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Robustness of Optimal Block Designs for Triallel Cross Experiments Against Exchange of a Cross

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SUMMARY

Robustness of optimal block designs for triallel cross experiments of Das and Gupta [4] is investigated using connectedness and efficiency criteria against exchange of a cross. The exchanged cross may have no line in common, one line in common or two lines in common with the substituted cross. Each of these aspects has been dealt with separately and it is found that the designs involving more than 9 lines are robust.

Key words : Triallel crosses, Exchange of cross, Robustness, Nested block designs.

1. Introduction

Triallel crosses form an important class of mating designs, which are used for studying the genetic properties of a set of inbred lines in plant breeding experiments. Let there be p inbred lines which give rise to $n_c = 3\,{}^pC_3$ possible crosses of the type $(i_1 \times i_2) \times i_3$, $i_1 \neq i_2 \neq i_3$, $i_1, i_2, i_3 = 1, 2, \ldots, p$. Rawlings and Cockerham [17] were the first to introduce mating designs for triallel crosses.

Triallel cross experiments are generally conducted using a completely randomized design (CRD), or a randomized complete block (RCB) design as environmental design involving n_c crosses. The total number of triallel crosses n_c becomes manifold with increasing value of p. As a result more resources are required for conducting experiments. Furthermore, accommodation of large number of crosses in a RCB design may result into large intra-block variances. In such situations, a sample of the complete triallel crosses, i.e. partial triallel crosses (PTC) introduced by Hinkelmann [9] can be used. Other authors who contributed in this field are Arora and Aggarwal [1, 2], Ceranka *et al.* [3] and Ponnuswamy and Srinivasan [16], etc. More details on triallel cross experiments can be found in Hinkelmann [10] and Narain [12].

Recently, following Gupta and Kageyama [8] and Dey and Midha [5], Das and Gupta [4] have constructed block designs for triallel crosses that start with p lines rather than n_c crosses in the experiment. This approach yields designs which are universally optimal in **D** (p, b, k), the class of connected block designs for triallel crosses in p lines with b blocks each of size k such that the total number of experimental units are $n < n_c$. Das and Gupta [4] used nested balanced block designs with sub-block size 3 in the construction of optimal block designs for triallel crosses. These optimal designs perform well under ideal conditions. However, disturbances may occur due to some reasons at different phases from planning to execution of an experiment. Exchange of a cross in one such aberration or discrepancy that is said to have occurred, if one of the crosses gets substituted by any of the remaining crosses during experimentation due to the following reasons

(i) In environmental design, a different cross has been applied in a block instead of what should have been, due to human mistake or due to erroneous tagging and labeling.

(ii) In environmental design, suppose one cross did not germinate due to incompatibility of the cross or for any other reason and was substituted by an already existing surplus cross.

(iii) If resource is not available for a particular cross, then exchange of that cross with any other surplus cross is necessitated.

Pearce [13] first reported such type of disturbances in general block design set up. Gomez and Gomez [7] have termed these disturbances as mechanical errors whereas Pearce [14] called these as errors in the application of the treatments. The presence of such discrepancy may affect the properties of the original design. Therefore, there is a need to use the designs that are insensitive to such types of disturbances. In this article, an attempt, therefore, has been made to investigate the robustness of optimal block designs for triallel crosses against exchange of a cross. In Section 2, preliminaries of block designs for triallel crosses have been discussed. The exchanged cross may be distinct in terms of the lines with the substituted cross or may have one line or two lines in common. Each of these has been dealt with separately in Section 3, 4 and 5 respectively. In each situation, relationship has been established between the information matrix of the resulting design and that of the original design. The eigenvalues of the information matrix of the resulting design have been obtained when the original design is variance balanced with respect to line effects. Robustness have been investigated using the connectedness criterion (see e.g., Ghosh [6]) and the efficiency criterion (see e.g., John [11]).

