, the identity for the quadratic form can be used to obtain three quadratic forms, namely total corrected sum of squares ($SS_T$), sum of squares due to lines ($SS_L$) and sum of squares due to error ($SS_E$). Now, based on Henderson's Method III (Searle *et al*., 1992), $SS_T$ can be partitioned into $SS_L$ and $SS_E$ as $SS_T = SS_L + SS_E$. Thus,

$$SS_T = \boldsymbol{y}'\boldsymbol{T}_1\boldsymbol{y} \quad (4.4.1.2)$$

$$SS_L = \boldsymbol{y}'[\boldsymbol{T}_1\boldsymbol{W}_4'(\boldsymbol{W}_4\boldsymbol{T}_1\boldsymbol{W}_4')^{-}\boldsymbol{W}_4\boldsymbol{T}_1]\boldsymbol{y} \quad (4.4.1.3)$$

$$SS_E = \boldsymbol{y}'\boldsymbol{T}_0\boldsymbol{y} \quad (4.4.1.4)$$

where $\boldsymbol{T}_1 = \boldsymbol{I} - \frac{\boldsymbol{11}'}{N}$ and $\boldsymbol{T}_0 = \boldsymbol{I} - (\boldsymbol{1} \quad \boldsymbol{W}_4')[(\boldsymbol{1} \quad \boldsymbol{W}_4')' \quad (\boldsymbol{1} \quad \boldsymbol{W}_4')]^{-}(\boldsymbol{1} \quad \boldsymbol{W}_4')'$.

$$E(SS_L) = \sigma_f^2 tr[\boldsymbol{T}_1\boldsymbol{W}_4'\boldsymbol{W}_4] + \sigma_e^2[rank(\boldsymbol{1} \quad \boldsymbol{W}_4') - rank(\boldsymbol{1})]$$

$$= \sigma_f^2 tr\left[(\boldsymbol{I} - \frac{\boldsymbol{11}'}{T})\boldsymbol{W}_4'\boldsymbol{W}_4\right] + \sigma_e^2[rank(\boldsymbol{1} \quad \boldsymbol{W}_4') - rank(\boldsymbol{1})]$$

$$= \sigma_f^2 tr\left[\boldsymbol{W}_4\boldsymbol{W}_4' - \frac{1}{T}(\boldsymbol{W}_4\boldsymbol{1})(\boldsymbol{W}_4\boldsymbol{1})'\right] + \sigma_e^2[N-1]$$

$$= \sigma_f^2 tr(\boldsymbol{G}_0) + \sigma_e^2[N-1]. \quad (4.4.1.5)$$

where $\boldsymbol{G}_0 = \left[\boldsymbol{G} - \frac{1}{T}\boldsymbol{pp}'\right]$.

$$E(SS_E) = (T-N)\sigma_e^2. \quad (4.4.1.6)$$

Thus, $E\begin{bmatrix}SS_L\\ SS_E\end{bmatrix} = \boldsymbol{L}\begin{pmatrix}\sigma_f^2\\ \sigma_e^2\end{pmatrix} = \boldsymbol{L}\sigma^2$. (4.4.1.7)

where $\boldsymbol{L} = \begin{bmatrix}tr(\boldsymbol{G}_0) & (N-1)\\ 0 & (T-N)\end{bmatrix}$ and $\sigma^2 = \begin{pmatrix}\sigma_f^2\\ \sigma_e^2\end{pmatrix}$.

Thus from (4.4.1.7) the unbiased estimator for $\sigma^2$ is obtained as

$$\widehat{\sigma^2} = \begin{pmatrix} \widehat{\sigma_f^2} \\ \widehat{\sigma_e^2} \end{pmatrix} = \boldsymbol{L}^{-1} \begin{bmatrix} SS_L \\ SS_E \end{bmatrix}, \tag{4.4.1.8}$$

where $\boldsymbol{L}^{-1} = \frac{1}{(T-N)tr(\boldsymbol{G}_0)} \begin{bmatrix} (T-N) & -(N-1) \\ 0 & tr(\boldsymbol{G}_0) \end{bmatrix}$.

Let $\boldsymbol{T}_3 = \boldsymbol{T}_1\boldsymbol{W}_4'$

$$= \left(\boldsymbol{I} - \frac{\boldsymbol{11}'}{N}\right)\boldsymbol{W}_4'$$

$= (\boldsymbol{W}_4' - \frac{\boldsymbol{1p}'}{N})$, and also we have

$$\boldsymbol{W}_4\boldsymbol{T}_1\boldsymbol{W}_4' = \boldsymbol{W}_4\left(\boldsymbol{I} - \frac{\boldsymbol{11}'}{N}\right)\boldsymbol{W}_4'$$

$$= \left(\boldsymbol{W}_4\boldsymbol{W}_4' - \frac{\boldsymbol{W}_4\boldsymbol{1}(\boldsymbol{W}_4\boldsymbol{1})'}{N}\right)$$

$$= \left(\boldsymbol{W}_4\boldsymbol{W}_4' - \frac{\boldsymbol{pp}'}{N}\right)$$

$$= \left(\boldsymbol{G} - \frac{\boldsymbol{pp}'}{N}\right) = \boldsymbol{G}_0 \text{ (say).}$$

Thus, we can have

$$SS_L = \boldsymbol{y}'\left[\boldsymbol{T}_1\boldsymbol{W}_4'\left(\boldsymbol{W}_4\boldsymbol{T}_1\boldsymbol{W}_4'\right)^{-}\boldsymbol{W}_4\boldsymbol{T}_1\right]\boldsymbol{y}$$

$$= \boldsymbol{y}'[\boldsymbol{T}_3\boldsymbol{G}_{\boldsymbol{0}}^{-}\boldsymbol{T}_3']\boldsymbol{y}.$$

Now, $\boldsymbol{G}_0\boldsymbol{1} = \left(\boldsymbol{W}_4\boldsymbol{W}_4' - \frac{\boldsymbol{pp}'}{N}\right)\boldsymbol{1}$

$$= \boldsymbol{W}_4\boldsymbol{W}_4'\boldsymbol{1} - \frac{\boldsymbol{pp}'}{N}\boldsymbol{1} = \boldsymbol{0},$$

which means that $rank(\boldsymbol{G}_0) \le N-1$, and since $rank(\boldsymbol{W_4}) = N$, we get $rank(\boldsymbol{G}_0) = (N-1)$.

### 4.4.2 Variance of estimates

In order to find the variances of estimates of the variance components we have to derive the dispersion matrix related to $\begin{bmatrix} SS_L \\ SS_E \end{bmatrix}$ as it is needed to study the sampling distribution of $\widehat{\sigma^2}$ in terms of variance and covariance.

Consider the matrix, $\boldsymbol{T}_4 = \boldsymbol{T}_3\boldsymbol{G}_{\boldsymbol{0}}^{-}\boldsymbol{T}'_3$. Then using the results on dispersion of quadratic forms, under the assumption of normality, we get

$$Var(SS_L) = Var(\boldsymbol{y}'\boldsymbol{T}_3\boldsymbol{G}_{\boldsymbol{0}}^{-}\boldsymbol{T}'_3\boldsymbol{y})$$

$$= Var(\boldsymbol{y}'\boldsymbol{T}_4\boldsymbol{y})$$

$$= 2tr(\boldsymbol{T}_4\boldsymbol{V}_{\boldsymbol{y}})^2,$$

$$= 2tr[\boldsymbol{T}_3\boldsymbol{G}_{\boldsymbol{0}}^{-}\boldsymbol{T}'_3(\sigma_f^2\boldsymbol{W}'_4\boldsymbol{W}_4 + \sigma_e^2\boldsymbol{I}_T)]^2$$

(where $\boldsymbol{V}_{\boldsymbol{y}}= D\left(\boldsymbol{y}/\sigma_f^2\,,\sigma_e^2\right) = \sigma_f^2\boldsymbol{W}'_4\boldsymbol{W}_4 + \sigma_e^2\boldsymbol{I}_T$)

$$= 2tr[\sigma_f^4\boldsymbol{T}_3\boldsymbol{G}_{\boldsymbol{0}}^{-}\boldsymbol{T}'_3\boldsymbol{W}'_4\boldsymbol{W}_4\boldsymbol{T}_3\boldsymbol{G}_{\boldsymbol{0}}^{-}\boldsymbol{T}'_3\boldsymbol{W}'_4\boldsymbol{W}_4 + \sigma_f^2\sigma_e^2\boldsymbol{T}_3\boldsymbol{G}_{\boldsymbol{0}}^{-}\boldsymbol{T}'_3\boldsymbol{W}'_4\boldsymbol{W}_4\boldsymbol{T}_3\boldsymbol{G}_{\boldsymbol{0}}^{-}\boldsymbol{T}'_3$$

$$+\sigma_f^2\sigma_e^2\boldsymbol{T}_3\boldsymbol{G}_{\boldsymbol{0}}^{-}\boldsymbol{T}'_3\boldsymbol{T}_3\boldsymbol{G}_{\boldsymbol{0}}^{-}\boldsymbol{T}'_3\boldsymbol{W}'_4\boldsymbol{W}_4 + \sigma_e^4\boldsymbol{T}_3\boldsymbol{G}_{\boldsymbol{0}}^{-}\boldsymbol{T}'_3\boldsymbol{T}_3\boldsymbol{G}_{\boldsymbol{0}}^{-}\boldsymbol{T}'_3]. \qquad (4.4.2.1)$$

