Experimental Designs in the Presence of Indirect Effects of Treatments

FINAL PROJECT REPORT

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FOREWORD

The ICAR-Indian Agricultural Statistics Research Institute is a premier Institute in the disciplines of Agricultural Statistics and Informatics and is engaged in conducting research, teaching and training in experimental designs, sample surveys, statistical modelling, crop forecasting, statistical genetics, bioinformatics and computer applications. The Institute has taken a lead in developing statistical software packages, expert systems and many online systems useful for Agricultural Research.

Designing an experiment is an essential component of any scientific investigation and refers to the specific manner in which the experiment is set up, conducted and the plans for data collection and analysis. Before an experiment is conducted, the experimental design has to be carefully planned to ensure that experimental objectives can be accomplished, assumptions required for hypotheses testing and data analysis are valid, randomization requirements are met and the experiment is reproducible.

Indirect effects are a serious problem in many of the field experiments and may be spatial or temporal in nature. The spatial effect is also known by different names in literature viz., interference effect, remote effect, competition effect, neighbour effects whereas the temporal effect is also known as crossover effect, carry over effect, residual effect. This project entitled **Experimental Designs in the Presence of Indirect Effects of Treatments** sponsored by Department of Science and Technology (DST) is a good initiative by the project team to develop designs for different situations incorporating indirect effects. These designs will be helpful to the students and researchers to conduct their research more effectively and draw valid conclusions by using appropriate designs suitable for their situations. The software **Web Generation of Experimental Design Balanced for Indirect Effects of Treatments (*WebDBIE*)** developed will help the experimenters in generating the randomized layout plans of the designs. The software is deployed at ICAR-IASRI website and is freely available to the users on the globe. I am happy to share that software developed has received copyright from Registrar, Copyright, Government of India.

I take this opportunity to complement the entire project team of Dr. Seema Jaggi, Dr. Cini Varghese, Dr. Eldho Varghese and Ms. Anu Sharma for doing a wonderful job and bringing out this valuable report. I thank DST for providing financial assistance and I look forward for more such collaborations from DST.

New Delhi

U.C. SUD
DIRECTOR, ICAR-IASRI

PREFACE

In designing of any scientific experiment, heterogeneity in the experimental material is an important aspect to be taken care of. Block designs are the most appropriate designs for controlling local variation over the experimental material by dividing the entire experimental material into groups/blocks such that the experimental units are homogeneous within a block. In agricultural field experiments, there may be situations where in order to control heterogeneity and conserve resources, the treatments are assessed using small, adjacent units i.e. plots in a block. The treatment applied to one experimental plot may affect the response on neighbouring plots as well as the response on the plot to which it is applied. Treatments such as fertilizer, irrigation, or pesticide may spread to adjacent plots causing neighbour effects. Thus, effects which occur in an experiment due to the units which are adjacent spatially to the unit being observed are called spatial indirect effects. When the experimental units are long lived or scarce, in order to study the effects of different treatments, the same unit is given the treatments over different periods. Treatments applied in a particular period may influence the response of the units not only in the period of their direct application but also in the subsequent periods. The treatments like drugs, feeds, leave carryover effects in the periods following the periods of their direct application. The effects which occur in an experiment due to the units which are adjacent temporally to the unit being observed are called temporal indirect effects.

Understanding the structure of these indirect effects helps in minimizing the bias in the treatment comparisons to a great extent so as to make more precise inference. It is thus important to include the indirect effects in the model to have the proper specification and obtain designs in the presence of indirect effects that are ordered over space or time. Neighbour Balanced Designs are used for the situations when spatial indirect effects are suspected from the treatments applied in the neighbouring experimental units whereas Crossover Designs are used when temporal indirect effects consisting of residual or carryover effects from the treatments applied in the previous period are present. These designs ensure that each treatment occurs adjacent to every other treatment spatially or temporarily same number of times.

In this report, several classes of designs have been developed for situation when the different types of indirect effect are suspected. Further, to enhance the application potential of these designs worldwide, software named 'Web Generation of Designs Balanced for Indirect Effects of Treatment (WebDBIE)' has been developed for the online generation and cataloguing of these designs. It is hoped that experimenters who conduct experiments globally will be immensely benefited by this study for planning and designing their experiments more efficiently.

The authors express their sincere thanks to the Director, ICAR-IASRI for his support and for providing all necessary facilities to carry out the research work successfully. The cooperation received from Head and other scientists of the Division of Design of Experiments, IASRI is thankfully acknowledged.

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Project Team

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Chapter I

INTRODUCTION AND REVIEW

1.1 Introduction

Designing an experiment is an essential component of any scientific investigation. Experimental design refers to the specific manner in which the experiment is set up, conducted and the plans for data collection and analysis. Specifically, experimental design has to do with the precise way different treatments are administered to experimental units (subjects, parts, plots, animals etc.) and what information is to be collected on the experimental units and other conditions that might affect the experimental units. Before an experiment is conducted, the experimental design must be carefully planned to ensure that experimental objectives can be accomplished, assumptions required for hypotheses testing and data analysis are valid, randomization requirements are met, and the experiment is reproducible.

There may arise experimental situations wherein the response from a unit may be affected by indirect effects from other units spatially or temporally belonging to the same cluster or group or block. It may be that the responses are correlated or that there is a systematic trend, or the response is affected by carryover effect from the units previously observed, or a neighbour effect from contiguous units. We may call these effects as indirect effects. Thus, indirect effects are effects which occur in an experiment due to the units which are adjacent (spatially or temporally) to the unit being observed. Given below are some **situations** where indirect effects can occur.

1.1.1 Spatial Indirect Effects

- In agricultural experiments under block design setting, where the blocks are made up of plots which can not be sufficiently isolated from each other, there could be spatial indirect effects coming from the treatments applied to the neighbouring plots.
- If the branches of a tree form plots while the tree serves as a block, spatial indirect effects may arise from the treatments applied to the neighbouring branches.
- In fertilizer trials, plants in an unfertilized plot may rob a share of the plants in a nearby heavily fertilized plot, thereby resulting in spatial indirect effects.
- In varietal trials, the yield of a variety may be depressed by more aggressive neighbouring varieties resulting in spatial indirect effects.
- In fungicide experiments, an unsprayed plot provides a source of spores which can infect neighbouring treated plots resulting in spatial indirect effects.
- In market studies, the sale of different brands on a store shelf may be affected by the brands in the neighbouring shelves.
- In the interpollination by natural hybridization of a group of genotypes, each clone has an equal chance of pollinating, or being pollinated by, any of the others.

1.1.2 Temporal Indirect Effects

Treatments applied in a particular period may influence the response of the units not only in the period of their direct application but also in the subsequent periods. The treatments leave carryover effects (temporal indirect effects) in the periods following the periods of their direct application. These situations may arise in following areas:

- Long-term agricultural field experiments
- Nutrition experiments with dairy cattle
- Clinical trials in medical research
- Psychological experiments

1.1.3 Spatial and Temporal Indirect Effects

- When treatments are applied sequentially to experimental units over time or space, there may be an unknown trend effect which can be expressed as a polynomial function of the order in which the observations are taken.
- The observations may be correlated in space or time because of neighbouring units or due the same units being observed over different time periods.

In such situations, it is important to include the indirect effects in the model to have the proper specification. The experimental units consisting of different treatments are to be arranged within each cluster or block that is ordered over space or time. It was, therefore, important to obtain designs in the presence of indirect effects and compute gain in efficiency of these designs over conventional designs. Following are some designs that are ordered over space or time:

- A block design for 3 treatments in 2 blocks with border plots spatially balanced for left and right neighbours

3	1	2	3	1
2	1	3	2	1

- A trend free block design for 7 treatments in 3 blocks

1	2	3	4	5	6	7
6	4	2	7	5	3	1
7	5	3	1	6	4	2

- A design for 4 treatments in 4 experimental units and 4 periods temporally balanced for carryover effects

Periods	Units			
	I	**II**	**III**	**IV**

1	1	2	3	4
2	4	1	2	3
3	2	3	4	1
4	3	4	1	2

1.2 Definition of the Problem

Indirect effects can arise in many situations as discussed earlier. Substantive amount of work has been done considering spatial/ temporal indirect effects. There are some experimental situations that have not been studied. Further, study incorporating both spatial as well as temporal indirect effects of treatments needs attention. There was a need to study and obtain designs addressing various issues with respect to the relationship between observations on units over time and/or space.

For easy accessibility of these designs by the experimenters, it was required that these designs are compiled and presented at one place. With the advancement in web technology, it was also desired to make available these designs on the web for quick reference. Web based experimental designs in the presence of indirect effects would be of great use for the researchers and students working in this area. It was also desired that the analytical procedure of these designs be illustrated for different situations for the experimenters. Considering the above situations, the broad objectives taken for this project are given below.

1.3 Objectives

1. To obtain designs in the presence of indirect (neighbour or carryover) effects of treatments for block/ row-column experimental situations.
2. To study designs considering more than one relationship between observations on units over time or space.
3. To generate web-enabled useful classes of designs in the presence of indirect effects of treatments.
4. To develop an online catalogue of designs incorporating indirect effects of treatments.
5. To illustrate the analytical procedures for designs in presence of indirect effects using live/ simulated data sets.

1.4 Review of Literature

1.4.1 International Status

- **Designs with Correlated Errors**

Papadakis (1937) adjusted the plot values by the analysis of covariance when covariates were taken as the residuals from neighbouring plots. This analysis resulted in substantial reduction in error variance. Papadakis method does not assume any specific correlation structure among the observations. The knowledge of correlation structure among the observations has been used by Williams (1952) in designing experiments. The designs for these situations were defined as *Neighbour Designs* by Rees (1967) for use in serology experiments. Atkinson (1969) showed that the properties of Papadakis estimator are very close to those of Maximum Likelihood (ML) estimators when applied to plots arranged in a row and underlying error structure is the first order auto-regressive [AR(1)] process. When observations are independent then using this method does not lead to a great increase in variance of the estimators, whereas failure to take into account the error structure leads to a considerable loss of precision. Bartlett (1978) suggested the use of correlation between neighbouring plots for the analysis of data from field experiments.

Balanced Incomplete Block (BIB) designs are known to be optimal for the elimination of one-way heterogeneity under equal error variances and additive models when the observations are independent. Kiefer and Wynn (1981) considered the effects of correlation on the efficiency of such designs. Kiefer and Wynn (1981) found the optimal designs among the class of block and Latin square designs when observations follow a moving average process. They used the technique of ordinary least squares. Martin (1982) studied the designs and their efficiencies when observations follow a stationary torus lattice process. Wilkinson *et al.* (1983) gave a good discussion of design and analysis of experiments for spatially correlated observations. Cheng (1983) developed an algorithm for construction of optimal BIB design for correlated observations.

Besag and Kempton (1986) described the applications of the use of neighbouring plot values in the analysis of agricultural field experiments. Kunert (1987) studied the neighbour balanced block designs for correlated errors. Russell and Eccleston (1987) gave algorithms for construction of optimal incomplete block designs when a known correlation structure is assumed for observations from plots in the same block. Jacroux (1998) constructed some efficient equineighboured incomplete block designs for block size 3 that are efficient when experimental units adjacent within blocks are correlated or when least squares Nearest Neighbour (NN) analysis is used to analyze data from an experiment because of suspected but unknown smooth trends within blocks. Martin and Eccleston (2001) gave an exchange-interchange algorithm for searching optimal and nearly optimal designs for specific dependence structure of observations. Pooladsaz and Martin (2005) investigated moving average type correlation structure of order g ($\geq$1) for $|\rho| \leq 0.50$. The bound for AR(1) correlation structure is considered as $|\rho| \leq 0.95$, as this correlation structure is not limiting by any condition. Uddin (2008) have studied MV-optimal block designs for correlated errors.