2. *Experimental Set-up*

Let d be a block design with b blocks each of size k for a triallel cross experiment in p inbred lines. Further, let r_t and s_i denote the number of replications of the t^{th} cross and the number of replications of i^{th} line in different crosses, respectively in d [t = 1, 2, ..., p(p − 1)(p − 2)/3; i = 1, 2, ..., p]. Evidently, $\Sigma r_t = bk$, $\Sigma s_i = 3bk$ and n = bk, the total number of observations. In a triallel cross experiment, the genotypic effect of the hybrid consists of single line effects, two line specific effects and three line specific effects. However, if we assume that for a partial triallel cross experiment (in which every line appears as half parent an equal number of times, say r_H, and every line appears as full parent an equal number of times, say r_F, and each of the crosses $(i_1 \times i_2) \times i_3$ appears at most once) the two line specific effects and three line specific effects are not of importance, still the line effects are of two types *viz.* effects as half parent and effect as full parent, i.e., the ordering of lines in a triallel cross is important. Some plant breeders argue that these ordering effects can also be averaged over line effects. Hence, in the present investigation, similar to Das and Gupta [4] we consider the situations where the ordering of lines in a triallel cross is not of importance and postulate the following linear additive fixed effect model for the observations

$$\mathbf{Y} = \mu \mathbf{1}_n + \Delta_1' \mathbf{g} + \Delta_2' \beta + \in \tag{2.1}$$

where $\mathbf{Y}$ is the $(n \times 1)$ vector of observed responses, μ is general mean, $\mathbf{g}$ and β are vectors of p line effects and b block effects, respectively, $\mathbf{1}_n$ is a n × 1 vector of unities. Δ_1' is the $(n \times p)$ observations vs lines design matrix, i.e. $(s, t)^{th}$ element of Δ_1' is 1 if s^{th} observation pertains to the t^{th} line and is zero otherwise. Δ_2' is the $(n \times b)$ observations vs blocks design matrix i.e. $(s, t)^{th}$ element of Δ_2' is 1 if s^{th} observation pertains to the t^{th} block and is zero otherwise and $\in$ is the random error which follows $N_n(\mathbf{0}, \sigma^2 \mathbf{I}_n)$.

The information matrix of the reduced normal equations for estimating linear functions of line effects, using d, under model (2.1) is

$$\mathbf{C}_d = \mathbf{G}_d - \mathbf{N}_d \mathbf{K}_d^{-1} \mathbf{N}_d' \tag{2.2}$$

where $\mathbf{G}_d = \Delta_1 \Delta_1' = (g_{dii'})$, $g_{dii} = s_{di}$ and for $i \neq i'$, $g_{dii'}$ is the number of crosses in d in which the lines i and i′ appear together. $\mathbf{N}_d = \Delta_1 \Delta_2' = ((n_{dij}))$, n_{dij} is the number of times the line i occurs in block j of d and $\mathbf{K}_d = \Delta_2 \Delta_2'$ is the diagonal matrix of block sizes. The design d will be connected if and only if rank $(\mathbf{C}_d)$ = p − 1. Henceforth, we shall consider only the connected designs.

We state below two theorems {Das and Gupta [4]} for better understanding of the optimality aspect of designs considered in **D** (p, b, k).

Theorem 1: For $d \in \mathbf{D}$ (p, b, k), tr $(\mathbf{C}_d) \le k^{-1}b$ $\{3k(k-1-2x) + p\,x\,(x+1)\}$, where $x = [3k/p]$, [.] being the integer valued function and tr (**A**) denotes the trace of the matrix **A**.

Corollary 1: For $d \in \mathbf{D}$ (p, b, k), if $3k/p \ge 1$ (i.e., $x \ge 1$), then

$$\text{tr}\,(\mathbf{C}_d) \le k^{-1}b\,\{3k(k-1-2x) + p\,x\,(x+1)\} \le 3bk\,(p-3)/p \le 3b(k-1)$$

And if $3k/p \le 1$, i.e. $x = 0$ then tr $(\mathbf{C}_d) \le 3b\,(k-1)$

Again, if $n_{dij} = 0$ or 1 for all i, j, then tr $(\mathbf{C}_d) = 3b(k-1)$

Theorem 2: Let $d \in \mathbf{D}$ (p, b, k), be a block design for triallel crosses satisfying

(i) tr $(\mathbf{C}_{d*}) = k^{-1}b\,\{3k(k-1-2x) + p\,x\,(x+1)\}$, and

(ii) $\mathbf{C}_{d*}$ is completely symmetric

Then d* is universally optimal in the relevant class of competing designs in **D** (p, b, k) and in particular is A-optimal.