Now, the first term in (4.4.2.1) is simplified as

$$2tr[\sigma_f^4\boldsymbol{T}_3\boldsymbol{G}_{\boldsymbol{0}}^{-}\boldsymbol{T}'_3\boldsymbol{W}'_4\boldsymbol{W}_4\boldsymbol{T}_3\boldsymbol{G}_{\boldsymbol{0}}^{-}\boldsymbol{T}'_3\boldsymbol{W}'_4\boldsymbol{W}_4]$$

$$= 2\sigma_f^4 tr[\boldsymbol{W}_4\boldsymbol{T}_3\boldsymbol{G}_{\boldsymbol{0}}^{-}\boldsymbol{T}'_3\boldsymbol{W}'_4\boldsymbol{W}_4\boldsymbol{T}_3\boldsymbol{G}_{\boldsymbol{0}}^{-}\boldsymbol{T}'_3\boldsymbol{W}'_4]$$

$$= 2\sigma_f^4 tr[(\boldsymbol{W}_4\boldsymbol{T}_3)\boldsymbol{G}_{\boldsymbol{0}}^{-}(\boldsymbol{W}_4\boldsymbol{T}_3)'(\boldsymbol{W}_4\boldsymbol{T}_3)\boldsymbol{G}_{\boldsymbol{0}}^{-}(\boldsymbol{W}_4\boldsymbol{T}_3)']$$

$$= 2\sigma_f^4 tr[\boldsymbol{G}_0\boldsymbol{G}_{\boldsymbol{0}}^{-}\boldsymbol{G}_0\boldsymbol{G}_0\boldsymbol{G}_{\boldsymbol{0}}^{-}\boldsymbol{G}_0]$$

$$= 2\sigma_f^4 tr[\boldsymbol{G}_{\boldsymbol{0}}^{\boldsymbol{2}}].$$

The second term in (4.4.2.1) is

$$2tr[\sigma_e^2\boldsymbol{T}_3\boldsymbol{G}_{\boldsymbol{0}}^{-}\boldsymbol{T}'_3\boldsymbol{W}'_4\boldsymbol{W}_4\boldsymbol{T}_3\boldsymbol{G}_{\boldsymbol{0}}^{-}\boldsymbol{T}'_3]$$

$$= 2\sigma_f^2\sigma_e^2 tr[\boldsymbol{T}'_3\boldsymbol{T}_3\boldsymbol{G}_{\boldsymbol{0}}^{-}\boldsymbol{T}'_3\boldsymbol{W}'_4\boldsymbol{W}_4\boldsymbol{T}_3\boldsymbol{G}_{\boldsymbol{0}}^{-}]$$

$$= 2\sigma_f^2\sigma_e^2 tr[(\boldsymbol{T}_3'\boldsymbol{T}_3)\boldsymbol{G}_{\mathbf{0}}^-(\boldsymbol{W}_4\boldsymbol{T}_3)'(\boldsymbol{W}_4\boldsymbol{T}_3)\boldsymbol{G}_{\mathbf{0}}^-]$$

$$= 2\sigma_f^2\sigma_e^2 tr[\boldsymbol{G}_0\boldsymbol{G}_{\mathbf{0}}^-\boldsymbol{G}_0\boldsymbol{G}_0\boldsymbol{G}_{\mathbf{0}}^-]$$

$$= 2\sigma_f^2\sigma_e^2 tr[\boldsymbol{G}_0\boldsymbol{G}_0\boldsymbol{G}_{\mathbf{0}}^-]$$

$$= 2\sigma_f^2\sigma_e^2 tr[\boldsymbol{G}_0\boldsymbol{G}_{\mathbf{0}}^-\boldsymbol{G}_0]$$

$$= 2\sigma_f^2\sigma_e^2 tr[\boldsymbol{G}_0].$$

The third term in (4.4.2.1) is expressed as

$$2tr[\sigma_f^2\sigma_e^2\boldsymbol{T}_3\boldsymbol{G}_{\mathbf{0}}^-\boldsymbol{T}_3'\boldsymbol{T}_3\boldsymbol{G}_{\mathbf{0}}^-\boldsymbol{T}_3'\boldsymbol{W}_4'\boldsymbol{W}_4]$$

$$= 2\sigma_f^2\sigma_e^2 tr[(\boldsymbol{W}_4\boldsymbol{T}_3)\boldsymbol{G}_{\mathbf{0}}^-(\boldsymbol{T}_3'\boldsymbol{T}_3)\boldsymbol{G}_{\mathbf{0}}^-(\boldsymbol{W}_4\boldsymbol{T}_3)']$$

$$= 2\sigma_f^2\sigma_e^2 tr[\boldsymbol{G}_0\boldsymbol{G}_{\mathbf{0}}^-\boldsymbol{G}_0\boldsymbol{G}_{\mathbf{0}}^-\boldsymbol{G}_0]$$

$$= 2\sigma_f^2\sigma_e^2 tr[\boldsymbol{G}_0\boldsymbol{G}_{\mathbf{0}}^-\boldsymbol{G}_0]$$

$$= 2\sigma_f^2\sigma_e^2 tr[\boldsymbol{G}_0].$$

The fourth term in (4.4.2.1) is simplified as

$$2tr[\sigma_e^4\boldsymbol{T}_3\boldsymbol{G}_{\mathbf{0}}^-\boldsymbol{T}_3'\boldsymbol{T}_3\boldsymbol{G}_{\mathbf{0}}^-\boldsymbol{T}_3']$$

$$= 2\sigma_e^4 tr[(\boldsymbol{T}_3'\boldsymbol{T}_3)\boldsymbol{G}_{\mathbf{0}}^-(\boldsymbol{T}_3'\boldsymbol{T}_3)\boldsymbol{G}_{\mathbf{0}}^-]$$

$$= 2\sigma_e^4 tr[\boldsymbol{G}_0\boldsymbol{G}_{\mathbf{0}}^-\boldsymbol{G}_0\boldsymbol{G}_{\mathbf{0}}^-]$$

$$= 2\sigma_e^4 tr[\boldsymbol{G}_0\boldsymbol{G}_{\mathbf{0}}^-]$$

$$= 2\sigma_e^4 rank[\boldsymbol{G}_0]$$

$$= 2(N-1)\sigma_e^4.$$

Thus, final expression after substituting the results is

$$Var(SS_L) = 2\left[\sigma_f^4 tr\left(\boldsymbol{G}_{\mathbf{0}}^{\mathbf{2}}\right) + 2\sigma_f^2\sigma_e^2 tr(\boldsymbol{G}_0) + 2(N-1)\sigma_e^4\right] \qquad (4.4.2.2)$$

We see that $\boldsymbol{W}_4\boldsymbol{T}_0 = \mathbf{0}$, and $\boldsymbol{T}_4\boldsymbol{T}_0 = \mathbf{0}$. Thus, we get

$$\boldsymbol{T}_4\boldsymbol{V}_{\boldsymbol{y}}\boldsymbol{T}_0 = \boldsymbol{T}_4(\sigma_f^2\boldsymbol{W}_4'\boldsymbol{W}_4 + \sigma_e^2\boldsymbol{I}_T)\boldsymbol{T}_0$$

$$= \boldsymbol{T}_4(\sigma_f^2 \boldsymbol{W}_4' \boldsymbol{W}_4 \boldsymbol{T}_0 + \sigma_e^2 \boldsymbol{I}_T) = \boldsymbol{0}.$$

Thus, we can get the following results

$$Cov(SS_L, SS_E) = Cov(\boldsymbol{y}'\boldsymbol{T}_4\boldsymbol{y},\ \boldsymbol{y}'\boldsymbol{T}_0\boldsymbol{y})$$

$$= 2tr(\boldsymbol{T}_4\boldsymbol{V}_{\boldsymbol{y}}\boldsymbol{T}_0\boldsymbol{V}_{\boldsymbol{y}}) = 0. \qquad (4.4.2.3)$$

$$Var(SS_E) = Var(\boldsymbol{y}'\boldsymbol{T}_0\boldsymbol{y})$$

$$= 2tr(\boldsymbol{T}_0\boldsymbol{V}_{\boldsymbol{y}})^2$$

$$= 2tr\{\boldsymbol{T}_0(\sigma_f^2 \boldsymbol{W}_4'\boldsymbol{W}_4 + \sigma_e^2 \boldsymbol{I}_T)\boldsymbol{T}_0(\sigma_f^2 \boldsymbol{W}_4'\boldsymbol{W}_4 + \sigma_e^2 \boldsymbol{I}_T)\}$$

$$= 2\sigma_e^4 tr(\boldsymbol{T}_0\boldsymbol{T}_0)$$

$$= 2\sigma_e^4 tr(\boldsymbol{T}_0)$$

$$= 2(T-N)\sigma_e^4. \qquad (4.4.2.4)$$

The final dispersion matrix of $D\begin{pmatrix} SS_L \\ SS_E \end{pmatrix}$ is obtained as follows:

$$\begin{bmatrix} 2\left[\sigma_f^4 tr(\boldsymbol{G}_0^2) + 2\sigma_f^2\sigma_e^2 tr(\boldsymbol{G}_0) + (N-1)\sigma_e^4\right] & \boldsymbol{0} \\ \boldsymbol{0} & 2(T-N)\sigma_e^4 \end{bmatrix}. \qquad (4.4.2.5)$$

The dispersion matrix related to $\begin{pmatrix} \widehat{\sigma_f^2} \\ \widehat{\sigma_e^2} \end{pmatrix}$ is given as

$$D(\widehat{\sigma^2}) = D\begin{pmatrix} \widehat{\sigma_f^2} \\ \widehat{\sigma_e^2} \end{pmatrix}$$

$$= \boldsymbol{L}^{-1}\left[D\begin{pmatrix} SS_L \\ SS_E \end{pmatrix}\right](\boldsymbol{L}^{-1})'$$

$$= 2\begin{pmatrix} \boldsymbol{d}_{11} & \boldsymbol{d}_{12} \\ \boldsymbol{d}_{21} & \boldsymbol{d}_{22} \end{pmatrix}, \qquad (4.4.2.6)$$

where $d_{11} = \frac{1}{(T-N)(tr(\boldsymbol{G}_0))^2}\left\{\begin{matrix} (T-N)\left(\sigma_f^4 tr(\boldsymbol{G}_0^2) + 2\sigma_f^2\sigma_e^2 tr(\boldsymbol{G}_0) + \sigma_e^4\right) \\ +(N-1)\sigma_e^4 \end{matrix}\right\}$,

$d_{12} = d_{21} = -\frac{(N-1)\sigma_e^4}{(T-N)tr(\boldsymbol{G}_0)}$ and $d_{22} = \frac{\sigma_e^4}{(T-N)}$.