- **Designs with Systematic Trend**

The study on systematic designs was initiated by Cox (1951). Bradley and Yeh (1980) first gave the theory of trend free block designs. They considered the situation where a common polynomial trend is assumed to exist over the plots in each block of a classical experimental design. Trend free block design was defined to be a design in which the adjusted sum of squares due to treatments in a model with trend effects remains the same as in the model without trend effects. Necessary and sufficient condition for the block design to be trend free was derived. Yeh and Bradley (1983) discussed the existence of trend free block design for specified trends under a homoscedastic model when each treatment is equally replicated.

Jacroux *et al*. (1995, 1997) developed some methods for identifying efficient designs when different blocks may have linear trend effect of different slope. Majumdar and Martin (2002) extended the above study for quadratic and cubic trend effects.

- **Designs with Neighbour/Interference Effects**

Agricultural experiments often show neighbour effects i.e., the response on a given plot is affected by the treatments on the neighbouring plots as well as by the treatment applied to that particular plot provided there are no gaps between plots. For example, in varietal trial, tall varieties may shade their neighbours. Sakai (1956, 1957) studied the effect of competition of plants and a varying number of competing and non competing individuals. Goodall (1960) studied the competition of marigold plants with one, two, four or eight neighbours arranged in a circle centered on a single plant and observed that the effect of neighbours was additive up to six neighbours at which stage the competition effects were complete and introduction of more neighbours had no effect. Hanson *et al.* (1961) have measured the competition using special design and further analysis for the design is also presented. Single-row yield plots of soybean genotypes have been used for his research.

Rees (1967) studied interference between neighbouring units under laboratory conditions on designing of plots to diffusion tests in virus research. Mathematical model was given by Mead (1967) to estimate of inter-plant competition based on correlation between neighbouring pairs of plants. Freeman (1967, 1969) used cyclic BIB designs for directional and non-directional seed orchards. Gomez (1972) undertook a study to investigate the extent of varietal competition and its effects on comparison between varieties in transplanted lowland rice. The analysis of variance was performed, separately for each variety, following the procedure of split-plot design with the two halves of the plot (each corresponding to one of the adjacent varieties) as main-plot treatments and the five row positions within each half as subplot treatments. The results showed a significant difference between row positions indicating the presence of varietal competition effect, and the presence of interaction between row position and adjacent variety indicated that the competition effect is affected by the adjacent variety. Martin (1973) developed beehive designs in which plants of two species are arranged on a hexagonal grid such that for one species the number of neighbouring plants of the second species varies between zero and six. These designs allow the experimenter to carry out the investigations in a much smaller area and each plant is either a recorded plant or a competing plant.

Breese and Hill (1973) have applied regression analysis to data from competition diallel experiments with plant species. The basic linear model for diallelic competition has been studied by Rawlings (1974) that corresponds to the model for the diallel cross mating design with parents included. Pearce and Moore (1976) investigated reduction of experimental error in perennial crops, using adjustment by neighbouring plots.

Gomez and Gomez (1976) have investigated the effects of two widely different nitrogen levels on adjacent plots separated by a 40 cm non-planted alley. Nitrogen (120 kg/ha) was applied to some plots and no fertilizer to some plots and 28-row plots were arranged systematically in an alternating series. There were a total of 16 plots and each plot was bordered on one side by plot of the same nitrogen level (control) and on the other side by plot of a different level. Dyke and Shelley (1976) introduced serial designs that allow the independent estimation of the effects of treatments to neighbouring plots and have constructed serial designs based on computer program. The aim of the experiment was to study interference between plots with regard to spread of disease. Designs were obtained in which treatments appeared repeatedly in one or more chosen orders.

Investigating the competition on individual plants, Mead (1979) observed that the response of individual plants varies depending on the number of competing plants. Further, it depends on differences in distances of the neighbours also. There is a difficulty in designing such experiments because when the distances of up to six neighbours are symmetrically varied, large number of possible patterns of neighbour arises. Freeman (1979) has obtained some two-dimensional designs balanced for nearest neighbours. Jenkyn *et al.* (1979) have investigated interactions between plots of spring barley with four spray treatments using the serially balanced design described by Dyke and Shelley (1976).

Veevers and Zafar (1980) have studied balanced designs for two-component competition experiments on a square lattice. A set of design strings has been presented, any of which generates an efficient balanced design. When an intra-variety competition experiment is performed using an arrangement of plants and spaces on a regular lattice of sites, it is desirable for some purposes for the design to be balanced. That is, equal number of plants is immediately surrounded by equal number of each of the possible number of species. For an equilateral triangular lattice this can be 0,1,…,6 spaces, defining seven treatments that can be applied to a plant. A set of nineteen distinct binary strings have been presented by Veevers (1982), any of which can be used to generate balanced design which is easy to implement.

Draper and Guttman (1980) have studied response surface models by incorporating overlap effects from neighbouring units. Kempton and Howes (1981) have used the neighbouring plot values in the analysis of variety trials. Kempton (1982) proposed a method of correcting for competition effects in yield trials by joint regression of plot yields on to the yields of neighbours. Estimation of the variety effects and competition coefficient along with tests of significance are described for a sugar-beet trial with single-row plots where competition effects are assumed to extend only to plants in immediately adjacent rows. Jenkyn *et al.* (1983) studied the effects of fungicide movement between plots in field experiments. The number of combinations for competition treatments increases rapidly as the number of treatments to be tested increases. Assuming that competition occurs only between the test plot and its immediate neighbouring plots on either side and that the effects are same for left or right-hand side arrangement, Lin *et al*. (1985) introduced a treatment sequence and computer aided non-random designs. For three treatments (cultivars), the total number of competition treatments or triplets is 18. Since each treatment consists of three plots (one test plot plus two neighbouring plots), if conventional randomized block design is to be used, the number of plots required for one replication would be three times the above figure. This is not only a wastage of resources (2/3 of the plots will be discarded) but also impractical because it is hard to find homogeneous fields for such numbers. Sequence of these triplets were constructed by a computer program in 20 plots (i. e., 18 inner plots and two border plots). Kempton *et al.* (1986) conducted series of experiments to investigate the effects of interplot competition on grain yield among hexaploid triticale cultivars of widely differing heights grown in 1.5 m wide plots using a design balanced for nearest neighbours.

Azais (1987) introduced a methodological study of the nature of competition on a given species. The plots were defined as lines of plants and treatments as genotypes. A line (plot) has only two neighbouring lines (plots). There will be no adjacencies (common neighbours) between plots of different blocks. Two assumptions viz. (a) the first order assumption in which variables measured on a line depend only on the genotype sown on that line (producer genotype) and the genotypes sown on the two neighbouring lines (the competitor genotypes), and (b) the symmetry assumption, in which the effect of a competitor does not depend on which side of the line it is sown, were introduced. Azais (1987) defined balancedness in block design for studying competition effects. A block design is balanced in the sense that every treatment has every other treatment appearing once as a right and once as a left neighbour. It means, every pair of treatments appears as neighbours equally often. Latin squares with this property were developed by Williams (1949). For even numbers, such Latin squares can be obtained easily whereas for odd numbers two Latin squares are to be used together to satisfy this property. Based on this property some methods of construction of neighbour balanced designs using Latin squares were attempted. When the neighbour effect is present on one side only, say from left-hand side, as caused by a prevailing wind, simply omit the right - hand border plot from every block. Randomization of these designs consists of allocation of a treatment number, randomizing the blocks and then randomizing each block independently by a circular interblock permutation. Langton (1990) have defined different types of edge effects on experimental plots and have shown that they are important in agroforestry experiments. Methods of eliminating edge effects either at the design or the analysis stage are described.

Azais *et al.* (1993) obtained series of designs that are balanced in t -1 blocks of size t and t blocks of size t -1, where t is the number of treatments. Azais *et al.* (1993) also developed a program that generates these designs. Designs consisting of factors balanced for neighbour effect of one factor were developed by Monod and Bailey (1993). In their setup the treatments were levels of two factors and competition is exerted only by the levels of one factor. They have developed designs optimal for the estimation of direct treatment effects and competition effects. It was assumed for analysis that the competition effects were directional and that direct effects and competition effects from one side are additive.

Block designs balanced for neighbour effects have been found quite useful by experimenters because they lead to simplified analysis. But these cannot be justified on statistical grounds unless they possess some optimal statistical properties as well. David and Kempton (1996) have investigated experimental designs that control interference so that it can be ignored in subsequent analysis. David and Kempton (1996) proposed modified alpha designs for controlling interplot competition in variety trials.

Azais and Druilhet (1997) have studied the optimality of neighbour balanced designs when neighbour effects are not taken into account in the analysis model. Raghavarao and Zhou (1998) have studied the universal optimality of 3-designs with unequal set sizes to estimate the effects in a competing effect model. Azais *et al.* (1998) have investigated the influence of design on validity and efficiency of neighbour methods by simulation study. Optimality of neighbour balanced designs obtained by Azais *et al.* (1993) when neighbour effects are present in the model has been studied by Druilhet (1999). Bailey (2003) studied designs for one-sided neighbour effects and have presented table of designs with different block sizes. Bailey and Druilhet (2004) studied the optimality of neighbour balanced block design when neighbour effects are present in the model.

- **Designs with Carryover Effects**

Simplest design balanced for carryover or residual effects is a Latin square design with rows representing periods and columns representing experimental units with the assumption that no residual effect of treatments is present. It is however, desirable to allow for the possibility that residual effects exist. The designs for such situations are known in literature as change over designs, switchover trials, crossover designs, time series designs, before after designs, repeated measurements designs and designs involving sequences of treatments.

Williams (1949) observed that the balanced repeated measurements designs for v treatments using v periods can be constructed in one Latin square for an even number of treatments and with two Latin squares for an odd number of treatments. Williams Latin square(s) are variance balanced designs in the sense that all elementary contrasts among direct effects of treatments are estimated with same variance and also all elementary contrasts among residual effects are estimated with same variance. These designs require that number of periods is equal to the number of treatments which is too severe a restriction when the number of treatment is large. Sometimes sufficient number of experimental units, required by a balanced design is not available. For such situations, a new class of designs called partially balanced repeated measurements designs was introduced by Patterson and Lucas (1962). Methods of constructing balanced, partially balanced, extra-period balanced and extra-period partially balanced repeated measurements designs were given by them. Another class of repeated measurements designs where treatment × period interactions are assumed to be present in place of residual effects has been given by Balaam (1968). These designs are obtained by taking all possible arrangements of v treatments as sequences of size two.