Consider a nested balanced block design d_n with parameters $v = p$, b_1, b_2, k_1, $k_2 = 3$. If we now identify the treatments of d_n as lines of a triallel cross experiment and perform crosses among lines appearing in the same sub-block of d_n, we get a block design d* for a triallel experiment involving p lines with $n = b_2$ crosses arranged in $b = b_1$ blocks, each of size $k = k_1/3$. The information matrix of d* is given by

$$\mathbf{C}_{d*} = (p-1)^{-1}\,k^{-1}b\,\{3k(k-1-2x) + p\,x\,(x+1)\}\,[\mathbf{I}_p - \mathbf{J}_p^{-1}] \qquad (2.3)$$

Thus, from (2.3) and Theorem 2, it is evident that $d^* \in \mathbf{D}$ (p, b, k), constructed using a nested balanced block design with parameters p, $b_1 = b$, $b_2 = bk$, $k_1 = 3k$, $k_2 = 3$ is universally optimal in **D** (p, b, k).

Now, we discuss the robustness of optimal triallel crosses in the sequel.

3. Robustness Against Exchange of Distinct Crosses

Without loss of generality, assume that the cross involving lines 1, 2 and 3 has been exchanged with the cross involving lines 4, 5 and 6 in block 1. Let d_E denote the resulting design. The incidence matrix of d in the partitioned form can be written as

$$\mathbf{N}_d = \begin{bmatrix} \mathbf{n}_1 & \mathbf{N}_p \\ \mathbf{u}_1 & \mathbf{N}_u \end{bmatrix} \qquad (3.1)$$

where

$\mathbf{n}_1$ is the 6 × 1 vector corresponding to affected lines vs affected block

$\mathbf{u}_1$ is the (p – 6) × 1 vector corresponding to unaffected lines vs affected block

$\mathbf{N}_p$ is the 6 × (b – 1) incidence matrix corresponding to affected lines vs unaffected blocks

$\mathbf{N}_u$ is the (p – 6) × (b – 1) incidence matrix corresponding to unaffected lines vs unaffected block

After interchange of a cross, the resulting incidence matrix is

$$\mathbf{N}_E = \begin{bmatrix} \mathbf{n}_{1E} & \mathbf{N}_p \\ \mathbf{u}_1 & \mathbf{N}_u \end{bmatrix} \tag{3.2}$$

with $\mathbf{n}_{1E} = \mathbf{n}_1 + \mathbf{e}$ and $\mathbf{e}' = [-1 \;\; -1 \;\; -1 \;\; 1 \;\; 1 \;\; 1]$

The $\mathbf{G}_d$ matrix defined in (2.2) in the partitioned form can be written as

$$\mathbf{G}_d = \begin{bmatrix} \mathbf{G}_{d1} & \mathbf{G}_{d2} \\ \mathbf{G}'_{d2} & \mathbf{G}_{d3} \end{bmatrix} \tag{3.3}$$

where

$\mathbf{G}_{d1}$ is the 6 × 6 matrix corresponding to affected lines and affected pairs of lines

$\mathbf{G}_{d2}$ is the 6 × (p – 6) matrix corresponding to pair of lines, having one line affected

$\mathbf{G}_{d3}$ is the (p – 6) × (p – 6) matrix corresponding to unaffected lines and unaffected pair of lines

After exchange, the resulting matrix $\mathbf{G}_E$ will be

$$\mathbf{G}_E = \mathbf{G}_d + \begin{bmatrix} \mathbf{F} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \text{ with} \tag{3.4}$$

$$\mathbf{F} = \begin{bmatrix} -1 & -1 & -1 & 0 & 0 & 0 \\ -1 & -1 & -1 & 0 & 0 & 0 \\ -1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

Here, we define $\mathbf{n}_l = [1\ 1\ 1\ x_1\ x_2\ x_3]'$, where x_i (i = 1, 2, 3) taking values 1 or 0 depending on the presence or absence of the lines 4, 5 and 6 in the affected block.