### 4.5 Robust designs for breeding experiments

An optimal or efficient design for triallel or tetra-allele cross experiment may not allow the estimation of all elementary treatment contrasts or may become inefficient due to missing observation. Hence, it is much important to obtain robust designs against missing observation. The connectedness and efficiency criteria of robustness have been considered here to characterize robust designs involving triallel and tetra-allele crosses.

Thus, a design for breeding experiments is said to be robust (considering the connectedness and efficiency criteria) against a missing observation, if remains connected and efficient even after the disturbances due to a missing observation.

#### 4.5.1 Robust designs for triallel cross experiments

The robustness of designs involving triallel crosses against missing observation can be studied by investigating the connectedness and efficiency criteria of connectedness. Let the original design be denoted as $d$ and the residual design after the missing observation as $d^*$. Let $\mathbf{C}_{d_{gca-half}}$ and $\mathbf{C}_{d_{gca-full}}$ are the information matrices related to half parents and that of full parents respectively under the original design $d$ and $\mathbf{C}_{{d^*}_{gca-half}}$ and $\mathbf{C}_{{d^*}_{gca-full}}$ are the information matrices related to half parents and that of full parents for the residual design $d^*$. We consider the following usual model setup given in chapter III for triallel cross without sca effects:

$$\boldsymbol{y} = \bar{y}\,\mathbf{1}_T + \boldsymbol{W}_1'\boldsymbol{h} + \boldsymbol{W}_2'\boldsymbol{g} + \boldsymbol{e},$$

where $\boldsymbol{h}$ and $\boldsymbol{g}$ represents orthogonal treatment contrasts, both having $(N-1)$ degrees of freedom and can be used for obtaining orthogonal estimates of function of gca effects of half and full parents.

##### 4.5.1.1 Criterion of connectedness

The design $d$ involving triallel cross is said to be robust against missing observation based on connectedness criterion if the residual design $d^*$ remains connected so that we can estimate all the elementary treatment contrasts pertaining to the

- gca effect of first kind *i.e.* gca effect of half parents, and
- gca effect of second kind *i.e.* gca effect of full parents.

Thus a design $d^*$ will be fulfilling this criterion if $\boldsymbol{h}^*$ and $\boldsymbol{g}^*$ representing the new set of orthogonal treatment contrasts are having $(N-1)$ degrees of freedom.

#### 4.5.1.2 Efficiency criterion

A robust design involving triallel cross must be efficient pertaining to the gca effects of half as well as full parents.

**Efficiency criterion for half parents**

The efficiency of the design $d^*$ in comparison to the design $d$ can be calculated as

$$E_h = \frac{\text{harmonic mean of non-zero eigenvalues of } \mathbf{C}_{d^*gca-half}}{\text{harmonic mean of non-zero eigenvalues of } \mathbf{C}_{d_{gca-half}}}.$$

**Efficiency criterion for full parents**

The efficiency of the design $d^*$ in comparison to the design $d$ can be calculated as

$$E_f = \frac{\text{harmonic mean of non-zero eigenvalues of } \mathbf{C}_{d^*gca-full}}{\text{harmonic mean of non-zero eigenvalues of } \mathbf{C}_{d_{gca-full}}}.$$

**List of robust $PT_rC$ designs**

The designs for $PT_rC$ plans have been investigated for robustness using the connectedness and efficiency criteria and a list of robust designs against missing observation which are connected and are having efficiency more than equal to 80 percent is given in Tables 4.5.1.1 to 4.5.1.4. These tables contain the parameters of the robust designs alongwith the degree of fractionation, efficiencies and the underlying method of construction used. There are two situations considered for both blocked and unblocked setup. In the first situation, corresponding to the missing cross other two crosses are also omitted to satisfy the structural symmetry property. In the second situation, all the crosses except the missing one are kept intact and the study is carried out.

A list of robust $PT_rC$ designs under unblocked situation and maintaining SSP is given here in Table 4.5.1.1. The designs are having low degree of fractionation and high efficiencies.

**Table 4.5.1.1 Robust designs against a missing observation with SSP under unblocked situation**

| **S.No.** | *N* | *T* | *f* | $E_h$ | $E_f$ | **Design/Association scheme/ Method used for construction** |
|---|---|---|---|---|---|---|
| **1** | 5 | 30 | 1.00 | 0.84 | 0.86 | MOLS |
| **2** | 5 | 60 | 2.00 | 0.94 | 0.94 | MOLS |
| **3** | 7 | 63 | 0.60 | 0.94 | 0.94 | MOLS |

| | | | | | | |
|---|---|---|---|---|---|---|
| **4** | 7 | 126 | 1.20 | 0.98 | 0.98 | MOLS |
| **5** | 10 | 30 | 0.08 | 0.99 | 0.92 | Triangular Association Scheme |
| **6** | 10 | 90 | 0.25 | 0.95 | 0.96 | Triangular Design |
| **7** | 11 | 165 | 0.33 | 0.98 | 0.98 | MOLS |
| **8** | 11 | 330 | 0.67 | 0.99 | 0.99 | MOLS |
| **9** | 13 | 39 | 0.05 | 0.99 | 0.95 | Cyclic Association Scheme |
| **10** | 13 | 234 | 0.27 | 0.99 | 0.99 | MOLS |
| **11** | 13 | 668 | 0.78 | 0.99 | 0.99 | MOLS |
| **12** | 15 | 60 | 0.04 | 0.85 | 0.88 | Triangular Association Scheme |
| **13** | 17 | 408 | 0.20 | 0.99 | 0.99 | MOLS |
| **14** | 17 | 816 | 0.40 | 0.99 | 0.99 | MOLS |
| **15** | 19 | 513 | 0.18 | 0.99 | 0.99 | MOLS |
| **16** | 19 | 1026 | 0.35 | 0.99 | 0.99 | MOLS |
| **17** | 21 | 105 | 0.03 | 0.94 | 0.95 | Triangular Association Scheme |
| **18** | 23 | 759 | 0.14 | 0.99 | 0.99 | MOLS |
| **19** | 23 | 1518 | 0.29 | 0.99 | 0.99 | MOLS |
| **20** | 28 | 168 | 0.02 | 0.97 | 0.97 | Triangular Association Scheme |

A list of robust $PT_rC$ designs under unblocked condition and without considering the SSP is given here in the Table 4.5.1.2. The efficiencies are slightly higher than the previous case where SSP was maintained.

**Table 4.5.1.2 Robust designs against a missing observation without SSP under unblocked situation**

| **S.No.** | $N$ | $T$ | $f$ | $E_h$ | $E_f$ | **Design/Association scheme/ Method used for construction** |
|---|---|---|---|---|---|---|
| **1** | 5 | 15 | 0.50 | 0.83 | 0.94 | Cyclic Association Scheme |
| **2** | 5 | 30 | 1.00 | 0.95 | 0.95 | MOLS |
| **3** | 5 | 60 | 2.00 | 0.98 | 0.98 | MOLS |
| **4** | 7 | 63 | 0.60 | 0.98 | 0.98 | MOLS |
| **5** | 7 | 126 | 1.20 | 0.99 | 0.99 | MOLS |
| **6** | 10 | 30 | 0.08 | 0.92 | 0.92 | Triangular Association Scheme |
| **7** | 10 | 90 | 0.25 | 0.99 | 0.98 | Triangular Design |

| | | | | | | |
|---|---|---|---|---|---|---|
| **8** | 11 | 165 | 0.33 | 0.99 | 0.99 | MOLS |
| **9** | 11 | 330 | 0.67 | 0.99 | 0.99 | MOLS |
| **10** | 13 | 39 | 0.05 | 0.93 | 0.93 | Cyclic Association Scheme |
| **11** | 13 | 234 | 0.27 | 0.99 | 0.99 | MOLS |
| **12** | 13 | 668 | 0.78 | 0.99 | 0.99 | MOLS |
| **13** | 15 | 60 | 0.04 | 0.97 | 0.97 | Triangular Association Scheme |
| **14** | 17 | 408 | 0.20 | 0.99 | 0.99 | MOLS |
| **15** | 17 | 816 | 0.40 | 0.99 | 0.99 | MOLS |
| **16** | 19 | 513 | 0.18 | 0.99 | 0.99 | MOLS |
| **17** | 19 | 1026 | 0.35 | 0.99 | 0.99 | MOLS |
| **18** | 21 | 105 | 0.03 | 0.98 | 0.98 | Triangular Association Scheme |
| **19** | 23 | 759 | 0.14 | 0.99 | 0.99 | MOLS |
| **20** | 23 | 1518 | 0.29 | 0.99 | 0.99 | MOLS |
| **21** | 28 | 168 | 0.02 | 0.99 | 0.99 | Triangular Association Scheme |

A list of robust $PT_rC$ designs under blocked situation and maintaining SSP is provided here in Table 4.5.1.3. The designs are having low degree of fractionation and higher efficiencies.