Berenblut (1964) developed a class of repeated measurements designs in which direct effects are orthogonal to first residuals as well as subject effects. These designs require v^2 units and 2v periods ($v \geq 2$) for testing v treatments. Patterson (1973) showed the Quenouille's cyclic method of construction can be extended to designs for any number v of treatments, 2v periods and v^2 units. Some of the restrictions imposed on the sequences of treatments by Berenblut have been improved by Patterson (1973). Design problems in the context of possible auto-correlations between observations were considered by Berenblut and Webb (1974). A design procedure was adopted involving the minimization of the generalized variance of estimates of parameters. Also, an explicit design was produced for the optimum settings of quantitative factors, under reasonably mild restrictions in block designs and time sequences. Takka and Armitage (1983) and Kunert (1985) have investigated models with auto-correlated errors, but they did not consider a residual term in these linear models. Azzalini and Giovagnoli (1987) derived conditions for optimality for models without nuisance parameters, and also with covariates and with block effects. Model for responses in a crossover experiment that includes direct and first-order carryover treatment effects, together with an auto correlated error term, was considered by Mathews (1987) when there are two treatments. A method for generating designs that minimize the variance of the estimated treatment effects was proposed. Afsarinejad (1990) gave methods for construction of circular balanced minimal RMDs with minimum number of units for the case when v is prime number provided some basic divisibility conditions hold. Kunert (1991) derived optimal two-treatment crossover designs for experiments with many periods, when the observations are correlated and first order residual effects are present. These designs are found to be highly efficient when the number of periods is large. Martin and Eccleston (1998) obtained some variance-balanced changeover designs from orthogonal arrays for many dependence structures. William and John (2007) have constructed some crossover designs with correlated errors.

1.4.2 National Status

- **Designs with Correlated Errors**

Dey and Chakravarty (1977) have obtained some classes of incomplete neighbour designs using the method of symmetric differences. Gill and Shukla (1985) studied the efficiency of block designs for auto-regressive error structure. Das (1991) have studied the optimality of block designs under heteroscedastic (unequal variances) model. Das *et al.* (1992) studied E-optimal block designs under heteroscedastic model. Shukla and Gill (1986) and Shukla (1994) have presented a detailed review of use of spatial information in the analysis of field experiments. A catalogue of neighbour balanced designs has been prepared in Parsad *et al.* (2004). Satpati (2006) developed the algorithm for computer aided search of efficient designs for various experimental settings, like block designs, nested block designs and change-over designs whenever the observations are dependent. Computer aided search of efficient nested incomplete block designs for correlated errors was done by Satpati *et al.* (2006). Satpati *et al.* (2007) have obtained the lower bound to the A-efficiency of block designs under correlated error structure and obtained efficient block designs for dependent observations through a computer-aided search.

Jaggi *et al.* (2008) studied experimental designs for the estimation of treatments when the observations are spatially correlated. Some methods of constructing block designs, block designs with neighbour effects and row-column designs have been developed for this situation. The performance of some existing designs has been studied under various correlation structures.

- **Designs with Systematic Trend**

Dhall (1986) studied the trend free incomplete block designs and prepared a catalogue of trend free BIB designs. Some trend free group divisible partially balanced incomplete block designs have also been given. Lal *et al.* (2005) obtained the condition for a general block design to be trend free under heteroscedastic model and prepared the catalogue of binary variance balanced block, balanced incomplete block and partially balanced incomplete block designs that are trend free. Sarkar (2008) studied the linear trend free designs under factorial setup and has given the computer aided search of these designs.

- **Designs with Neighbour/Interference Effects**

Subramanyam (1991) used 16 units in a sequence for four treatments. To reduce the number of triplets, triplets of the type AAA and ABA or BAB were used. Of the v^2 treatments, ${}^{(v+1)}C_2$ were chosen and arranged in v blocks, each comprising of $3({}^{v+1}C_2)$ units. Gill (1993) studied the design and analysis of field experiments in the presence of local and remote treatment effects. The design criteria considered were optimality and balance for estimating local and remote effects. Bhaumik (1995) has studied optimality in the competing effect model in the class of binary designs and has shown that for estimating the treatments and competing effects together, a 3-design is universally optimal. Kumar (1995) has also investigated the design and analysis of experiments for investigating competition effects among neighbouring units. The treatments were arranged in the form of sequences, each treatment being used both for estimating test treatment effect as well as the neighbour effects. The construction and analysis of designs based on these sequences making use of a factorial nature has been studied.

Jaggi and Gupta (2003) investigated the problem of competition in agricultural experiments and have obtained some complete/incomplete balanced and partially balanced block designs for these situations. They also studied the optimality of block designs with competition effects both in complete and incomplete block settings under fixed effects model and obtained some optimal block designs. Tomar (2003) has developed some methods of constructing block designs balanced for neighbouring competition effects.

Tomar *et al.* (2005) studied the problem of competition and obtained some methods of constructing totally balanced incomplete block designs for competition effects. Jaggi and Tomar (2005) investigated the robustness of neighbour balanced complete block designs when the observations within a block are correlated. The A-efficiency for direct and neighbour effects of the treatments under AR(1) and NN correlation structure has been computed and tabulated for some cases under one-sided neighbour effects and two-sided neighbour effects when generalized least square estimation is used. Jaggi *et al.* (2006) have obtained some series of block designs partially balanced for neighbouring competition effects. Pateria (2006) obtained various classes of neighbour balanced designs and studied their optimal properties under fixed/ mixed effects model.

Jaggi *et al.* (2007) studied optimal complete block designs for neighbouring competition effects. Tomar (2007) studied some aspects of neighbour balanced block designs for correlated observations. Pateria *et al.* (2007) proposed a series of incomplete non-circular block designs for competition effects.

Sarika (2008) studied response surface designs incorporating neighbour effects. Sarika *et al.* (2008a) studied first order response surface model with neighbour effects from immediate left and right neighbouring units and the conditions have been derived for the orthogonal estimation of coefficients of this model. The variance of estimated response has also been obtained and conditions for first order response surface model with neighbour effects to be rotatable have been obtained. A method of obtaining designs satisfying the derived conditions was given. Second order response surface model in which the experimental units experience the neighbour effects has also been studied by Sarika *et al.* (2008b). Abeynayake (2008) studied neighbour balanced designs for making test treatments vs control comparisons under two-sided neighbour effect model and has discussed the robustness of balanced block designs against missing observation.

- **Designs with Carryover Effects**

Extra period designs have more significant use when higher precision is required for the estimation of carryover or residual and cumulative effects. Subsequently, these designs were studied by Saha (1970). He also obtained some classes of repeated measurements designs with number of periods more than two for estimating treatment × period interaction by taking a complete set of mutually orthogonal Latin squares. Sharma (1975) gave a simple method of construction of Williams Latin Square Designs balanced for the first order residual effects of treatments. Dey and Balachandran (1976) obtained a class of totally balanced designs by making designs circular by adding a pre-period consisting of the treatments in the last period. For v treatments a class of Repeated Measurements Designs (RMDs) balanced for first residual effects was given by Sharma (1981) using v experimental units and 2v periods. Sharma (1982) gave designs by repeating the treatments of the last period in the added extra period. Chawla (1983) developed some series of two factor symmetrical and asymmetrical split type factorial RMDs. Bora (1984, 1985) obtained the necessary and sufficient conditions for the designs to be balanced in presence of auto-correlated errors. Yadav (1990) obtained some methods of construction of control balanced RMDs.

Vijaya (1992) investigated two-treatment RMDs in the presence of time trends. Varghese and Sharma (2000) constructed a series of totally balanced RMDs considering the presence of first order residual effects of treatments with parameters (v, p = 2v-1, n = v) and compared their efficiencies with some of the existing designs. Sharma *et al.* (2002) established universal optimality of circular balanced change-over designs allowing the estimation of first and second order residual effects of treatments under an additive fixed effects model. A class of circular balanced change-over designs with parameters (v, p = 3v, n = v^2) has been shown to be universally optimal for the estimation of direct, first order as well as second order residual effects.

Sharma *et al*. (2003) gave a general method of construction of minimal balanced RMDs for odd number of treatments with parameters (v, p = (v+1)/2, n = 2v) along with an outline for their analysis. These designs are basically partially balanced designs for estimating direct and residual treatment effects. Bhattacharyya (2006) obtained efficient two-treatment RMDs when the errors are auto-correlated. A class of two-period totally balanced trend-free RMDs considering presence of first order residual effects has been developed by Gharde (2007). Gharde (2007) obtained two new classes of minimal strongly balanced RMDs assuming the presence of first order residual effects useful for the situations where experimental units or periods are scarce.

1.5 Scope of the Study

Proper planning, designing and analysis of experiments for testing the significance of treatment comparisons under different situations, results in drawing correct and valid inferences about the treatment effects. One of the requirements in the designing and analysis of data from comparative experiments is that the observations are independent. However, there may be experimental situations wherein the response from a unit may be affected by other units spatially or temporally belonging to the same cluster or group or block. It may be that the responses are correlated or that there is a systematic trend, or the response is affected by carryover effect from the units previously observed, or a neighbour effect from contiguous units. Under this situation, it is important that these indirect effects be included in the model to have the proper specification. The project aimed to study designs in the presence of indirect effects of treatments for different experimental situations. Designs considering more than one relationship between observations on units over time or space would be studied. For easy accessibility of these designs by the experimenters, it was required that these designs are compiled and presented at one place. Web-enabled useful classes of designs in the presence of indirect effects of treatments were required to be generated. An online catalogue of designs incorporating indirect effects of treatments has to be developed. The analytical procedures for designs in presence of indirect effects using live/ simulated data sets have to be illustrated.

The project report is summarized in six chapters. An introduction to indirect effects and a brief review of work done in the area of experimental designs in the presence of indirect effects is given in the first chapter. Chapter II deals with several classes of design developed which are balanced/optimal in the presence of spatial indirect effects. Chapter III explains methodology as well as the different classes of designs developed which are balanced/optimal for spatial indirect effects in the presence of systematic trend. Several classes of Row-columns designs suitable for tackling the problem of spatial indirect effects are given in Chapter IV. Chapter V gives a methodology as well as the designs developed for experimental situations where temporal indirect effect is suspected and also gives a class of designs which are balanced with respect to both spatial and temporal indirect effects. A web solution named Web Generation of Designs Balanced for Indirect Effects Treatments (*Web DBIE*) developed for the generation of designs balanced for spatial/temporal indirect effects is explained in details in Chapter VI. The report ends with a summary and references.

Chapter II

BLOCK DESIGNS BALANCED FOR SPATIAL INDIRECT EFFECTS

2.1 Block Designs Balanced for Spatial Indirect Effects from Neighbouring Experimental Units at Distance 2

The spatial indirect effects i.e. the interference effects may not only arise from the treatments applied to the immediate neighbouring units but also from the treatments applied to the units at higher distance. Hence, interference may not be restricted to immediately adjacent units but may extend further, as with the spread of inoculum in disease screening trials. Block designs balanced for interference effects at higher distance are thus needed. Iqbal *et al.* (2006), Mingyao *et al.* (2007), Akhtar and Ahmed (2009) etc. highlighted some aspects of block designs with interference effects from the neighbouring units at higher distance.

Here, we have considered block model with interference effects arising from neighbouring units on both sides (left and right) at distance 2. The case of one-sided (say, left) interference effects from the neighbouring units up to distance 2 has been considered as a particular case. Methods of constructing series of complete/ incomplete block designs balanced for interference effects up to distance 2 have been discussed and their characterization properties have been investigated.