Now, the resulting information matrix can be expressed as

$$\mathbf{C}_E = \mathbf{C}_d - \mathbf{A}_E$$

where

$$\mathbf{A}_E = k^{-1}\begin{bmatrix} \mathbf{n}_l\mathbf{e}' + \mathbf{e}\mathbf{n}_l' + \mathbf{e}\mathbf{e}' - k\mathbf{F} & \mathbf{e}\mathbf{u}_l' \\ \mathbf{u}_l\mathbf{e}' & \mathbf{0} \end{bmatrix} \quad (3.5)$$

Substituting $\mathbf{n}_l$, $\mathbf{e}$ and $\mathbf{F}$ in (3.5), we get

$$\mathbf{A}_E = k^{-1}\begin{bmatrix} \mathbf{X} & \mathbf{T}' \\ \mathbf{T} & \mathbf{Z} \end{bmatrix}$$

where

$$\mathbf{X} = \begin{bmatrix} k-1 & k-1 & k-1 & -x_1 & -x_2 & -x_3 \\ k-1 & k-1 & k-1 & -x_1 & -x_2 & -x_3 \\ k-1 & k-1 & k-1 & -x_1 & -x_2 & -x_3 \\ -x_1 & -x_2 & -x_3 & 2x_1+1-k & x_1+x_2+1-k & x_1+x_3+1-k \\ -x_1 & -x_2 & -x_3 & x_1+x_2+1-k & 2x_2+1-k & x_2+x_3+1-k \\ -x_1 & -x_2 & -x_3 & x_1+x_3+1-k & x_2+x_3+1-k & 2x_3+1-k \end{bmatrix}$$

$\mathbf{T} = \mathbf{u}_l\mathbf{e}'$ and $\mathbf{Z} = \mathbf{0}_{(p-6)\times(p-6)}$, the null matrix

One can easily see that $\mathbf{A}_E$ is symmetric with row and column sum zero and $\mathbf{A}_E$ commutes with information matrix $\mathbf{C}_d$ of a variance balanced block design for triallel crosses. Therefore, the eigenvalues of resulting information matrix $(\mathbf{C}_E)$ can be obtained by subtracting the eigenvalues of $\mathbf{A}_E$ from that of $\mathbf{C}_d$. Thus, the eigenvalues of $\mathbf{C}_E$ are

(i) $\theta_0 = 0$ with multiplicity 1

(ii) $\theta_{1E} = \mu$ with multiplicity $(p-3)$

(iii) $\theta_{2E} = (\mu - \theta_1)$, and

(iv) $\theta_{3E} = (\mu - \theta_2)$ (3.6)

Here, $\mu = 3b(k-1)/(p-1)$ is the unique non-zero eigenvalue of $\mathbf{C}_d$. For design d_E to be connected, $\text{rank}(\mathbf{C}_E)$ should be $p-1$. In other words $\mu \neq \theta_i$, $i = 1, 2$. It is difficult to obtain the conditions for which $\mu \neq \theta_i$ in general. Hence, we shall study the particular designs.

Furthermore, for a connected design, the efficiency of the resulting design (d_E) relative to the original design (d) is seen to be

$$E = \frac{(p-1)\mu^{-1}}{\left[(p-3)\theta_{1E}^{-1} + \theta_{2E}^{-1} + \theta_{3E}^{-1}\right]} \tag{3.7}$$

The eigenvalues of $\mathbf{A}_E$ are obtained by solving

$$|\mathbf{A}_E - \theta\,\mathbf{I}| = 0 \tag{3.8}$$

which simplifies to

$$\theta^3 - 2\theta^2(x_1 + x_2 + x_3) - \theta(Q^2 + 6Q + v) + Q\,v = 0 \tag{3.9}$$

where $Q = 3k - 3 - x_1 - x_2 - x_3$

and $v = 2\,[x_1^2 + x_2^2 + x_3^2] - 2\,[x_1 x_2 + x_1 x_3 + x_2 x_3]$

Equation (3.9) is cubic in θ. Therefore, we have considered here only two cases, namely $x_1 = x_2 = x_3 = 1$ and $x_1 = x_2 = x_3 = 0$. For both the cases, the cubic equation reduces to a quadratic, giving the eigenvalues of $\mathbf{A}_E$ as

$$\theta_1, \theta_2 = \{(x_1 + x_2 + x_3) \pm [(x_1 + x_2 + x_3)^2 + Q^2 + 6Q]^{1/2}\} / k \tag{3.10}$$