**Table 4.5.1.3 Robust designs against a missing observation with SSP under blocked situation**

| **S.No.** | $N$ | $b$ | $k$ | $T$ | $f$ | $E_h$ | $E_f$ | **Design/Association scheme/ Method used for construction** |
|---|---|---|---|---|---|---|---|---|
| 1 | 5 | 2 | 15 | 30 | 1.00 | 0.81 | 0.84 | MOLS |
| 2 | 5 | 4 | 15 | 60 | 2.00 | 0.93 | 0.93 | MOLS |
| 3 | 7 | 3 | 21 | 63 | 0.60 | 0.93 | 0.94 | MOLS |
| 4 | 7 | 6 | 21 | 126 | 1.20 | 0.97 | 0.97 | MOLS |
| 5 | 8 | 2 | 24 | 48 | 0.29 | 0.88 | 0.90 | Lattice Design |
| 6 | 9 | 9 | 12 | 108 | 0.43 | 0.95 | 0.96 | Kronecker Product |
| 7 | 11 | 5 | 33 | 165 | 0.33 | 0.98 | 0.98 | MOLS |
| 8 | 11 | 10 | 33 | 330 | 0.67 | 0.99 | 0.99 | MOLS |
| 9 | 12 | 18 | 12 | 216 | 0.33 | 0.98 | 0.98 | Kronecker Product |
| 10 | 13 | 3 | 52 | 156 | 0.18 | 0.98 | 0.98 | BIBD |
| 11 | 13 | 6 | 39 | 234 | 0.27 | 0.98 | 0.99 | MOLS-2 |

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 12 | 13 | 12 | 39 | 668 | 0.78 | 0.99 | 0.99 | MOLS-1 |
| 13 | 15 | 30 | 12 | 360 | 0.26 | 0.99 | 0.99 | Kronecker Product |
| 14 | 17 | 8 | 51 | 408 | 0.20 | 0.99 | 0.99 | MOLS-2 |
| 15 | 17 | 16 | 51 | 816 | 0.40 | 0.99 | 0.99 | MOLS-1 |
| 16 | 18 | 2 | 180 | 360 | 0.15 | 0.99 | 0.99 | Lattice Design |
| 17 | 19 | 9 | 57 | 513 | 0.18 | 0.99 | 0.99 | MOLS-2 |
| 18 | 19 | 18 | 57 | 1026 | 0.35 | 0.99 | 0.99 | MOLS-1 |
| 19 | 23 | 11 | 69 | 759 | 0.14 | 0.99 | 0.99 | MOLS-2 |
| 20 | 23 | 22 | 69 | 1518 | 0.29 | 0.99 | 0.99 | MOLS-1 |

A list of robust $PT_rC$ designs under a blocked situation without maintaining SSP is given here in Table 4.5.1.4. It can be seen that the efficiencies are slightly higher than the case where SSP is maintained.

**4.5.1.4 Robust designs against a missing observation without SSP under blocked situation**

| S.No. | **N** | **b** | **k** | **T** | **f** | **$E_h$** | **$E_f$** | **Design/Association scheme/ Method used for construction** |
|---|---|---|---|---|---|---|---|---|
| 1 | 5 | 2 | 15 | 30 | 1.00 | 0.95 | 0.95 | MOLS |
| 2 | 5 | 4 | 15 | 60 | 2.00 | 0.98 | 0.98 | MOLS |
| 3 | 7 | 3 | 7 | 21 | 0.20 | 0.86 | 0.86 | BIBD |
| 4 | 7 | 3 | 21 | 63 | 0.60 | 0.98 | 0.98 | MOLS |
| 5 | 7 | 6 | 21 | 126 | 1.20 | 0.99 | 0.99 | MOLS |
| 6 | 8 | 2 | 24 | 48 | 0.29 | 0.97 | 0.97 | Lattice Design |
| 7 | 9 | 9 | 12 | 108 | 0.43 | 0.99 | 0.99 | Kronecker Product |
| 8 | 9 | 4 | 9 | 36 | 0.14 | 0.94 | 0.94 | Lattice Design |
| 9 | 10 | 5 | 6 | 30 | 0.08 | 0.97 | 0.78 | Triangular Design |
| 10 | 11 | 5 | 33 | 165 | 0.33 | 0.99 | 0.99 | MOLS |
| 11 | 11 | 10 | 33 | 330 | 0.67 | 0.99 | 0.99 | MOLS |
| 12 | 12 | 18 | 12 | 216 | 0.33 | 0.99 | 0.99 | Kronecker Product |
| 13 | 13 | 3 | 52 | 156 | 0.18 | 0.99 | 0.99 | BIBD |
| 14 | 13 | 6 | 39 | 234 | 0.27 | 0.99 | 0.99 | MOLS |
| 15 | 13 | 12 | 39 | 668 | 0.78 | 0.99 | 0.99 | MOLS |
| 16 | 15 | 30 | 12 | 360 | 0.26 | 0.99 | 0.99 | Kronecker Product |

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 17 | 17 | 8 | 51 | 408 | 0.20 | 0.99 | 0.99 | MOLS |
| 18 | 17 | 16 | 51 | 816 | 0.40 | 0.99 | 0.99 | MOLS |
| 19 | 18 | 2 | 180 | 360 | 0.15 | 0.99 | 0.99 | Lattice Design |
| 20 | 19 | 9 | 57 | 513 | 0.18 | 0.99 | 0.99 | MOLS |
| 21 | 19 | 18 | 57 | 1026 | 0.35 | 0.99 | 0.99 | MOLS |
| 22 | 23 | 11 | 69 | 759 | 0.14 | 0.99 | 0.99 | MOLS |
| 23 | 23 | 22 | 69 | 1518 | 0.29 | 0.99 | 0.99 | MOLS |

### 4.5.2 Robust design for tetra-allele cross experiments

In a similar manner, the robustness of designs involving tetra-allele crosses against missing observation can be studied by investigating the connectedness and efficiency criteria.

**List of robust $PT_eC$ designs**

Designs constructed for tetra-allele cross plan under blocked and unblocked situations has been considered and investigated for robustness against a missing observation. The list of robust designs alongwith the efficiencies is given in the Tables 4.5.2.1 and 4.5.2.2. The parameters of the designs alongwith the degree of fractionation and efficiencies have been tabulated.

A list of robust $PT_eC$ designs against missing observations under unblocked situation is given in the Table 4.5.2.1.

**Table 4.5.2.1 Robust tetra-allele cross designs constructed using MOLS against a missing observation under unblocked situation**

| **S.No.** | *N* | *T* | *f* | *E* |
|---|---|---|---|---|
| **1** | 5 | 10 | 0.67 | 0.83 |
| **2** | 7 | 21 | 0.20 | 0.93 |
| **3** | 11 | 55 | 0.06 | 0.98 |
| **4** | 13 | 78 | 0.04 | 0.98 |
| **5** | 17 | 136 | 0.02 | 0.99 |
| **6** | 19 | 171 | 0.01 | 0.99 |
| **7** | 23 | 253 | 0.01 | 0.99 |

A list of robust $PT_eC$ under blocked situation is given in the Table 4.5.2.2. The designs are having low degree of fractionation alongwith higher efficiencies and thus, can be used in case an observation is missing without much loss of precision.

**Table 4.5.2.2 Robust tetra-allele cross designs constructed using MOLS against a missing observation under blocked situation**

| **S.No.** | *N* | *b* | *k* | *T* | *f* | *E* |
|---|---|---|---|---|---|---|
| **1** | 5 | 2 | 5 | 10 | 0.67 | 0.80 |
| **2** | 7 | 3 | 7 | 21 | 0.20 | 0.92 |
| **3** | 11 | 5 | 11 | 55 | 0.06 | 0.98 |
| **4** | 13 | 6 | 13 | 78 | 0.04 | 0.98 |
| **5** | 17 | 8 | 17 | 136 | 0.02 | 0.99 |
| **6** | 19 | 9 | 19 | 171 | 0.01 | 0.99 |
| **7** | 23 | 11 | 23 | 253 | 0.01 | 0.99 |

# Chapter-5 DISCUSSION

## 5.1 Introduction

The major objective of breeding is releasing new hybrids with enhanced genetic potential, which is based on information on general and specific combining abilities, information related to variance components and predictor of yielding capacities of crosses. Further, designs used for attaining these goals should be robust against disturbances like missing observation(s).

In this study, optimal and efficient class of designs involving higher order crosses (triallel and tetra-allele) have been obtained considering a fixed effects model. Robustness of these designs against missing observation has also been studied. Further, under a random effects model set up, the Best Linear Unbiased Predictor (BLUP) for yielding capacity of cross has been obtained. The unbiased estimates of variance components alongwith sampling distribution have also been obtained.