2.1.1 Experimental Setup and Model

We consider a class of block designs with v treatments, b blocks and n experimental units. The size of the j^{th} (j = 1, 2,…, b) block is k_j and the s^{th} (s = 1, 2, …, v) treatment is replicated r_s times. Let Y_{ij} be the response from the i^{th} plot (i = 1, 2,…, k_j) in the j^{th} block. It is assumed that the experiment is conducted in small plots in well separated blocks with no guard areas between the plots in a block. The blocks are circular i.e., the treatment on the immediate left border plot is same as the treatment on the right end inner plot of the block and treatment on the left border plot at distance 2 (leaving one plot from the first plot of the block) is same as the treatment on the second last inner plot from the right side. Similarly, treatments on the immediate right border plot is same as the treatment on the left end inner plot of the block and treatment on the right border plot at distance 2 (leaving one plot from the last plot of the block) is same as the treatment on the second last inner plot from the left side. Following fixed effects additive model is considered for analyzing a block design with interference effects up to distance 2:

$$\mathbf{Y} = \mu\mathbf{1} + \mathbf{\Delta}'\boldsymbol{\tau} + \mathbf{\Delta}_1'\boldsymbol{\delta} + \mathbf{\Delta}_2'\boldsymbol{\gamma} + \mathbf{\Delta}_3'\boldsymbol{\alpha} + \mathbf{\Delta}_4'\boldsymbol{\eta} + \mathbf{D}'\boldsymbol{\beta} + \mathbf{e}, \qquad \dots (2.1.1)$$

where $\mathbf{Y}$ is a $n \times 1$ vector of observations, μ is the general mean, $\mathbf{1}$ is a $n \times 1$ vector of ones, $\mathbf{\Delta}'$ is a $n \times v$ matrix of observations versus direct treatments, $\boldsymbol{\tau}$ is a $v \times 1$ vector of direct treatment effects, $\mathbf{\Delta}_1'$ is a n × v matrix of observations versus interference effect from treatment on the immediate left neighbour units i.e. treatment at distance 1, $\boldsymbol{\delta}$ is v × 1 vector of left neighbour interference effects at distance 1, $\mathbf{\Delta}_2'$ is a n × v incidence matrix of observations versus interference effect from treatment on the immediate right neighbour units i.e. treatment at distance 1, $\boldsymbol{\gamma}$ is v × 1 right neighbour interference effects at distance 1, $\mathbf{\Delta}_3'$ is a n × v incidence matrix of observations versus interference effect from left neighbour treatments at distance 2 (leaving one plot) $\boldsymbol{\alpha}$ is v × 1 vector of left interference effects at distance 2, $\mathbf{\Delta}_4'$ is a n × v incidence matrix of observations versus interference effect from right neighbour treatments at distance 2, $\boldsymbol{\eta}$ is v × 1 vector of right neighbour interference effects at distance 2, $\mathbf{D}'$ is a n × b incidence matrix of observations versus blocks, $\boldsymbol{\beta}$ is a b × 1 vector of block effects and $\mathbf{e}$ is a n × 1 vector of errors with $\mathbf{e} \sim N(\mathbf{0}, \sigma^2 \mathbf{I}_n)$.

Let,

$\mathbf{\Delta\Delta}_1' = \mathbf{M}_1$, v × v incidence matrix of direct treatments versus immediate left neighbour treatments,

$\mathbf{\Delta\Delta}_2' = \mathbf{M}_2$, v × v incidence matrix of direct treatments versus immediate right neighbour treatments

$\mathbf{\Delta}_1\mathbf{\Delta}_2' = \mathbf{M}_3$, v × v incidence matrix of immediate left neighbour treatments versus immediate right neighbour treatments,

$\mathbf{\Delta\Delta}_3' = \mathbf{M}_4$, v × v incidence matrix of direct treatments versus left neighbour treatments at distance 2,

$\mathbf{\Delta\Delta}_4' = \mathbf{M}_5$, v × v incidence matrix of direct treatments versus right neighbour treatments at distance 2,

$\mathbf{\Delta}_1\mathbf{\Delta}_3' = \mathbf{M}_6$, v × v incidence matrix of immediate left neighbour treatments versus left neighbour treatments at distance 2,

$\mathbf{\Delta}_1\mathbf{\Delta}_4' = \mathbf{M}_7$, v × v incidence matrix of immediate left neighbour treatments versus right neighbour treatments at distance 2,

$\mathbf{\Delta}_2\mathbf{\Delta}_3' = \mathbf{M}_8$, v × v incidence matrix of immediate right neighbour treatments versus left neighbour treatments at distance 2,

$\mathbf{\Delta}_2\mathbf{\Delta}_4' = \mathbf{M}_9$, v × v incidence matrix of immediate right neighbour treatments versus right neighbour treatments at distance 2,

$\mathbf{\Delta}_3\mathbf{\Delta}_4' = \mathbf{M}_{10}$, v × v incidence matrix of left neighbour treatments at distance 2 versus right neighbour treatments at distance 2,

$\mathbf{\Delta D}' = \mathbf{N}_1$, v × b incidence matrix of direct treatments versus blocks,

$\mathbf{\Delta}_1\mathbf{D}' = \mathbf{N}_2$, v × b incidence matrix of immediate left neighbour treatments versus blocks,

$\mathbf{\Delta}_2\mathbf{D}' = \mathbf{N}_3$, v × b incidence matrix of immediate right neighbour treatments versus blocks,

$\mathbf{\Delta}_3\mathbf{D}' = \mathbf{N}_4$, v × b incidence matrix of left neighbour treatments at distance 2 versus blocks,

$\mathbf{\Delta}_4\mathbf{D}' = \mathbf{N}_5$, v × b incidence matrix of right neighbour treatments at distance 2 versus blocks,

$\mathbf{r} = (r_1, r_2, ..., r_v)'$ be the v × 1 replication vector of direct treatments with r_s (s = 1, 2, …,v) being the number of times the s^{th} treatment appears in the design,

$\mathbf{r}_1 = (r_{11}, r_{12}, ..., r_{1v})'$ be the v × 1 replication vector of the immediate left neighbour treatments with r_{1s} being the number of times the treatments in the design has s^{th} treatment as immediate left neighbour,

$\mathbf{r}_2 = (r_{21}, r_{22}, ..., r_{2v})'$ be the v × 1 replication vector of the immediate right neighbour treatments with r_{2s} being the number of times the treatments in the design has s^{th} treatment as immediate right neighbour,

$\mathbf{r}_3 = (r_{31}, r_{32}, ..., r_{3v})'$ be the v × 1 replication vector of the left neighbour treatments at distance 2 with r_{3s} being the number of times the treatments in the design has s^{th} treatment as left neighbour at distance 2,

$\mathbf{r}_4 = (r_{41}, r_{42}, ..., r_{4v})'$ be the v × 1 replication vector of the right neighbour treatments at distance 2 with r_{4s} being the number of times the treatments in the design has s^{th} treatment as right neighbour at distance 2,

$\mathbf{k} = (k_1, k_2, ..., k_b)'$ be the b × 1 replication vector of the block sizes

$\mathbf{R}_\tau$ = diag (r_1, r_2, … , r_v), $\mathbf{R}_\delta$ = diag (r_{11},r_{12}, … , r_{1v}), $\mathbf{R}_\gamma$ = diag (r_{21},r_{22}, … , r_{2v}), $\mathbf{R}_\alpha$ = diag (r_{31},r_{32}, … , r_{3v}), $\mathbf{R}_\eta$ = diag (r_{41},r_{42}, … , r_{4v}), $\mathbf{K}$ = diag (k_1, k_2, …, k_b),

$\mathbf{T} = [T_1, T_2, ..., T_v]'$, where T_s is the total of s^{th} direct treatments,

$\mathbf{L}_1 = [L_{11}, L_{12}, ..., L_{1v}]'$, where L_{1s} is the total of all the treatments that have s^{th} treatment as immediate left neighbour,

$\mathbf{R}_1 = [R_{11}, R_{12}, ..., R_{1v}]'$, where R_{1s} is the total of all the treatments that have s^{th} treatment as immediate right neighbour,

$\mathbf{L}_2 = [L_{21}, L_{22}, ..., L_{2v}]'$, where L_{2s} is the total of all the treatments that have s^{th} treatment as left neighbour at distance 2,

$\mathbf{R}_2 = [R_{21}, R_{22}, ..., R_{2v}]'$, where R_{2s} is the total of all the treatments that have s^{th} treatment as right neighbour at distance 2,

$\mathbf{B} = [B_1, B_2, ..., B_b]'$, where B_j is the j^{th} block total and **G** = Grand total.

The above model is rewritten as follows by writing parameter of interest first:

$$\mathbf{Y} = \mathbf{\Delta}'\boldsymbol{\tau} + \mathbf{\Delta}_1'\boldsymbol{\delta} + \mathbf{\Delta}_2'\boldsymbol{\gamma} + \mathbf{\Delta}_3'\boldsymbol{\alpha} + \mathbf{\Delta}_4'\boldsymbol{\eta} + \mu\mathbf{1} + \mathbf{D}'\boldsymbol{\beta} + \mathbf{e} \qquad \text{… (2.1.2)}$$

This can also be written as:

$$\mathbf{Y} = \mathbf{X}_1\boldsymbol{\theta}_1 + \mathbf{X}_2\boldsymbol{\theta}_2 + \mathbf{e},$$

where

$$\mathbf{X}_1 = \begin{bmatrix} \boldsymbol{\Delta}' & \boldsymbol{\Delta}_1' & \boldsymbol{\Delta}_2' & \boldsymbol{\Delta}_3' & \boldsymbol{\Delta}_4' \end{bmatrix}, \ \mathbf{X}_2 = \begin{bmatrix} \mathbf{1} & \mathbf{D}' \end{bmatrix},$$

$$\boldsymbol{\theta}_1 = \begin{bmatrix} \boldsymbol{\tau}' & \boldsymbol{\delta}' & \boldsymbol{\gamma}' & \boldsymbol{\alpha}' & \boldsymbol{\eta}' \end{bmatrix}'$$

and $\boldsymbol{\theta}_2 = \begin{bmatrix} \mu & \boldsymbol{\beta}' \end{bmatrix}'$.

Minimization of the residual sum of squares with respect to $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$ leads to the normal equations:

$$\begin{aligned} \mathbf{X}_1'\mathbf{X}_1\hat{\boldsymbol{\theta}}_1 + \mathbf{X}_1'\mathbf{X}_2\hat{\boldsymbol{\theta}}_2 &= \mathbf{X}_1'\mathbf{Y} \\ \mathbf{X}_2'\mathbf{X}_1\hat{\boldsymbol{\theta}}_1 + \mathbf{X}_2'\mathbf{X}_2\hat{\boldsymbol{\theta}}_2 &= \mathbf{X}_2'\mathbf{Y} \end{aligned} \qquad \ldots (2.1.3)$$

Solving the above equations, reduced normal equations for vector of parameters $\boldsymbol{\theta}_1$ is $\mathbf{C}\,\hat{\boldsymbol{\theta}}_1 = \mathbf{Q}$, where

$$\mathbf{C} = \mathbf{X}_1'\mathbf{X}_1 - \mathbf{X}_1'\mathbf{X}_2(\mathbf{X}_2'\mathbf{X}_2)^{-}\mathbf{X}_2'\mathbf{X}_1, \qquad \ldots (2.1.4)$$

is the information matrix of $\boldsymbol{\theta}_1$,

$$\mathbf{Q} = \mathbf{X}_1'\mathbf{Y} - \mathbf{X}_1'\mathbf{X}_2(\mathbf{X}_2'\mathbf{X}_2)^{-}\mathbf{X}_2'\mathbf{Y}, \qquad \ldots (2.1.5)$$

is the vector of adjusted treatment totals. $(\mathbf{X}_2'\mathbf{X}_2)^{-}$ is a g-inverse of $(\mathbf{X}_2'\mathbf{X}_2)$ such that $\mathbf{X}_2'\mathbf{X}_2(\mathbf{X}_2'\mathbf{X}_2)^{-}\mathbf{X}_2'\mathbf{X}_2 = \mathbf{X}_2'\mathbf{X}_2$.