Case I: $x_1 = 0, x_2 = 0, x_3 = 0$

The non-zero eigenvalues of $\mathbf{A}_E$ are

$$\theta_1, \theta_2 = \pm 3\sqrt{1 - \frac{1}{k^2}} \tag{3.11}$$

Case II: $x_1 = 1, x_2 = 1, x_3 = 1$

In this case, the non-zero eigenvalues of $\mathbf{A}_E$ are

$$\theta_1 = 3 \text{ and } \theta_2 = 3(k-2)/k \tag{3.12}$$

A catalogue of all variance balanced block designs for diallel crosses obtainable from Families 1 to 4 of Das and Gupta [4] has been prepared for $p \leq 30$ and is given in Table 1 in the Appendix. The connectedness of d_E

obtained from d given in Table 1 has been examined individually and empirically. It is found that all the designs in Table 1 (except the design with parameters $p = 9$, $b = 4$, $k = 3$ in Case II) remains connected after the exchange of a distinct cross in both the cases. The relative efficiencies of optimal designs for triallel crosses which remains connected after exchange of a cross have been obtained using (3.7). It is observed that the relative efficiencies of the block design for triallel crosses with parameters $p = 9$, $b = 12$, $k = 2$ are 0.8889 and 0.8000 in Case-I and Case-II respectively. Further, the relative efficiencies of the designs with parameters $p = 15$, $b = 7$, $k = 5$; $p = b = 7$, $k = 2$ and $p = b = 10$, $k = 3$ are 0.9479, 0.5000 and 0.9290 respectively in Case-II. For rest of the designs the relative efficiency is greater than or equal to 0.9500. Thus, the designs with $p > 9$ (except the designs with $p = 15$, $b = 7$, $k = 5$ and $p = b = 10$, $k = 3$ for the cases discussed above) are robust according to efficiency criterion.

4. Robustness Against Exchange of Crosses having One Line in Common

Without loss of generality, we assume that the cross having lines 1, 2 and 5 has been exchanged with the cross having lines 3, 4 and 5 in block 1 so that the line 5 is common between them. The incidence matrix of d in the partitioned form can be written as

$$\mathbf{N}_d = \begin{bmatrix} \mathbf{n}_1 & \mathbf{N}_p \\ \mathbf{u}_1 & \mathbf{N}_u \end{bmatrix} \tag{4.1}$$

where

$\mathbf{n}_1$ is the 5×1 vector of involved lines vs affected block

$\mathbf{u}_1$ is the $(p-5) \times 1$ vector of uninvolved lines vs affected block

$\mathbf{N}_p$ is the $5 \times (b-1)$ matrix corresponding to involved lines vs unaffected blocks

$\mathbf{N}_u$ is the $(p-5) \times (b-1)$ matrix corresponding to uninvolved lines vs unaffected blocks

As in Section 3, here $\mathbf{n}_1$ can be written as

$$\mathbf{n}_1 = [1 \;\; 1 \;\; x_1 \; x_2 \;\; 1]'$$

where x_1, x_2 take value 1or 0 according as the line 3 or 4 is present or absent in block 1.

Here also, $\mathbf{n}_{1E} = \mathbf{n}_1 + \mathbf{e}$, with $\mathbf{e}' = [-1 \;\; -1 \;\; -1 \;\; 1 \;\; 0]$

The matrices $\mathbf{G}_{d1}$, $\mathbf{G}_{d2}$ and $\mathbf{G}_{d3}$ have the same meanings as defined earlier in (3.3) but are of order (5 × 5), 5 × (p – 5) and (p – 5) × (p – 5), respectively. Here, **F** given in (3.4) is

$$\mathbf{F} = \begin{bmatrix} -1 & -1 & 0 & 0 & -1 \\ -1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 & 0 \end{bmatrix} \tag{4.2}$$