A brief discussion on various research findings related to methodology, designs obtained and their characterization properties is given here.

## 5.2 Triallel and tetra-allele cross plans incorporating sca effects

Under a fixed effects model including sca effects for triallel crosses, the estimates of general and specific combining abilities have been obtained. A class of designs for triallel crosses arranged in blocks has been obtained using triangular association scheme and information matrices, eigenvalues, variance factors, efficiency factors and degree of fractionation have been derived.

Another method of constructing partial triallel cross designs has also been developed using various types of lattice designs viz., square, rectangular, circular and cubic lattice. This method gives designs for triallel crosses under blocked set up for a wide range of parameters. Both the proposed classes of designs are having low degree of fractionation with high efficiencies. The main restriction of first series of designs is that these designs are available only for cases where the number of lines is of the particular form $N = \frac{n(n-1)}{2}$. The second series is available for many combinations and can be used in conjunction with the first method to fill the gaps of designs not available for particular parameters.

The third method, of constructing partial triallel cross plans is based on Kronecker product of incidence matrices. The incidence matrices of two BIB designs with small block sizes are used as input designs to construct designs with desired combination of parameters. The method is very simple and gives partial triallel cross plans. The plans obtained with this method are also having low degree of fractionation and high efficiencies and can be used when there is scarcity of resources. This method along with the other two methods can provide designs for almost all sets of parametric combinations. With an adequate knowledge of block designs, all the proposed crossing plans can easily be constructed. These plans can save a lot of resources and time of the breeders. A list of parameters and efficiency factor for the designs developed by these three methods is given in Tables 4.2.1.3.1 to 4.2.1.3.3 of Chapter 4.

Under a restricted model including lower order sca effects for tetra-allele crosses, conditions of orthogonality have been derived for a block design such that the contrasts pertaining to the gca effects and sca effects are estimated free from each other, after eliminating the other nuisance factors. Conditions for universal optimality of block designs for tetra-allele cross experiments have been derived and various related results regarding the existence of optimal designs have been deduced. A method of construction of optimal block designs has been described and a class of optimal designs based on Mutually Orthogonal Latin Squares have been obtained. The proposed class of designs satisfy the conditions of optimality of equireplicated design for estimating gca effects while including sca effects in the underlying model. The existence of the above optimal class of designs does not mean that there are no other such classes but further work in this area is needed for determining more optimal classes of designs because this method is not for all ranges of parameters.

### 5.3 Prediction of combining ability effects

The Best Linear Unbiased Predictor (BLUP) for predicting the unobserved combining ability effects together with general mean effect in tetra-allele cross design has been obtained. For predicting the yielding capacity of the crosses from the sample of inbred lines, where observations are taken on some random variables and the value of some other related random variables that cannot be observed is to be predicted. BLUP is useful for unbiased prediction of the yielding capacity of the crosses from the sample of inbred lines under random effects model.

A lower bound of mean square error of prediction has been also derived which can be further used to find optimal designs regarding for prediction of yielding capacity of crosses. This bound can be used to characterize various classes of optimal designs.

**5.4 Variance components estimates**

The variance component estimates along with their large sample variances, using mixed linear model approach in tetra-allele crosses has been obtained using Henderson Method III. BLUP of the unobserved line effects are used for ranking the value of inbred lines, which will increase the productivity of future generation. But the prediction of line effects depends on good estimates of variance components related to line effects and error. Thus, variance components are of much interest for breeders. Besides this, the genetic parameter heritability on which the breeding policies depends is also a function of variance components.

**5.5 Robust design for breeding experiments**

Robustness of designs for triallel cross experiments have been investigated using connectedness and efficiency criteria against missing observation under unblocked situation (Table 4.5.1.1 of Chapter 4) and maintaining the structural symmetry property (SSP). The designs are connected when SSP is maintained, only if the number of lines is more than 5. Moreover, the designs are having good efficiencies for estimating the contrasts pertaining to gca effects of half as well as full parents.

In another approach, the SSP is neglected and robustness of newly developed and previously available designs for triallel cross experiments have been investigated using connectedness and efficiency criteria against missing observation under unblocked situation and a list of such efficient designs has been given in Table 4.5.1.2. The main advantage over the previous one is that the designs can be used for lower number of lines also as they remain connected. Further, the efficiencies are much higher.

Under blocked set up too, robustness of designs for triallel cross experiments with SSP has been investigated using connectedness and efficiency criteria against missing observation. The designs given in Table 4.5.1.3 can be advantageously used in the situation where the number of lines or the crosses is high and thus homogeneity cannot be maintained. These designs, with smaller block sizes are more appropriate for the situation.

Also, robustness of designs for triallel cross experiments have been investigated using connectedness and efficiency criteria against missing observation without accounting for SSP property. These designs are robust and efficient against a missing observation. A list of parameters of such designs alongwith the degree of fractionation and efficiencies can be seen in Table 4.5.1.4.

Furthermore, robustness of designs for tetra-allele cross experiments have been investigated using connectedness and efficiency criteria against missing observation under unblocked situation The designs are having low degree of fractionation alongwith high efficiency factors. These can be used as robust designs for tetra-allele cross experiments against a missing observation. These designs are catalogued in Table 4.5.2.1 alongwith efficiency factor.

Again, under blocked situation, robustness designs for triallel cross experiments have been investigated using connectedness and efficiency criteria against missing observation. The designs given in Table 4.5.2.2 can be used with increased precision of estimates as we are getting designs with crosses arranged in blocks.

Overall, Tables (4.5.1.1-4.5.1.4) and Tables (4.5.2.1-4.5.2.2) give lists of good efficient designs that are robust in performance, if an observation is missing, by any disturbance in the experiment. These designs are advocated to the breeders who deal with higher order crossing plans as in such trials, there is a fair chance of missing observation.

# SUMMARY AND CONCLUSIONS

One of the major objectives of plant and animal breeding is to develop new hybrids with enhanced genetic potential. For this purpose information on combining abilities, variance component estimates alongwith their distributions, predictor of yielding capacities of crosses and robust designs are needed by the breeders. Triallel and tetra-allele crosses are considered for the investigatory study as they can provide more information on specific combining abilities as compared to diallel crosses and are having better individual as well as population buffering mechanisms due to wider genetic base.

An introduction to importance of various types of designs used in breeding along with their merits and demerits has been given in Chapter I. The importance of using different types of models and their applications has been also described. In this chapter, the motivation under which the objectives are undertaken along with the scope of the thesis has been explained.

In Chapter II of the thesis, a critical review of the research work related to the objectives has been given. Section wise review of research work related to triallel and tetra-allele cross designs, mating designs incorporating specific combining ability effects, prediction of combining ability effects, variance components estimates and robust design for breeding experiments has been elucidated.

The material and methods related to the research work are described in the Chapter III of the thesis. In this chapter, various fixed effects models for triallel and tetra-allele cross experiments have been described. The random effects model for tetra-allele cross experiments without specific combining ability effects have been described which is to be used for the pupose of Best Linear Unbiased Prediction of yielding capacity of crosses and for variance components estimation. The definition of Best Linear Unbiased Predictor and its properties has been discussed. The concept of robustness along with the connectedness and efficiency criteria has been discussed and the procedure of investigating the robustness of designs against missing observation has been described. The various definitions and designs used in this thesis have been also given in this chapter.

The first section of Chapter IV of this thesis dedicates to higher order mating designs including specific combining abililies. Under a fixed effects model including sca effects for triallel crosses, the estimates of general and specific combining abilities have been obtained. A class of designs for triallel cross experiments with crosses arranged in blocks have been obtained and various characterization properties including information matrices, eigenvalues, variance factors, efficiency factors and degree of fractionation have been derived. Another two classes are obtained based on lattice designs and Kronecker product of incidence matrices. The main restriction of first method is that they are only available for a particular set of parameters but the other two methods are available for wide range of parameters. All the three methods are having low degree of fractionation alongwith higher efficiencies and can be used together to provide designs for almost all set of parametric combinations. Under a restricted model including lower order specific combining ability effects for tetra-allele crosses, the orthogonal estimates of general and specific combining ability effects under a block design set up have been obtained. A general method of construction of optimal block designs has been described and a class of design based on Mutually Orthogonal Latin Squares have been obtained. These designs along with lower degree of fractionation also satisfy the conditions of universal optimality of equireplicated designs for estimating gca effects, including sca effects in the model. In the second section of Chapter IV, the Best Linear Unbiased Predictor for predicting the unobserved combining ability effects together with general mean effect in tetra-allele cross design have been obtained. A lower bound of mean square error of prediction has been also derived which can be further used to find optimal designs. The third section of chapter IV deals with variance components. The variance component estimates along with their large sample variances, using mixed linear model approach in tetra-allele crosses has been obtained using Henderson Method III. In these two sections, a random effect model is used. In last section of Chapter IV robustness of designs for triallel and tetra-allele cross experiments have been investigated using connectedness and efficiency criteria against a missing observation under unblocked and blocked setup. These robust designs have been tabulated with efficiency factors. The list of efficient designs with lower degree of fractionation for triallel and tetra-allele cross experiments under blocked and unblocked setup can be used as robust designs against a missing observation.