Here,

$$\mathbf{X}_1'\mathbf{X}_1 = \begin{bmatrix} \boldsymbol{\Delta}\boldsymbol{\Delta}' & \boldsymbol{\Delta}\boldsymbol{\Delta}_1' & \boldsymbol{\Delta}\boldsymbol{\Delta}_2' & \boldsymbol{\Delta}\boldsymbol{\Delta}_3' & \boldsymbol{\Delta}\boldsymbol{\Delta}_4' \\ \boldsymbol{\Delta}_1\boldsymbol{\Delta}' & \boldsymbol{\Delta}_1\boldsymbol{\Delta}_1' & \boldsymbol{\Delta}_1\boldsymbol{\Delta}_2' & \boldsymbol{\Delta}_1\boldsymbol{\Delta}_3' & \boldsymbol{\Delta}_1\boldsymbol{\Delta}_4' \\ \boldsymbol{\Delta}_2\boldsymbol{\Delta}' & \boldsymbol{\Delta}_2\boldsymbol{\Delta}_1' & \boldsymbol{\Delta}_2\boldsymbol{\Delta}_2' & \boldsymbol{\Delta}_2\boldsymbol{\Delta}_3' & \boldsymbol{\Delta}_2\boldsymbol{\Delta}_4' \\ \boldsymbol{\Delta}_3\boldsymbol{\Delta}' & \boldsymbol{\Delta}_3\boldsymbol{\Delta}_1' & \boldsymbol{\Delta}_3\boldsymbol{\Delta}_2' & \boldsymbol{\Delta}_3\boldsymbol{\Delta}_3' & \boldsymbol{\Delta}_3\boldsymbol{\Delta}_4' \\ \boldsymbol{\Delta}_4\boldsymbol{\Delta}' & \boldsymbol{\Delta}_4\boldsymbol{\Delta}_1' & \boldsymbol{\Delta}_4\boldsymbol{\Delta}_2' & \boldsymbol{\Delta}_4\boldsymbol{\Delta}_3' & \boldsymbol{\Delta}_4\boldsymbol{\Delta}_4' \end{bmatrix} = \begin{bmatrix} \mathbf{R}_\tau & \mathbf{M}_1 & \mathbf{M}_2 & \mathbf{M}_4 & \mathbf{M}_5 \\ \mathbf{M}_1' & \mathbf{R}_\delta & \mathbf{M}_3 & \mathbf{M}_6 & \mathbf{M}_7 \\ \mathbf{M}_2' & \mathbf{M}_3' & \mathbf{R}_\gamma & \mathbf{M}_8 & \mathbf{M}_9 \\ \mathbf{M}_4' & \mathbf{M}_6' & \mathbf{M}_8' & \mathbf{R}_\alpha & \mathbf{M}_{10} \\ \mathbf{M}_5' & \mathbf{M}_7' & \mathbf{M}_9' & \mathbf{M}_{10}' & \mathbf{R}_\eta \end{bmatrix},$$

$$\mathbf{X}_1'\mathbf{X}_2 = \begin{bmatrix} \boldsymbol{\Delta}\mathbf{1} & \boldsymbol{\Delta}\mathbf{D}' \\ \boldsymbol{\Delta}_1\mathbf{1} & \boldsymbol{\Delta}_1\mathbf{D}' \\ \boldsymbol{\Delta}_2\mathbf{1} & \boldsymbol{\Delta}_2\mathbf{D}' \\ \boldsymbol{\Delta}_3\mathbf{1} & \boldsymbol{\Delta}_3\mathbf{D}' \\ \boldsymbol{\Delta}_4\mathbf{1} & \boldsymbol{\Delta}_4\mathbf{D}' \end{bmatrix} = \begin{bmatrix} \mathbf{r} & \mathbf{N}_1 \\ \mathbf{r}_1 & \mathbf{N}_2 \\ \mathbf{r}_2 & \mathbf{N}_3 \\ \mathbf{r}_3 & \mathbf{N}_4 \\ \mathbf{r}_4 & \mathbf{N}_5 \end{bmatrix} \text{ and } \mathbf{X}_2'\mathbf{X}_2 = \begin{bmatrix} \mathbf{1}'\mathbf{1} & \mathbf{1}'\mathbf{D}' \\ \mathbf{D}\mathbf{1} & \mathbf{D}\mathbf{D}' \end{bmatrix} = \begin{bmatrix} n & \mathbf{k}' \\ \mathbf{k} & \mathbf{K} \end{bmatrix}.$$

Thus, $(\mathbf{X}_2'\mathbf{X}_2)^{-} = \begin{bmatrix} 0 & \mathbf{0}' \\ \mathbf{0} & \mathbf{K}^{-1} \end{bmatrix}$.

The 5v × 5v symmetric, nonnegative definite joint information matrix for estimating the direct effects of treatment and interference effects from the neighbouring units up to distance 2 is obtained as:

$$\mathbf{C} = \begin{bmatrix} \mathbf{R}_\tau - \mathbf{N}_1\mathbf{K}^{-1}\mathbf{N}_1' & \mathbf{M}_1 - \mathbf{N}_1\mathbf{K}^{-1}\mathbf{N}_2' & \mathbf{M}_2 - \mathbf{N}_1\mathbf{K}^{-1}\mathbf{N}_3' & \mathbf{M}_4 - \mathbf{N}_1\mathbf{K}^{-1}\mathbf{N}_4' & \mathbf{M}_5 - \mathbf{N}_1\mathbf{K}^{-1}\mathbf{N}_5' \\ \mathbf{M}_1' - \mathbf{N}_2\mathbf{K}^{-1}\mathbf{N}_1' & \mathbf{R}_\delta - \mathbf{N}_2\mathbf{K}^{-1}\mathbf{N}_2' & \mathbf{M}_3 - \mathbf{N}_2\mathbf{K}^{-1}\mathbf{N}_3' & \mathbf{M}_6 - \mathbf{N}_2\mathbf{K}^{-1}\mathbf{N}_4' & \mathbf{M}_7 - \mathbf{N}_2\mathbf{K}^{-1}\mathbf{N}_5' \\ \mathbf{M}_2' - \mathbf{N}_3\mathbf{K}^{-1}\mathbf{N}_1' & \mathbf{M}_3' - \mathbf{N}_3\mathbf{K}^{-1}\mathbf{N}_2' & \mathbf{R}_\gamma - \mathbf{N}_3\mathbf{K}^{-1}\mathbf{N}_3' & \mathbf{M}_8 - \mathbf{N}_3\mathbf{K}^{-1}\mathbf{N}_4' & \mathbf{M}_9 - \mathbf{N}_3\mathbf{K}^{-1}\mathbf{N}_5' \\ \mathbf{M}_4' - \mathbf{N}_4\mathbf{K}^{-1}\mathbf{N}_1' & \mathbf{M}_6' - \mathbf{N}_4\mathbf{K}^{-1}\mathbf{N}_2' & \mathbf{M}_8' - \mathbf{N}_4\mathbf{K}^{-1}\mathbf{N}_3' & \mathbf{R}_\alpha - \mathbf{N}_4\mathbf{K}^{-1}\mathbf{N}_4' & \mathbf{M}_{10} - \mathbf{N}_4\mathbf{K}^{-1}\mathbf{N}_5' \\ \mathbf{M}_5' - \mathbf{N}_5\mathbf{K}^{-1}\mathbf{N}_1' & \mathbf{M}_7' - \mathbf{N}_5\mathbf{K}^{-1}\mathbf{N}_2' & \mathbf{M}_9' - \mathbf{N}_5\mathbf{K}^{-1}\mathbf{N}_3' & \mathbf{M}_{10}' - \mathbf{N}_5\mathbf{K}^{-1}\mathbf{N}_4' & \mathbf{R}_\eta - \mathbf{N}_5\mathbf{K}^{-1}\mathbf{N}_5' \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{C}_{11} & \mathbf{C}_{12} \\ \mathbf{C}_{21} & \mathbf{C}_{22} \end{bmatrix}, \qquad \ldots (2.1.6)$$

where

$$\mathbf{C}_{11} = \mathbf{R}_\tau - \mathbf{N}_1\mathbf{K}^{-1}\mathbf{N}_1',$$

$$\mathbf{C}_{12} = \begin{bmatrix} \mathbf{M}_1 - \mathbf{N}_1\mathbf{K}^{-1}\mathbf{N}_2' & \mathbf{M}_2 - \mathbf{N}_1\mathbf{K}^{-1}\mathbf{N}_3' & \mathbf{M}_4 - \mathbf{N}_1\mathbf{K}^{-1}\mathbf{N}_4' & \mathbf{M}_5 - \mathbf{N}_1\mathbf{K}^{-1}\mathbf{N}_5' \end{bmatrix},$$

$$\mathbf{C}_{22} = \begin{bmatrix} \mathbf{R}_\delta - \mathbf{N}_2\mathbf{K}^{-1}\mathbf{N}_2' & \mathbf{M}_3 - \mathbf{N}_2\mathbf{K}^{-1}\mathbf{N}_3' & \mathbf{M}_6 - \mathbf{N}_2\mathbf{K}^{-1}\mathbf{N}_4' & \mathbf{M}_7 - \mathbf{N}_2\mathbf{K}^{-1}\mathbf{N}_5' \\ \mathbf{M}_3' - \mathbf{N}_3\mathbf{K}^{-1}\mathbf{N}_2' & \mathbf{R}_\gamma - \mathbf{N}_3\mathbf{K}^{-1}\mathbf{N}_3' & \mathbf{M}_8 - \mathbf{N}_3\mathbf{K}^{-1}\mathbf{N}_4' & \mathbf{M}_9 - \mathbf{N}_3\mathbf{K}^{-1}\mathbf{N}_5' \\ \mathbf{M}_6' - \mathbf{N}_4\mathbf{K}^{-1}\mathbf{N}_2' & \mathbf{M}_8' - \mathbf{N}_4\mathbf{K}^{-1}\mathbf{N}_3' & \mathbf{R}_\alpha - \mathbf{N}_4\mathbf{K}^{-1}\mathbf{N}_4' & \mathbf{M}_{10} - \mathbf{N}_4\mathbf{K}^{-1}\mathbf{N}_5' \\ \mathbf{M}_7' - \mathbf{N}_5\mathbf{K}^{-1}\mathbf{N}_2' & \mathbf{M}_9' - \mathbf{N}_5\mathbf{K}^{-1}\mathbf{N}_3' & \mathbf{M}_{10}' - \mathbf{N}_5\mathbf{K}^{-1}\mathbf{N}_4' & \mathbf{R}_\eta - \mathbf{N}_5\mathbf{K}^{-1}\mathbf{N}_5' \end{bmatrix}.$$

The information matrix for estimating the direct effects of treatment can be obtained as follows:

$$\mathbf{C}_\tau = \mathbf{C}_{11} - \mathbf{C}_{12}\mathbf{C}_{22}^{-}\mathbf{C}_{21} \qquad \ldots (2.1.7)$$

where $\mathbf{C}^{-}_{22}$ is a g-inverse of $\mathbf{C}_{22}$. Similarly the information matrix for estimating the interference effects from the neighbouring units up to distance 2 can be obtained.