Substitution of $\mathbf{n}_1$, **e** and **F** in (3.5), yields

$$\mathbf{A}_E = k^{-1}\begin{bmatrix} \mathbf{X} & \mathbf{T}' \\ \mathbf{T} & \mathbf{Z} \end{bmatrix} \tag{4.3}$$

where

$$\mathbf{X} = \begin{bmatrix} -1+k & -1+k & -x_1 & -x_2 & -1+k \\ -1+k & -1+k & -x_1 & -x_2 & -1+k \\ -x_1 & -x_1 & 2x_1+1-k & x_1+x_2+1-k & 1-k \\ -x_2 & -x_2 & x_1+x_2+1-k & 2x_2+1-k & 1-k \\ -1+k & -1+k & 1-k & 1-k & 0 \end{bmatrix}$$

$$\mathbf{T} = \mathbf{u}_1\mathbf{e}' \text{ and } \mathbf{Z} = \mathbf{0}_{(p-5)\times(p-5)}$$

It can be seen here that $\mathbf{A}_E$ satisfies all the properties described earlier in Section 3.

Now, the solution of the equation

$$|\mathbf{A}_E - \theta\,\mathbf{I}| = 0 \tag{4.4}$$

gives the non-zero eigenvalues of $\mathbf{A}_E$ as

$$\theta_1, \theta_2 = \left\{(x_1 + x_2) \pm \left[(x_1+x_2)^2 + c\right]^{1/2}\right\} \Big/ k \tag{4.5}$$

where $c = 2(x_1 - x_2)^2 + 4\,(3k - 3 - x_1 - x_2) + (x_1 + x_2 + 2 - 2k) + 4\,(1 - k)^2$

Depending upon the 1, 0 values of x_i (i = 1, 2), all the optimal block designs for triallel crosses have been classified into 3 cases which are presented below along with the non-zero eigenvalues θ_1, θ_2 of $\mathbf{A}_E$.

Case I: $x_1 = 0, x_2 = 0$

$$\theta_1,\ \theta_2 = \pm\left[4\ \left(2k^2 - k - 1\right)\right]^{1/2}/k$$

Case II: $x_1 = 0, x_2 = 1$

$$\theta_1,\ \theta_2 = \left[1 \pm \left(8k^2 - 8k\right)\right]^{1/2}/k$$

Case III: $x_1 = 1, x_2 = 1$

$$\theta_1,\ \theta_2 = \left[2 \pm \left(8k^2 - 12k + 4\right)\right]^{1/2}/k \tag{4.6}$$

As earlier, here the connectedness of d_E has been proved for particular designs given in Table 1 by equating θ_i $(i = 1, 2)$ with μ, the non-zero eigenvalue of that particular design, d. Relative efficiencies for connected designs in different cases have been computed using (3.7). It is observed that the relative efficiency of the block design for triallel crosses with parameters $p = b = 19$, $k = 3$ is 0.8463 in Case-I. In Case II, the relative efficiencies for the designs with parameters $p = 9$, $b = 12$, $k = 2$ and $p = b = 7$, $k = 2$ are 0.8889 and 0.7317 respectively. In case-III, the relative efficiencies of the designs with parameters $p = 9$, $b = 4$, $k = 3$; $p = 9$, $b = 12$, $k = 2$ and $p = b = 7$, $k = 2$ were found to be 0.4000, 0.8506 and 0.6393 respectively. For rest of the designs the relative efficiency is greater than or equal to 0.9500. Thus, the designs with $p > 9$ (except the designs with $p = 19$, $b = 19$, $k = 3$ for the Case I) are robust according to efficiency criterion.

5. *Robustness Against Exchange of Crosses having Two Lines in Common*

Without loss of generality, we assume that the cross having lines 1, 3 and 4 has been exchanged with the cross having lines 2, 3 and 4 in block 1, so that lines 3 and 4 are common. The incidence matrix of d can be written as

$$\mathbf{N}_d = \begin{bmatrix} \mathbf{n}_1 & \mathbf{N}_p \\ \mathbf{u}_1 & \mathbf{N}_u \end{bmatrix} \tag{5.1}$$

where $\mathbf{n}_1$, $\mathbf{u}_1$, $\mathbf{N}_p$ and $\mathbf{N}_u$ are of order (4×1), $(p - 4) \times 1$, $4 \times (b - 1)$ and $(p-4) \times (b-1)$ respectively and have the meaning as in (3.1). Here, $\mathbf{n}_{1E} = \mathbf{n}_1 + \mathbf{e}$ with $\mathbf{e}' = [-1\ \ 1\ \ 0\ \ 0]$. Let $\mathbf{n}_1 = [1\ \ x_1\ \ 1\ \ 1]$ with $x_1 = 1$, if line 2 is present in block 1 and 0 otherwise. Also, the matrix $\mathbf{F}$ given in (3.4) takes the form

$$F = \begin{bmatrix} -1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 1 \\ -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{bmatrix}$$