In Chapter V, a general discussion on various methodologies developed and designs obtained has been given. Finally, the thesis concludes with a brief summary, abstract and bibliography. SAS codes have been provided in the Annexure after the bibliography.

An attempt has been made to fill the larger gaps existing in literature. Considering the increasing popularity of higher order crosses among breeders, there is a need to obtain more classes of designs to meet their requirements. A class of optimal design based on MOLS have been obtained. More classes of optimal designs are needed to be characterized. BLUP for predicting the yielding capacity of crosses, variance component estimates along with their large sample variances, has been obtained under unblocked situation. Hence, there is a need to work for blocked situation. Robustness of designs has been investigated against a missing observation. There is need to investigate for other disturbances also.

# ABSTRACT

The major objective of a breeding programme is to release new hybrids with enhanced genetic potential, which is based on information on general and specific combining abilities, information related to variance components and predictor of yielding capacities of crosses. Further, designs used for attaining these goals should be robust against disturbances like missing observation. Higher order crosses like triallel and tetra-allele crosses are considered as they, being genetically more viable and consistent performers, can provide more information on combining abilities. Under a fixed effects model including specific combining abilities, the estimates of combining abilities have been obtained for both triallel and tetra-allele crosses. A method of construction of partial triallel crosses arranged in blocks has been obtained based on two-associate triangular association scheme alongwith the information matrices, eigenvalues, variance factors, efficiency factor and degree of fractionation. Another two methods of constructing partial triallel cross designs have also been obtained using various types of Lattice designs and Kronecker product of incidence matrices respectively. All these methods are efficient alongwith lower degree of fractionation. A class of orthogonal tetra-allele cross designs for estimating contrasts pertaining to general combining ability effects has been obtained under a reduced model including lower order specific combining ability effects using mutually orthogonal Latin squares. The obtained class satisfies the condition of optimality for eqiureplicated designs for tetra-allele crosses and are having low degree of fractionation. Also, under a random effect model excluding specific combining ability effects for tetra-allele crosses, the Best Linear Unbiased Predictor (BLUP) for yielding capacity of cross has been obtained. A lower bound to mean square prediction error for characterizing optimal class of designs has been obtained. The lower bound so obtained is important in finding optimal designs. The unbiased estimates of variance components alongwith sampling distribution have been obtained following Henderson Method III. These parameters can be further used for obtaining the estimates of genetic parameters. The robustness of designs against missing observation using connectedness and efficiency criteria has been studied and a list of efficient robust designs for triallel and tetra-allele crosses has been tabulated. A list of robust and efficient designs with lower degree of fractionation is very much helpful for the breeders in situations of missing observation. Programs have been written in SAS [PROC IML] software for computing efficiency factor of the designs involving triallel crosses for estimating gca effects to investigate the robustness of designs against missing observation by calculating the canonical efficiency.

# सार

प्रजनन का प्रमुख उद्देश्य बढ़ी हुई आनुवंशिक क्षमता वाले संकर जारी करना होता है, जो कि सामान्य एवं विशिष्ट संयोजन क्षमताओं की जानकारी, प्रसरण घटकों और क्रॉस की उपज क्षमता के पूर्वसूचक से संबन्धित जानकारी पर आधारित होता है। इसके अतिरिक्त, इन लक्ष्यों को प्राप्त करने हेतु उपयोग की जाने वाली अभिकल्पनाएं लुप्त उक्ति जैसी अव्यवस्था के समक्ष मजबूत होनी चाहिए। त्रि-पथ एवं चार-पथ क्रॉस जैसे उच्च-स्तरीय क्रॉस, द्वि-पथ क्रॉस की तुलना मे आनुवांशिक रूप से अधिक जीवक्षम और स्थिर प्रदर्शन करने वाले होते हैं, संयोजक क्षमता की अधिक जानकारी प्रदान करते हैं और व्यापक आनुवांशिक आधार होने के कारण व्यक्तिगत एवं जनसंख्या के रूप में बेहतर बफरिंग तंत्र को प्रदर्शित करते हैं। विशिष्ट संयोजन क्षमताओं को सम्मिलित कर एक निश्चित प्रभाव मॉडल के अंतर्गत त्रि-पथ एवं चार-पथ क्रॉसों के लिए संयोजन क्षमताओं के अनुमान ज्ञात किए गए हैं। द्वि-सहभागी त्रिकोणीय योजना पर आधारित खंडों में व्यवस्थित आंशिक त्रि-पथ क्रॉसों की निर्माण विधि के साथ में सूचना आव्यूह, आइगेन मूल्य, प्रसरण कारक, दक्षता कारक एवं विभाजन की मात्रा प्राप्त की गई है। लैटिस अभिकल्पनाओं एवं व्यापकता आव्यूहों के गुणन का प्रयोग कर आंशिक त्रि-पथ क्रॉसों की अन्य दो निर्माण विधियाँ भी प्राप्त की गई हैं। निचले क्रम की विशिष्ट संयोजन क्षमताओं को सम्मिलित कर एक कमतर मॉडल के अंतर्गत चार-पथ क्रॉसों के लिए सामान्य संयोजन क्षमताओं से संबन्धित कंट्रास्टों के अनुमान हेतु ओर्थोगोनल चार-पथ क्रॉस अभिकल्पनाओं की एक इष्टतम श्रेणी प्राप्त की गई है। विशिष्ट संयोजन क्षमताओं को छोड़ कर एक अनियमित-प्रभाव मॉडल के अंतर्गत चार-पथ क्रॉसों के लिए उपज की छमता हेतु BLUP प्राप्त किया गया है। अभिकल्पनाओं के इष्टतम वर्गों के विशेषीकरण हेतु MSE के लिए एक न्यून सीमा भी प्राप्त की गई है। अनियमित-प्रभाव मॉडल के अंतर्गत हेंडरसन विधि III का प्रयोग कर सैंपलिंग वितरण सहित प्रसरण घटकों के निष्पक्ष अनुमान प्राप्त किए गए हैं। सन्युक्तता एवं दक्षता मापदंड का प्रयोग कर लुप्त-उक्ति की स्थिति में अभिकल्पनाओं की मजबूती ज्ञात की गई है एवं त्रि-पथ एवं चार-पथ क्रॉसों के लिए ऐसी मजबूत अभिकल्पनाओं को तलिकाबद्ध भी किया गया है। SAS [PROC IML] सॉफ्टवेयर में सामान्य संयोजन क्षमताओं से संबन्धित कंट्रास्टों के अनुमान हेतु दक्षता कारक एवं लुप्त उक्ति की स्थिति में अभिकल्पनाओं की मजबूती ज्ञात करने के लिए कनोनिकल दक्षता कारक के परिकलन हेतु प्रोग्राम लिखे गए हैं।

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# Annexure-I

**SAS code for computing variance factors of the design involving triallel crosses for estimating gca effects for half parents as well as full parents under unblocked set-up**

```
Data Triallel;
input line1 line2 line3;
cards;
1	2	5
1	3	6
1	4	7
2	3	8
2	4	9
3	4	10
1	5	2
1	6	3
1	7	4
5	6	8
;
run;
prociml;
usetriallel;
read all into cross;
print xx;
k=max(cross[ ,1]);
kk=max(cross[ ,2]);
l=max(k,kk);
ll=comb(l,2);
print l;
printll;
m=j(nrow(cross),1,1);
print cross;
x=j(nrow(cross),max(cross),0);
k=1;
doi=1tonrow(cross);
do j=1toncol(cross)-1;
if cross[i,j]>0then
x[k,cross[i,j]]=1;
end;
k=k+1;
end;
x=x/2;
print x;
z=j(nrow(cross),max(cross),0);
k=1;
doi=1tonrow(cross);
if cross[i,3]>0then
```

```
z[k,cross[i,3]]=1;
k=k+1;
end;
z=z;
print z;
x2=m;
c11=(x`*x)-(x`*x2)*ginv(x2`*x2)*(x2`*x);
c12=(x`*z)-(x`*x2)*ginv(x2`*x2)*(x2`*z);
c22=(z`*z)-(z`*x2)*ginv(x2`*x2)*(x2`*z);
c_mat=(c11||c12)//(c12`||c22);
*print c_mat;
c_halfparent=c11-c12*ginv(c22)*c12`;
c_fullparent=c22-c12`*ginv(c11)*c12;
printc_halfparent;
printc_fullparent;
l=nrow(c_halfparent);
ll=comb(nrow(c_halfparent),2);
contrast=j(ll,l,0);
k=1;
doi=1to l-1;
do j=ito l-1;
contrast[k,i]=1;
contrast[k,j+1]=-1;
k=k+1;
end;
end;
*print contrast;
varcov_halfparent=contrast*ginv(c_halfparent)*contrast`;
varcov_fullparent=contrast*ginv(c_fullparent)*contrast`;
printvar_halfparent;
printvar_fullparent;
var_halfparent=j(ll,1,0);
doi= 1toll;
var_halfparent[i,1]=varcov_halfparent[i,i];
end;
ave_var_halfparent=var_halfparent[+, ]/nrow(var_halfparent);
printvar_halfparent;
printave_var_halfparent;
var_fullparent=j(ll,1,0);
doi= 1toll;
var_fullparent[i,1]=varcov_fullparent[i,i];
end;
printvar_fullparent;
ave_var_fullparent=var_fullparent[+, ]/nrow(var_fullparent);
printave_var_fullparent;
quit;
```