From equation (2.1.5), the vector of adjusted treatment totals can be obtained as follows:

$$\mathbf{Q} = \begin{bmatrix} \mathbf{T} - \mathbf{N}_1\mathbf{K}^{-1}\mathbf{B} \\ \mathbf{L}_1 - \mathbf{N}_2\mathbf{K}^{-1}\mathbf{B} \\ \mathbf{R}_1 - \mathbf{N}_3\mathbf{K}^{-1}\mathbf{B} \\ \mathbf{L}_2 - \mathbf{N}_4\mathbf{K}^{-1}\mathbf{B} \\ \mathbf{R}_2 - \mathbf{N}_5\mathbf{K}^{-1}\mathbf{B} \end{bmatrix}. \qquad \ldots (2.1.8)$$

Following are some general definitions associated with block designs with interference effects from the neighbouring units up to distance 2:

Definition 2.1.1: A block design is *balanced for interference effects* from the neighbouring units up to distance 2, if every treatment has every other treatment as both left and right neighbour up to distance 2 constant number of times (say, μ_1). Further, a block design with both sided interference effects is *strongly balanced* if each treatment has every treatment including itself as both left and right neighbours up to distance 2 a constant number of times (say μ_2). μ_1 and μ_2 may or may not be equal

Definition 2.1.2: A block design with interference effects from neighbouring units up to distance 2 is called *variance balanced* if the variance of any estimated elementary contrast among the direct effects is constant (say V_1), the variance of any estimated elementary contrast among the interference effects arising from the immediate left neighbouring units is constant (say V_2), the variance of any estimated elementary contrast among the interference effects arising from the immediate right neighbouring units is constant (say V_3), the variance of any estimated elementary contrast among the interference effects arising from the left neighbouring units at distance 2 is constant (say V_4) and the variance of any estimated elementary contrast among the interference effects arising from the right neighbouring units at distance 2 is constant (say V_5). A block design is *totally balanced* if $V_1 = V_2 = V_3 = V_4 = V_5$.

2.1.2 Block Designs Balanced for Interference Effects

In this section, some methods of constructing complete and incomplete circular balanced and strongly balanced block designs with interference effects from the neighbouring units up to distance 2 have been described.

Method 2.1.2.1: Let there be v (prime) treatments labeled as 0, 1, 2, …, v-1. A series of complete circular block design strongly balanced for interference effects up to distance 2 can be obtained by developing the blocks of the design as follows for all q = 0, 1, …, (v-1) and p = 1, 2, …, (v-1)/2:

q, q + p, q + 2p,…, q + (v-2)p, q + (v-1)p, q + (v-2)p,…, q + 2p, q + p, q, (modulo v)

The parameters of the designs are v, b = v(v-1)/2, r = (v-1)(2v-1)/2, k = 2v-1, μ_1 = v-1, and μ_2 = (v-1)/2.

For this class of designs,

$$\mathbf{R}_\tau = \mathbf{R}_\delta = \mathbf{R}_\gamma = \mathbf{R}_\alpha = \mathbf{R}_\eta = \frac{(\text{v-1})(2\text{v}-1)}{2}\mathbf{I}_\text{v}, \mathbf{K} = \text{k}\,\mathbf{I}_\text{b} = (2\text{v}-1)\mathbf{I}_\text{b},$$

$$\mathbf{M}_1 = \mathbf{M}_2 = \mathbf{M}_3 = \mathbf{M}_4 = \mathbf{M}_5 = \mathbf{M}_6 = \mathbf{M}_7 = \mathbf{M}_8 = \mathbf{M}_9 = \mathbf{M}_{10} = \frac{(\text{v-1})}{2}(2\mathbf{11}' - \mathbf{I}_\text{v}),$$

$$\mathbf{N}_\text{u}\mathbf{N}'_{\text{u}'} = \frac{(\text{v-1})}{2}\left[\mathbf{I}_\text{v} + 4(\text{v-1})\mathbf{11}'\right], \text{u, u}' = 1, 2, \ldots 5 \qquad \ldots (2.1.9)$$

Hence, from equation (2.1.9), we can write:

$$\mathbf{R}_\tau - \mathbf{N}_1\mathbf{K}^{-1}\mathbf{N}_1' = \mathbf{R}_\delta - \mathbf{N}_2\mathbf{K}^{-1}\mathbf{N}_2' = \mathbf{R}_\gamma - \mathbf{N}_3\mathbf{K}^{-1}\mathbf{N}_3'$$
$$= \mathbf{R}_\alpha - \mathbf{N}_4\mathbf{K}^{-1}\mathbf{N}_4' = \mathbf{R}_\eta - \mathbf{N}_5\mathbf{K}^{-1}\mathbf{N}_5' = \frac{2\text{v}(\text{v-1})^2}{(2\text{v}-1)}\left[\mathbf{I}_\text{v} - \frac{\mathbf{11}'}{\text{v}}\right] \quad \ldots (2.1.10)$$

and

$$\mathbf{M}_1 - \mathbf{N}_1\mathbf{K}^{-1}\mathbf{N}_2' = \mathbf{M}_2 - \mathbf{N}_1\mathbf{K}^{-1}\mathbf{N}_3' = \mathbf{M}_3 - \mathbf{N}_2\mathbf{K}^{-1}\mathbf{N}_3' = \mathbf{M}_4 - \mathbf{N}_1\mathbf{K}^{-1}\mathbf{N}_4'$$
$$= \mathbf{M}_5 - \mathbf{N}_1\mathbf{K}^{-1}\mathbf{N}_5' = \mathbf{M}_6 - \mathbf{N}_2\mathbf{K}^{-1}\mathbf{N}_4' = \mathbf{M}_7 - \mathbf{N}_2\mathbf{K}^{-1}\mathbf{N}_5' = \mathbf{M}_8 - \mathbf{N}_3\mathbf{K}^{-1}\mathbf{N}_4' \;. \quad \ldots (2.1.11)$$
$$= \mathbf{M}_9 - \mathbf{N}_3\mathbf{K}^{-1}\mathbf{N}_5' = \mathbf{M}_{10} - \mathbf{N}_4\mathbf{K}^{-1}\mathbf{N}_5' = \frac{\text{v}(\text{v-1})}{(2\text{v}-1)}\left[\frac{\mathbf{11}'}{\text{v}} - \mathbf{I}_\text{v}\right]$$

Thus, from Equation (2.1.6), the joint information matrix for estimating the direct as well as interference effects from the neighbouring units up to distance 2 can be obtained as:

$$\mathbf{C} = \begin{bmatrix} \frac{2\text{v}(\text{v-1})^2}{(2\text{v}-1)}\left(\mathbf{I}_\text{v}-\frac{\mathbf{11}'}{\text{v}}\right) & \frac{\text{v}(\text{v-1})}{(2\text{v}-1)}\left(\frac{\mathbf{11}'}{\text{v}}-\mathbf{I}_\text{v}\right) & \frac{\text{v}(\text{v-1})}{(2\text{v}-1)}\left(\frac{\mathbf{11}'}{\text{v}}-\mathbf{I}_\text{v}\right) & \frac{\text{v}(\text{v-1})}{(2\text{v}-1)}\left(\frac{\mathbf{11}'}{\text{v}}-\mathbf{I}_\text{v}\right) & \frac{\text{v}(\text{v-1})}{(2\text{v}-1)}\left(\frac{\mathbf{11}'}{\text{v}}-\mathbf{I}_\text{v}\right) \\ \frac{\text{v}(\text{v-1})}{(2\text{v}-1)}\left(\frac{\mathbf{11}'}{\text{v}}-\mathbf{I}_\text{v}\right) & \frac{2\text{v}(\text{v-1})^2}{(2\text{v}-1)}\left(\mathbf{I}_\text{v}-\frac{\mathbf{11}'}{\text{v}}\right) & \frac{\text{v}(\text{v-1})}{(2\text{v}-1)}\left(\frac{\mathbf{11}'}{\text{v}}-\mathbf{I}_\text{v}\right) & \frac{\text{v}(\text{v-1})}{(2\text{v}-1)}\left(\frac{\mathbf{11}'}{\text{v}}-\mathbf{I}_\text{v}\right) & \frac{\text{v}(\text{v-1})}{(2\text{v}-1)}\left(\frac{\mathbf{11}'}{\text{v}}-\mathbf{I}_\text{v}\right) \\ \frac{\text{v}(\text{v-1})}{(2\text{v}-1)}\left(\frac{\mathbf{11}'}{\text{v}}-\mathbf{I}_\text{v}\right) & \frac{\text{v}(\text{v-1})}{(2\text{v}-1)}\left(\frac{\mathbf{11}'}{\text{v}}-\mathbf{I}_\text{v}\right) & \frac{2\text{v}(\text{v-1})^2}{(2\text{v}-1)}\left(\mathbf{I}_\text{v}-\frac{\mathbf{11}'}{\text{v}}\right) & \frac{\text{v}(\text{v-1})}{(2\text{v}-1)}\left(\frac{\mathbf{11}'}{\text{v}}-\mathbf{I}_\text{v}\right) & \frac{\text{v}(\text{v-1})}{(2\text{v}-1)}\left(\frac{\mathbf{11}'}{\text{v}}-\mathbf{I}_\text{v}\right) \\ \frac{\text{v}(\text{v-1})}{(2\text{v}-1)}\left(\frac{\mathbf{11}'}{\text{v}}-\mathbf{I}_\text{v}\right) & \frac{\text{v}(\text{v-1})}{(2\text{v}-1)}\left(\frac{\mathbf{11}'}{\text{v}}-\mathbf{I}_\text{v}\right) & \frac{\text{v}(\text{v-1})}{(2\text{v}-1)}\left(\frac{\mathbf{11}'}{\text{v}}-\mathbf{I}_\text{v}\right) & \frac{2\text{v}(\text{v-1})^2}{(2\text{v}-1)}\left(\mathbf{I}_\text{v}-\frac{\mathbf{11}'}{\text{v}}\right) & \frac{\text{v}(\text{v-1})}{(2\text{v}-1)}\left(\frac{\mathbf{11}'}{\text{v}}-\mathbf{I}_\text{v}\right) \\ \frac{\text{v}(\text{v-1})}{(2\text{v}-1)}\left(\frac{\mathbf{11}'}{\text{v}}-\mathbf{I}_\text{v}\right) & \frac{\text{v}(\text{v-1})}{(2\text{v}-1)}\left(\frac{\mathbf{11}'}{\text{v}}-\mathbf{I}_\text{v}\right) & \frac{\text{v}(\text{v-1})}{(2\text{v}-1)}\left(\frac{\mathbf{11}'}{\text{v}}-\mathbf{I}_\text{v}\right) & \frac{\text{v}(\text{v-1})}{(2\text{v}-1)}\left(\frac{\mathbf{11}'}{\text{v}}-\mathbf{I}_\text{v}\right) & \frac{2\text{v}(\text{v-1})^2}{(2\text{v}-1)}\left(\mathbf{I}_\text{v}-\frac{\mathbf{11}'}{\text{v}}\right) \end{bmatrix}$$

The information matrix for estimating the direct effects of treatment is

$$\mathbf{C}_\tau = \frac{2\text{v}(\text{v-1})(\text{v-3})}{(2\text{v}-5)}\left(\mathbf{I}_\text{v} - \frac{\mathbf{11}'}{\text{v}}\right),\ \text{v} > 3. \quad \ldots (2.1.12)$$

The variance of estimated elementary contrast pertaining to direct effects of treatments is thus obtained as

$$\text{V}\left(\hat{\tau}_\text{s}\text{-}\hat{\tau}_{\text{s}'}\right) = \text{V}_1 = \sigma^2\,\frac{(2\text{v}-5)}{\text{v}(\text{v-1})(\text{v-3})};\ \forall\,\text{s},\text{s}' = 1, 2, \ldots, \text{v} \quad \ldots (2.1.13)$$

Similarly, the information matrix for estimating the immediate left interference effects, immediate right interference effects, left interference effects of treatments at distance 2 and right interference effects at distance 2 from the neighbouring units are obtained as

$$\mathbf{C}_\delta = \mathbf{C}_\gamma = \mathbf{C}_\alpha = \mathbf{C}_\eta = \frac{2\text{v}(\text{v-1})(\text{v-3})}{(2\text{v}-5)}\left(\mathbf{I}_\text{v} - \frac{\mathbf{11}'}{\text{v}}\right),\ \text{v} > 3. \quad \ldots (2.1.14)$$

The design is thus *variance balanced* for estimating the contrast pertaining to direct effects of treatments and interference effects from the neighbouring units up to distance 2. Also, since $V_1 = V_2 = V_3 = V_4 = V_5$ the series obtained is *totally balanced* for estimating the contrasts pertaining to direct effects of treatments and interference effects up to distance 2.