Substituting the values of $\mathbf{n}_1$, $\mathbf{e}$ and $\mathbf{F}$ in (3.5), we get

$$\mathbf{A}_E = k^{-1}\begin{bmatrix} \mathbf{X} & \mathbf{T}' \\ \mathbf{T} & \mathbf{Z} \end{bmatrix} \tag{5.2}$$

with

$$\mathbf{X} = \begin{bmatrix} -1+k & -x_1 & -1+k & -1+k \\ -x_1 & 2x_1+1-k & 1-k & 1-k \\ -1+k & 1-k & 0 & 0 \\ -1+k & 1-k & 0 & 0 \end{bmatrix}$$

$\mathbf{T} = \mathbf{u}_1\mathbf{e}'$ and $\mathbf{Z} = \mathbf{0}_{(p-4)\times(p-4)}$

The solution of

$$|\mathbf{A}_E - \theta\,\mathbf{I}| = 0 \tag{5.3}$$

gives the non-zero eigenvalues (θ_1, θ_2) of $\mathbf{A}_E$ as

$$\theta_1, \theta_2 = [x_1 \pm (x_1^2 + c)^{1/2}]\,/k \tag{5.4}$$

where $c = 2(3k - 3 - x_1) + 4(k-1)^2 + (x_1 + 1 - k)^2$

The values of θ_1, θ_2 for different cases are given below

Case I: $x_1 = 0$

$$\theta_1,\ \theta_2 = \pm\left[\left(5k^2 - 4k - 1\right)\right]^{1/2}\Big/k$$

Case II: $x_1 = 1$

$$\theta_1,\ \theta_2 = \pm\left[\left(5k^2 - 6k + 1\right)\right]^{1/2}\Big/k$$

Like earlier cases, in this section also, the connectedness of d_E has been studied empirically for particular designs given in Table 1. The relative efficiencies have been computed for connected designs for different cases using (3.7). It is observed that in Case I the relative efficiencies of the designs with parameters p = 9, b = 4, k = 3 and p = b = 7, k = 2 are 0.8596 and 0.9120 respectively. In Case II, the relative efficiencies for the designs with parameters p = 9, b = 4, k = 3; p = 9, b = 12, k = 2 and p = b = 7, k = 2 are 0.8000, 0.9283 and 0.8438 respectively. For rest of the designs the relative efficiency is greater than or equal to 0.9500. Thus, the designs with p > 9 are robust according to the efficiency criterion.

Hence, we may conclude that all the universally optimal designs for triallel crosses with p > 9 considered here are robust according to the efficiency criterion.

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APPENDIX

Table 1: Variance balanced block designs for triallel crosses for $p \leq 30$ obtainable from Das and Gupta [4]

Sl.No.	Parameters			Source
	p	b	k	
1	7	7	2	S1-F4
2	9	4	3	S1-F1
3	9	12	2	S1-F3
4	10	10	3	S1-P5(i)
5	10	40	3	S1-F2
6	13	13	4	S1-F4
7	13	52	3	S1-F2
8	15	35	4	S1-F3
9	15	7	5	S1-F1
10	16	16	5	S1-P13
11	16	112	5	S1-F2
12	19	19	3	S1-P6
13	19	19	6	S1-F4
14	19	76	3	S1-F2
15	21	70	3	S1-F3
16	21	70	4	S1-F3
17	21	70	6	S1-F3
18	21	10	7	S1-F1
19	25	100	3	S1-F2
20	25	25	8	S1-F4
21	27	117	8	S1-F3
22	27	13	9	S1-F1

S1 : Das and Gupta [4]

F# : Family # of Designs

P : Table 3 of Preece [15]