# Annexure-II

**SAS code for computing canonical efficiency factor of the design involving triallel crosses for estimating gca effects for half parents as well as full parents under blocked set-up.**

```
%let r1=6;/*replication of half parents*/
%let r2=3;/*replication of full parents*/
dataTriallel;
input Block line1 line2 line3;
cards;
1	1	2	3
1	4	5	6
1	7	8	9
1	10	11	12
1	13	14	15
1	16	17	18
1	19	20	21
1	22	23	24
1	25	26	27
1	1	3	2
;
run;
prociml;
usetriallel;
read all into xx;
/*print xx;*/
cross=xx[ ,2]||xx[ ,3]||xx[ ,4];
m=j(nrow(cross),1,1);
/*print cross;*/
x=j(nrow(cross),max(cross),0);
k=1;
doi=1tonrow(cross);
do j=1toncol(cross)-1;
if cross[i,j]>0then
x[k,cross[i,j]]=1;
end;
k=k+1;
end;
z=j(nrow(cross),max(cross),0);
k=1;
doi=1tonrow(cross);
if cross[i,3]>0then
z[k,cross[i,3]]=1;
k=k+1;
end;
block=j(nrow(xx[ ,1]),max(xx[ ,1]),0);
k=1;
doi=1tonrow(xx[ ,1]);
if xx[i,1]>0then
block[k,xx[i,1]]=1;
k=k+1;
end;
x2=m||block;
c11=(x`*x)-(x`*x2)*ginv(x2`*x2)*(x2`*x);
```

```
c12=(x`*z)-(x`*x2)*ginv(x2`*x2)*(x2`*z);
c22=(z`*z)-(z`*x2)*ginv(x2`*x2)*(x2`*z);
c_mat=(c11||c12)//(c12`||c22);
c_halfparent=c11-c12*ginv(c22)*c12`;
c_fullparent=c22-c12`*ginv(c11)*c12;
l=nrow(c_halfparent);
ll=comb(nrow(c_halfparent),2);
contrast=j(ll,l,0);
k=1;
doi=1to l-1;
do j=ito l-1;
contrast[k,i]=1;
contrast[k,j+1]=-1;
k=k+1;
end;
end;
ginv_hp=ginv(c_halfparent);
ginv_fp=ginv(c_fullparent);
varcov_halfparent=contrast*ginv(c_halfparent)*contrast`;
varcov_fullparent=contrast*ginv(c_fullparent)*contrast`;
var_halfparent=j(ll,1,0);
doi= 1toll;
var_halfparent[i,1]=varcov_halfparent[i,i];
end;
ave_var_halfparent=var_halfparent[+, ]/nrow(var_halfparent);
var_fullparent=j(ll,1,0);
doi= 1toll;
var_fullparent[i,1]=varcov_fullparent[i,i];
end;
ave_var_fullparent=var_fullparent[+, ]/nrow(var_fullparent);
eigH=eigval(c_halfparent);
printeigH;
eigF=eigval(c_fullparent);
printeigF;
eigH1=eigH[loc(eigH>0.0000001),];/*positive eigen values*/
eigF1=eigF[loc(eigF>0.0000001),];/*positive eigen values*/
eigH2=eigH1/&r1;
eigF2=eigF1/&r2;
eigH3=1/eigH2;
eigF3=1/eigF2;
CanEffFacH=nrow(eigH3)/sum(eigH3);
CanEffFacF=nrow(eigF3)/sum(eigF3);
printCanEffFacH;
printCanEffFacF;

quit;
```

# Annexure-III

**SAS code for computing efficiency of the disturbed design involving triallel crosses for estimating gca effects for half parents as well as full parents under unblocked set-up to investigate the robustness of designs against missing observation using the connectedness and efficiency criteria.**

```
Data Triallel;
input line1 line2 line3;
cards;

1     2     8
2     3     8
3     4     8
4     5     8
5     6     8
6     7     8
7     1     8
1     3     8
3     5     8
5     7     8
;

run;

proc iml;
use triallel;
read all into cross;
/*print xx;*/
k=max(cross[ ,1]);
kk=max(cross[ ,2]);
l=max(k,kk);
ll=comb(l,2);

*print l;
*print ll;

m=j(nrow(cross),1,1);

/*print cross;*/

x=j(nrow(cross),max(cross),0);
k=1;
doi=1tonrow(cross);
do j=1toncol(cross)-1;
if cross[i,j]>0then
x[k,cross[i,j]]=1;
end;
k=k+1;
end;

*print x;
```

```
z=j(nrow(cross),max(cross),0);
k=1;
doi=1tonrow(cross);
if cross[i,3]>0then
z[k,cross[i,3]]=1;
k=k+1;
end;

*print z;

x2=m;
c11=(x`*x)-(x`*x2)*ginv(x2`*x2)*(x2`*x);
c12=(x`*z)-(x`*x2)*ginv(x2`*x2)*(x2`*z);
c22=(z`*z)-(z`*x2)*ginv(x2`*x2)*(x2`*z);
c_mat=(c11||c12)//(c12`||c22);

*print c_mat;

c_halfparent=c11-c12*ginv(c22)*c12`;
c_fullparent=c22-c12`*ginv(c11)*c12;

*print c_halfparent;
*print c_fullparent;

l=nrow(c_halfparent);
ll=comb(nrow(c_halfparent),2);
contrast=j(ll,l,0);
k=1;
doi=1to l-1;
do j=ito l-1;
contrast[k,i]=1;
contrast[k,j+1]=-1;
k=k+1;
end;
end;

*print contrast;

varcov_halfparent=contrast*ginv(c_halfparent)*contrast`;
varcov_fullparent=contrast*ginv(c_fullparent)*contrast`;

*print var_halfparent;
*print var_fullparent;

var_halfparent=j(ll,1,0);
doi= 1toll;
var_halfparent[i,1]=varcov_halfparent[i,i];
end;
ave_var_halfparent=var_halfparent[+, ]/nrow(var_halfparent);

*print var_halfparent;
*print ave_var_halfparent;

var_fullparent=j(ll,1,0);
doi= 1toll;
```

```
var_fullparent[i,1]=varcov_fullparent[i,i];
end;

*print var_fullparent;
ave_var_fullparent=var_fullparent[+, ]/nrow(var_fullparent);
*print ave_var_fullparent;

eig=eigval(c_halfparent);
eig1=eig[loc(eig>0.0000001),];/*positive eigen values*/
hm_half_1=harmean(eig1);
print hm_half_1;
eig=eigval(c_fullparent);
eig1=eig[loc(eig>0.0000001),];/*positive eigen values*/
hm_full_1=harmean(eig1);
print hm_full_1;

/**********CHECKING Robustness (without changing the number of crosses)*****/
/* Enter the number of crosses to be deleted*/

editTriallel;
delete all
where((line1=1&&line2=2&&line3=8)|(line1=2&&line2=8&&line3=1)|(line1=8&&line2=1&&line3
=2));   /*change the lines to see the robustness*/
closeTriallel;
usetriallel;
read all intocross_n;
k=max(cross_n[ ,1]);
kk=max(cross_n[ ,2]);
l=max(k,kk);
ll=comb(l,2);

*print l;
*print ll;

m_n=j(nrow(cross_n),1,1);
x_n=j(nrow(cross_n),max(cross_n),0);
k=1;
doi=1tonrow(cross_n);
do j=1toncol(cross_n)-1;
ifcross_n[i,j]>0then
x_n[k,cross_n[i,j]]=1;
end;
k=k+1;
end;

*print x_n;

z_n=j(nrow(cross_n),max(cross_n),0);
k=1;
doi=1tonrow(cross_n);
ifcross_n[i,3]>0then
z_n[k,cross_n[i,3]]=1;
k=k+1;
end;
```

```
*print z_n;
x2_n=m_n;
c11_n=(x_n`*x_n)-(x_n`*x2_n)*ginv(x2_n`*x2_n)*(x2_n`*x_n);
c12_n=(x_n`*z_n)-(x_n`*x2_n)*ginv(x2_n`*x2_n)*(x2_n`*z_n);
c22_n=(z_n`*z_n)-(z_n`*x2_n)*ginv(x2_n`*x2_n)*(x2_n`*z_n);
c_mat_n=(c11_n||c12_n)//(c12_n`||c22_n);

*print c_mat_n;

c_halfparent_n=c11_n-c12_n*ginv(c22_n)*c12_n`;
c_fullparent_n=c22_n-c12_n`*ginv(c11_n)*c12_n;
eig_n=eigval(c_halfparent_n);
eig1_n=eig_n[loc(eig_n>0.0000001),];/*positive eigen values*/
hm_half_1_n=harmean(eig1_n);
print hm_half_1_n;
eig_n=eigval(c_fullparent_n);
eig1_n=eig_n[loc(eig_n>0.0000001),];/*positive eigen values*/
hm_full_1_n=harmean(eig1_n);
print hm_full_1_n;
Robustness_half=hm_half_1_n/hm_half_1;
Robustness_full=hm_full_1_n/hm_full_1;
printRobustness_halfRobustness_full;
quit;
```