Example 2.1.2.1: For v = 5, following is a strongly balanced complete block design with interference effects up to distance 2 from the neighbouring unit with v = 5, b = 10, r = 18, k = 9, $\mu_1 = 4$, $\mu_2 = 2$:

0	1	2	3	4	3	2	1	0
1	2	3	4	0	4	3	2	1
2	3	4	0	1	0	4	3	2
3	4	0	1	2	1	0	4	3
4	0	1	2	3	2	1	0	4
0	2	4	1	3	1	4	2	0
1	3	0	2	4	2	0	3	1
2	4	1	3	0	3	1	4	2
3	0	2	4	1	4	2	0	3
4	1	3	0	2	0	3	1	4

Remark 2.1.2.1: For the above class of design, when interference effects from only one side say, left neighbouring units are considered, the information matrices for estimating direct effects and information matrices for estimating interference effects from the neighbouring units up to distance 2 can be obtained as:

$$\mathbf{C}_\tau = \mathbf{C}_\delta = \mathbf{C}_\gamma = \frac{2\text{v}(\text{v-1})(\text{v-2})}{(2\text{v}-3)}\left(\mathbf{I}_\text{v} - \frac{\mathbf{11}'}{\text{v}}\right), \text{ v} > 2. \qquad \ldots (2.1.15)$$

A series of incomplete block design strongly balanced for interference effects can also be obtained by developing the blocks of the design as follows for all q = 0, 1, …, (v-1) and p = 1, 2, …, (v-1)/2:

q, q + p, q + 2p,…, q + (v-3)p, q + (v-2)p, q + (v-3)p,…, q + 2p, q + p, q, (modulo v)

The parameters of the designs are v, b = v(v-1)/2, r = (v-1)(2v-3)/2, k = 2v-3, μ_1 = v-2, and μ_2 = (v-1)/2. For this class of designs:

$$\mathbf{R}_\tau = \mathbf{R}_\delta = \mathbf{R}_\gamma = \mathbf{R}_\alpha = \mathbf{R}_\eta = \frac{(\text{v-1})(2\text{v}-3)}{2}\mathbf{I}_\text{v}, \mathbf{K} = \text{k}\,\mathbf{I}_\text{b} = (2\text{v}-3)\mathbf{I}_\text{b},$$

$$\mathbf{M}_1 = \mathbf{M}_2 = \mathbf{M}_3 = \mathbf{M}_4 = \mathbf{M}_5 = \mathbf{M}_6 = \mathbf{M}_7 = \mathbf{M}_8 = \mathbf{M}_9 = \mathbf{M}_{10} = (\text{v-2})\mathbf{11}' - \frac{(\text{v-3})}{2}\mathbf{I}_\text{v},$$

$$\mathbf{N}_\text{u}\mathbf{N}'_{\text{u}'} = 2(\text{v-2})^2\,\mathbf{11}' + \frac{(5\text{v}-9)}{2}\mathbf{I}_\text{v}', \text{ u, u}' = 1, 2, \ldots 5 \qquad \ldots (2.1.16)$$

Hence, we obtain

$$\begin{aligned}&\mathbf{R}_\tau - \mathbf{N}_1\mathbf{K}^{-1}\mathbf{N}_1' = \mathbf{R}_\delta - \mathbf{N}_2\mathbf{K}^{-1}\mathbf{N}_2' = \mathbf{R}_\gamma - \mathbf{N}_3\mathbf{K}^{-1}\mathbf{N}_3' \\ &= \mathbf{R}_\alpha - \mathbf{N}_4\mathbf{K}^{-1}\mathbf{N}_4' = \mathbf{R}_\eta - \mathbf{N}_5\mathbf{K}^{-1}\mathbf{N}_5' = \frac{2\mathrm{v}(\mathrm{v}\text{-}2)^2}{(2\mathrm{v}-3)}\left[\mathbf{I}_\mathrm{v} - \frac{\mathbf{11}'}{\mathrm{v}}\right]\end{aligned} \qquad \ldots (2.1.17)$$

and

$$\begin{aligned}&\mathbf{M}_1 - \mathbf{N}_1\mathbf{K}^{-1}\mathbf{N}_2' = \mathbf{M}_2 - \mathbf{N}_1\mathbf{K}^{-1}\mathbf{N}_3' = \mathbf{M}_3 - \mathbf{N}_2\mathbf{K}^{-1}\mathbf{N}_3' \\ &= \mathbf{M}_4 - \mathbf{N}_1\mathbf{K}^{-1}\mathbf{N}_4' = \mathbf{M}_5 - \mathbf{N}_1\mathbf{K}^{-1}\mathbf{N}_5' = \mathbf{M}_6 - \mathbf{N}_2\mathbf{K}^{-1}\mathbf{N}_4' \\ &= \mathbf{M}_7 - \mathbf{N}_2\mathbf{K}^{-1}\mathbf{N}_5' = \mathbf{M}_8 - \mathbf{N}_3\mathbf{K}^{-1}\mathbf{N}_4' = \mathbf{M}_9 - \mathbf{N}_3\mathbf{K}^{-1}\mathbf{N}_5' \;. \\ &= \mathbf{M}_{10} - \mathbf{N}_4\mathbf{K}^{-1}\mathbf{N}_5' = \frac{\mathrm{v}(\mathrm{v}\text{-}2)}{(2\mathrm{v}-3)}\left(\frac{\mathbf{11}'}{\mathrm{v}} - \mathbf{I}_\mathrm{v}\right)\end{aligned} \qquad \ldots (2.1.18)$$

Thus, the joint information matrix for estimating the direct as well as interference effects from the neighbouring units up to distance 2 can be obtained as

$$\mathbf{C} = \begin{bmatrix} \frac{2\mathrm{v}(\mathrm{v}\text{-}2)^2}{(2\mathrm{v}-3)}\left(\mathbf{I}_\mathrm{v} - \frac{\mathbf{11}'}{\mathrm{v}}\right) & \frac{\mathrm{v}(\mathrm{v}\text{-}2)}{(2\mathrm{v}-3)}\left(\frac{\mathbf{11}'}{\mathrm{v}} - \mathbf{I}_\mathrm{v}\right) & \frac{\mathrm{v}(\mathrm{v}\text{-}2)}{(2\mathrm{v}-3)}\left(\frac{\mathbf{11}'}{\mathrm{v}} - \mathbf{I}_\mathrm{v}\right) & \frac{\mathrm{v}(\mathrm{v}\text{-}2)}{(2\mathrm{v}-3)}\left(\frac{\mathbf{11}'}{\mathrm{v}} - \mathbf{I}_\mathrm{v}\right) & \frac{\mathrm{v}(\mathrm{v}\text{-}2)}{(2\mathrm{v}-3)}\left(\frac{\mathbf{11}'}{\mathrm{v}} - \mathbf{I}_\mathrm{v}\right) \\ \frac{\mathrm{v}(\mathrm{v}\text{-}2)}{(2\mathrm{v}-3)}\left(\frac{\mathbf{11}'}{\mathrm{v}} - \mathbf{I}_\mathrm{v}\right) & \frac{2\mathrm{v}(\mathrm{v}\text{-}2)^2}{(2\mathrm{v}-3)}\left(\mathbf{I}_\mathrm{v} - \frac{\mathbf{11}'}{\mathrm{v}}\right) & \frac{\mathrm{v}(\mathrm{v}\text{-}2)}{(2\mathrm{v}-3)}\left(\frac{\mathbf{11}'}{\mathrm{v}} - \mathbf{I}_\mathrm{v}\right) & \frac{\mathrm{v}(\mathrm{v}\text{-}2)}{(2\mathrm{v}-3)}\left(\frac{\mathbf{11}'}{\mathrm{v}} - \mathbf{I}_\mathrm{v}\right) & \frac{\mathrm{v}(\mathrm{v}\text{-}2)}{(2\mathrm{v}-3)}\left(\frac{\mathbf{11}'}{\mathrm{v}} - \mathbf{I}_\mathrm{v}\right) \\ \frac{\mathrm{v}(\mathrm{v}\text{-}2)}{(2\mathrm{v}-3)}\left(\frac{\mathbf{11}'}{\mathrm{v}} - \mathbf{I}_\mathrm{v}\right) & \frac{\mathrm{v}(\mathrm{v}\text{-}2)}{(2\mathrm{v}-3)}\left(\frac{\mathbf{11}'}{\mathrm{v}} - \mathbf{I}_\mathrm{v}\right) & \frac{2\mathrm{v}(\mathrm{v}\text{-}2)^2}{(2\mathrm{v}-3)}\left(\mathbf{I}_\mathrm{v} - \frac{\mathbf{11}'}{\mathrm{v}}\right) & \frac{\mathrm{v}(\mathrm{v}\text{-}2)}{(2\mathrm{v}-3)}\left(\frac{\mathbf{11}'}{\mathrm{v}} - \mathbf{I}_\mathrm{v}\right) & \frac{\mathrm{v}(\mathrm{v}\text{-}2)}{(2\mathrm{v}-3)}\left(\frac{\mathbf{11}'}{\mathrm{v}} - \mathbf{I}_\mathrm{v}\right) \\ \frac{\mathrm{v}(\mathrm{v}\text{-}2)}{(2\mathrm{v}-3)}\left(\frac{\mathbf{11}'}{\mathrm{v}} - \mathbf{I}_\mathrm{v}\right) & \frac{\mathrm{v}(\mathrm{v}\text{-}2)}{(2\mathrm{v}-3)}\left(\frac{\mathbf{11}'}{\mathrm{v}} - \mathbf{I}_\mathrm{v}\right) & \frac{\mathrm{v}(\mathrm{v}\text{-}2)}{(2\mathrm{v}-3)}\left(\frac{\mathbf{11}'}{\mathrm{v}} - \mathbf{I}_\mathrm{v}\right) & \frac{2\mathrm{v}(\mathrm{v}\text{-}2)^2}{(2\mathrm{v}-3)}\left(\mathbf{I}_\mathrm{v} - \frac{\mathbf{11}'}{\mathrm{v}}\right) & \frac{\mathrm{v}(\mathrm{v}\text{-}2)}{(2\mathrm{v}-3)}\left(\frac{\mathbf{11}'}{\mathrm{v}} - \mathbf{I}_\mathrm{v}\right) \\ \frac{\mathrm{v}(\mathrm{v}\text{-}2)}{(2\mathrm{v}-3)}\left(\frac{\mathbf{11}'}{\mathrm{v}} - \mathbf{I}_\mathrm{v}\right) & \frac{\mathrm{v}(\mathrm{v}\text{-}2)}{(2\mathrm{v}-3)}\left(\frac{\mathbf{11}'}{\mathrm{v}} - \mathbf{I}_\mathrm{v}\right) & \frac{\mathrm{v}(\mathrm{v}\text{-}2)}{(2\mathrm{v}-3)}\left(\frac{\mathbf{11}'}{\mathrm{v}} - \mathbf{I}_\mathrm{v}\right) & \frac{\mathrm{v}(\mathrm{v}\text{-}2)}{(2\mathrm{v}-3)}\left(\frac{\mathbf{11}'}{\mathrm{v}} - \mathbf{I}_\mathrm{v}\right) & \frac{2\mathrm{v}(\mathrm{v}\text{-}2)^2}{(2\mathrm{v}-3)}\left(\mathbf{I}_\mathrm{v} - \frac{\mathbf{11}'}{\mathrm{v}}\right) \end{bmatrix}$$