# Annexure-IV

**SAS code for computing canonical efficiency of the disturbed design involving triallel crosses for estimating gca effects for half parents as well as full parents under blocked set-up to investigate the robustness of designs against missing observation using the connectedness and efficiency criteria.**

```
Data Triallel;
input Block line1 line2 line3;
cards;
1    1    2    3
1    2    3    4
1    3    4    5
1    4    5    6
1    5    6    7
1    6    7    8
1    7    8    9
1    8    9    10
1    9    10   11
1    10   11   12
;

run;

proc iml;
usetriallel;
read all into xx;

/*print xx;*/

cross=xx[ ,2]||xx[ ,3]||xx[ ,4];
m=j(nrow(cross),1,1);

/*print cross;*/

x=j(nrow(cross),max(cross),0);
k=1;
doi=1tonrow(cross);
do j=1toncol(cross)-1;
if cross[i,j]>0then
x[k,cross[i,j]]=1;
end;
k=k+1;
end;

*print x;
z=j(nrow(cross),max(cross),0);
k=1;
doi=1tonrow(cross);
if cross[i,3]>0then
z[k,cross[i,3]]=1;
k=k+1;
end;
```

```
*print z;

block=j(nrow(xx[ ,1]),max(xx[ ,1]),0);
k=1;
doi=1tonrow(xx[ ,1]);
if xx[i,1]>0then
block[k,xx[i,1]]=1;
k=k+1;
end;

*print block;

x2=m||block;
c11=(x`*x)-(x`*x2)*ginv(x2`*x2)*(x2`*x);
c12=(x`*z)-(x`*x2)*ginv(x2`*x2)*(x2`*z);
c22=(z`*z)-(z`*x2)*ginv(x2`*x2)*(x2`*z);
c_mat=(c11||c12)//(c12`||c22);

*print c_mat;

c_halfparent=c11-c12*ginv(c22)*c12`;
c_fullparent=c22-c12`*ginv(c11)*c12;

*print c_halfparent;
*print c_fullparent;

eig=eigval(c_halfparent);
printeig;
eig1=eig[loc(eig>0.0000001),];/*positive eigen values*/
hm_half_1=harmean(eig1);
print hm_half_1;

eig=eigval(c_fullparent);
eig1=eig[loc(eig>0.0000001),];/*positive eigen values*/
hm_full_1=harmean(eig1);

print hm_full_1;


/**********CHECKING Robustness (without changing the number of crosses)*****/

/* Enter the number of crosses to be deleted*/
Edit Triallel;
delete all
where((line1=1&&line2=2&&line3=3)|(line1=1&&line2=3&&line3=2)|(line1=2&&line2=3&&line3
=1));   /*change the lines to see the robustness*/
closeTriallel;
usetriallel;
read all intoxx_n;

/*print xx_n;*/

cross_n=xx_n[ ,2]||xx_n[ ,3]||xx_n[ ,4];
m_n=j(nrow(cross_n),1,1);
```

```
*print cross_n;

x_n=j(nrow(cross_n),max(cross_n),0);
k=1;
doi=1tonrow(cross_n);
do j=1toncol(cross_n)-1;
ifcross_n[i,j]>0then
x_n[k,cross_n[i,j]]=1;
end;
k=k+1;
end;

*print x_n;

z_n=j(nrow(cross_n),max(cross_n),0);
k=1;
doi=1tonrow(cross_n);
ifcross_n[i,3]>0then
z_n[k,cross_n[i,3]]=1;
k=k+1;
end;

*print z_n;

block_n=j(nrow(xx_n[ ,1]),max(xx_n[ ,1]),0);
k=1;
doi=1tonrow(xx_n[ ,1]);
ifxx_n[i,1]>0then
block_n[k,xx_n[i,1]]=1;
k=k+1;
end;

*print block_n;

x2_n=m_n||block_n;
c11_n=(x_n`*x_n)-(x_n`*x2_n)*ginv(x2_n`*x2_n)*(x2_n`*x_n);
c12_n=(x_n`*z_n)-(x_n`*x2_n)*ginv(x2_n`*x2_n)*(x2_n`*z_n);
c22_n=(z_n`*z_n)-(z_n`*x2_n)*ginv(x2_n`*x2_n)*(x2_n`*z_n);
c_mat_n=(c11_n||c12_n)//(c12_n`||c22_n);
*print c_mat_n;
c_halfparent_n=c11_n-c12_n*ginv(c22_n)*c12_n`;
c_fullparent_n=c22_n-c12_n`*ginv(c11_n)*c12_n;

eig_n=eigval(c_halfparent_n);
eig1_n=eig_n[loc(eig_n>0.0000001),];/*positive eigen values*/
hm_half_1_n=harmean(eig1_n);

print hm_half_1_n;

eig_n=eigval(c_fullparent_n);
eig1_n=eig_n[loc(eig_n>0.0000001),];/*positive eigen values*/
hm_full_1_n=harmean(eig1_n);

print hm_full_1_n;
```

```
Robustness_half=hm_half_1_n/hm_half_1;
Robustness_full=hm_full_1_n/hm_full_1;

Print Robustness_halfRobustness_full;
rankh1 = round (trace(ginv(c_halfparent)*c_halfparent));
print rankh1;
rankf1 = round (trace(ginv(c_fullparent)*c_fullparent));
print rankf1;
rankh2 = round (trace(ginv(c_halfparent_n)*c_halfparent_n));
print rankh2;
rankf2 = round (trace(ginv(c_fullparent_n)*c_fullparent_n));
print rankf2;

quit;
```

# Annexure-V

**SAS code for computing canonical efficiency of the disturbed design involving tetra-allele crosses for estimating gca effects under blocked set-up to investigate the robustness of designs against missing observation using the connectedness and efficiency criteria.**

```
Data Tetrallele;
input block line1 line2 line3 line4;
cards;
1	1	2	3	4
1	1	3	2	4
1	1	4	2	3
1	1	2	3	5
1	1	3	2	5
1	1	5	2	3
1	1	2	4	5
1	1	4	2	5
1	1	5	2	4
1	1	3	4	5
;
run;
prociml;
useTetrallele;
read all into xx;
cross=xx[ ,2]||xx[ ,3]||xx[ ,4]||xx[ ,5];
m=j(nrow(cross),1,1);
x1=j(nrow(cross),max(cross),0);
k=1;
doi=1tonrow(cross);
do j=1toncol(cross);
if cross[i,j]>0then
x1[k,cross[i,j]]=1;
end;
k=k+1;
end;
block=j(nrow(xx[ ,1]),max(xx[ ,1]),0);
k=1;
doi=1tonrow(xx[ ,1]);
if xx[i,1]>0then
block[k,xx[i,1]]=1;
k=k+1;
end;
*print block;
x2=m||block;
*print x1 x2;

c_mat=(x1`*x1)-(x1`*x2)*ginv(x2`*x2)*(x2`*x1);
printc_mat;
eig=eigval(c_mat);
eig1=eig[loc(eig>0.0000001),];/*positive eigen values*/
hm=harmean(eig1);
printhm;
```

```
/**********CHECKING Robustness (without changing the number of crosses)*****/

/* Enter the number of crosses to be deleted*/
editTetrallele;
delete all where(line1=1&&line2=2&&line3=3&&line4=4);   /*change the lines to see the
robustness*/
closeTetrallele;
useTetrallele;
read all intoxx_n;
cross_n=xx_n[ ,2]||xx_n[ ,3]||xx_n[ ,4]||xx_n[ ,5];
*print cross_n;
m_n=j(nrow(cross_n),1,1);
x1_n=j(nrow(cross_n),max(cross_n),0);
k=1;
doi=1tonrow(cross_n);
do j=1toncol(cross_n);
ifcross_n[i,j]>0then
x1_n[k,cross_n[i,j]]=1;
end;
k=k+1;
end;
*print x1;
block_n=j(nrow(xx_n[ ,1]),max(xx_n[ ,1]),0);
k=1;
doi=1tonrow(xx_n[ ,1]);
ifxx_n[i,1]>0then
block_n[k,xx_n[i,1]]=1;
k=k+1;
end;
*print block_n;
x2_n=m_n||block_n;
*print x1_n x2_n;

c_mat_n=(x1_n`*x1_n)-(x1_n`*x2_n)*ginv(x2_n`*x2_n)*(x2_n`*x1_n);
printc_mat_n;
eig=eigval(c_mat_n);
eig1=eig[loc(eig>0.0000001),];/*positive eigen values*/
hm_n=harmean(eig1);
printhm_n;
Robustness=hm_n/hm;
print Robustness;

quit;
```

# Annexure-VI

**SAS code for computing information matrices related to gca effects under blocked set-up for tetra-allele crosses.**

```
Data Tetrallele;
input block line1 line2 line3 line4;
cards;
1	1	2	3	4
1	5	6	7	8
2	1	3	5	7
2	2	4	6	8
3	1	2	3	4
3	5	6	7	8
4	1	3	5	7
4	2	4	6	8

;
run;
prociml;
useTetrallele;
read all into xx;
cross=xx[ ,2]||xx[ ,3]||xx[ ,4]||xx[ ,5];
m=j(nrow(cross),1,1);
x1=j(nrow(cross),max(cross),0);
k=1;
doi=1tonrow(cross);
do j=1toncol(cross);
if cross[i,j]>0then
x1[k,cross[i,j]]=1;
end;
k=k+1;
end;
block=j(nrow(xx[ ,1]),max(xx[ ,1]),0);
k=1;
doi=1tonrow(xx[ ,1]);
if xx[i,1]>0then
block[k,xx[i,1]]=1;
k=k+1;
end;
*print block;
x2=m||block;

*print x1 x2;

c_mat=(x1`*x1)-(x1`*x2)*ginv(x2`*x2)*(x2`*x1);

printc_mat;

quit;
```