The information matrix for estimating the direct effects of treatment is

$$\mathbf{C}_\tau = \frac{2\mathrm{v}(\mathrm{v}\text{-}2)(\mathrm{v}\text{-}4)}{(2\mathrm{v}-7)}\left(\mathbf{I}_\mathrm{v} - \frac{\mathbf{11}'}{\mathrm{v}}\right), \ \mathrm{v} > 4. \qquad \ldots (2.1.19)$$

Similarly, the information matrix for estimating the immediate left interference effects, immediate right interference effects, left interference effects at distance 2 and right interference effects at distance 2 respectively from the neighbouring units are obtained as

$$\mathbf{C}_\delta = \mathbf{C}_\gamma = \mathbf{C}_\alpha = \mathbf{C}_\eta = \frac{2\mathrm{v}(\mathrm{v}\text{-}2)(\mathrm{v}\text{-}4)}{(2\mathrm{v}-7)}\left(\mathbf{I}_\mathrm{v} - \frac{\mathbf{11}'}{\mathrm{v}}\right), \ \mathrm{v} > 4. \qquad \ldots (2.1.20)$$

The series of design thus obtained is also *totally balanced* for estimating the contrast pertaining to direct effects of treatments and interference effects from the neighbouring units up to distance 2.

Example 2.1.2.2: For v = 5, following is a strongly balanced incomplete block design with interference effects up to distance 2 from the neighbouring units with v = 5, b = 10, r = 14, k = 7, μ_1= 3, μ_2 = 2:

0	1	2	3	2	1	0
1	2	3	4	3	2	1
2	3	4	0	4	3	2
3	4	0	1	0	4	3
4	0	1	2	1	0	4
0	2	4	1	4	2	0
1	3	0	2	0	3	1
2	4	1	3	1	4	2
3	0	2	4	2	0	3
4	1	3	0	3	1	4

Remark 2.1.2.2: For the above class of designs, when interference effects from only one side say, left neighbouring units are considered, the information matrices for estimating direct effects and information matrices for estimating interference effects from the neighbouring units up to distance 2 are obtained as:

$$\mathbf{C}_\tau = \mathbf{C}_\delta = \mathbf{C}_\gamma = \frac{2v(v\text{-}2)(v-3)}{(2v-5)}\left(\mathbf{I}_v - \frac{\mathbf{11}'}{v}\right), \; v > 3. \qquad \ldots (2.1.21)$$

Method 2.1.2.2: The designs obtained by Azais *et al.* (1993) in v-1 blocks of size v (> 5) each are shown to be balanced for interference effects up to distance 2. These designs for v (prime) treatments are obtained by arranging the contents of the v-1 complete blocks of the design by writing the treatments in systematic order within a block with a difference of 1,2,…,v-1 between the treatments (modulo v) in the consecutive blocks. The first block is formed by taking the difference of one between treatments, the second block by taking the difference of two and so on, the (v-1)th block by taking the difference of (v-1). The series of complete block design is balanced for interference effect up to distance 2 with parameters v = k, b = (v-1) = r, μ_1= 1.

For this class of designs:

$$\mathbf{R}_\tau = \mathbf{R}_\delta = \mathbf{R}_\gamma = \mathbf{R}_\alpha = \mathbf{R}_\eta = (v\text{-}1)\mathbf{I}_v, \mathbf{K} = k\,\mathbf{I}_b = v\,\mathbf{I}_b,$$

$$\mathbf{M}_1 = \mathbf{M}_2 = \mathbf{M}_3 = \mathbf{M}_4 = \mathbf{M}_5 = \mathbf{M}_6 = \mathbf{M}_7 = \mathbf{M}_8 = \mathbf{M}_9 = \mathbf{M}_{10} = (\mathbf{11}' - \mathbf{I}_v),$$

$$\mathbf{N}_u \mathbf{N}'_{u'} = (v\text{-}1)\mathbf{11}', \; u, u' = 1, 2, \ldots 5 \qquad \ldots (2.1.22)$$

Hence,

$$\mathbf{R}_\tau - \mathbf{N}_1\mathbf{K}^{-1}\mathbf{N}'_1 = \mathbf{R}_\delta - \mathbf{N}_2\mathbf{K}^{-1}\mathbf{N}'_2 = \mathbf{R}_\gamma - \mathbf{N}_3\mathbf{K}^{-1}\mathbf{N}'_3 = \mathbf{R}_\alpha - \mathbf{N}_4\mathbf{K}^{-1}\mathbf{N}'_4$$

$$= \mathbf{R}_\eta - \mathbf{N}_5\mathbf{K}^{-1}\mathbf{N}'_5 = (v\text{-}1)\left[\mathbf{I}_v - \frac{\mathbf{11}'}{v}\right] \qquad \ldots (2.1.23)$$

and

$$\mathbf{M}_1 - \mathbf{N}_1\mathbf{K}^{-1}\mathbf{N}_2' = \mathbf{M}_2 - \mathbf{N}_1\mathbf{K}^{-1}\mathbf{N}_3' = \mathbf{M}_3 - \mathbf{N}_2\mathbf{K}^{-1}\mathbf{N}_3' = \mathbf{M}_4 - \mathbf{N}_1\mathbf{K}^{-1}\mathbf{N}_4'$$
$$= \mathbf{M}_5 - \mathbf{N}_1\mathbf{K}^{-1}\mathbf{N}_5' = \mathbf{M}_6 - \mathbf{N}_2\mathbf{K}^{-1}\mathbf{N}_4' = \mathbf{M}_7 - \mathbf{N}_2\mathbf{K}^{-1}\mathbf{N}_5' = \mathbf{M}_8 - \mathbf{N}_3\mathbf{K}^{-1}\mathbf{N}_4' . \qquad \ldots (2.1.24)$$
$$= \mathbf{M}_9 - \mathbf{N}_3\mathbf{K}^{-1}\mathbf{N}_5' = \mathbf{M}_{10} - \mathbf{N}_4\mathbf{K}^{-1}\mathbf{N}_5' = \left[\frac{\mathbf{11}'}{\mathrm{v}} - \mathbf{I}_{\mathrm{v}}\right]$$

The joint information matrix for estimating the direct as well as interference effects from the neighbouring units up to distance 2 can be obtained as:

$$\mathbf{C} = \begin{bmatrix} (\text{v-1})\left(\mathbf{I}_{\mathrm{v}} - \frac{\mathbf{11}'}{\mathrm{v}}\right) & \left(\frac{\mathbf{11}'}{\mathrm{v}} - \mathbf{I}_{\mathrm{v}}\right) & \left(\frac{\mathbf{11}'}{\mathrm{v}} - \mathbf{I}_{\mathrm{v}}\right) & \left(\frac{\mathbf{11}'}{\mathrm{v}} - \mathbf{I}_{\mathrm{v}}\right) & \left(\frac{\mathbf{11}'}{\mathrm{v}} - \mathbf{I}_{\mathrm{v}}\right) \\ \left(\frac{\mathbf{11}'}{\mathrm{v}} - \mathbf{I}_{\mathrm{v}}\right) & (\text{v-1})\left(\mathbf{I}_{\mathrm{v}} - \frac{\mathbf{11}'}{\mathrm{v}}\right) & \left(\frac{\mathbf{11}'}{\mathrm{v}} - \mathbf{I}_{\mathrm{v}}\right) & \left(\frac{\mathbf{11}'}{\mathrm{v}} - \mathbf{I}_{\mathrm{v}}\right) & \left(\frac{\mathbf{11}'}{\mathrm{v}} - \mathbf{I}_{\mathrm{v}}\right) \\ \left(\frac{\mathbf{11}'}{\mathrm{v}} - \mathbf{I}_{\mathrm{v}}\right) & \left(\frac{\mathbf{11}'}{\mathrm{v}} - \mathbf{I}_{\mathrm{v}}\right) & (\text{v-1})\left(\mathbf{I}_{\mathrm{v}} - \frac{\mathbf{11}'}{\mathrm{v}}\right) & \left(\frac{\mathbf{11}'}{\mathrm{v}} - \mathbf{I}_{\mathrm{v}}\right) & \left(\frac{\mathbf{11}'}{\mathrm{v}} - \mathbf{I}_{\mathrm{v}}\right) \\ \left(\frac{\mathbf{11}'}{\mathrm{v}} - \mathbf{I}_{\mathrm{v}}\right) & \left(\frac{\mathbf{11}'}{\mathrm{v}} - \mathbf{I}_{\mathrm{v}}\right) & \left(\frac{\mathbf{11}'}{\mathrm{v}} - \mathbf{I}_{\mathrm{v}}\right) & (\text{v-1})\left(\mathbf{I}_{\mathrm{v}} - \frac{\mathbf{11}'}{\mathrm{v}}\right) & \left(\frac{\mathbf{11}'}{\mathrm{v}} - \mathbf{I}_{\mathrm{v}}\right) \\ \left(\frac{\mathbf{11}'}{\mathrm{v}} - \mathbf{I}_{\mathrm{v}}\right) & \left(\frac{\mathbf{11}'}{\mathrm{v}} - \mathbf{I}_{\mathrm{v}}\right) & \left(\frac{\mathbf{11}'}{\mathrm{v}} - \mathbf{I}_{\mathrm{v}}\right) & \left(\frac{\mathbf{11}'}{\mathrm{v}} - \mathbf{I}_{\mathrm{v}}\right) & (\text{v-1})\left(\mathbf{I}_{\mathrm{v}} - \frac{\mathbf{11}'}{\mathrm{v}}\right) \end{bmatrix}$$

The information matrix for estimating the direct effects of treatments is

$$\mathbf{C}_\tau = \frac{\mathrm{v}(\text{v-5})}{(\text{v-4})}\left(\mathbf{I}_{\mathrm{v}} - \frac{\mathbf{11}'}{\mathrm{v}}\right), \ \mathrm{v} > 5. \qquad \ldots (2.1.25)$$

Similarly, the information matrix for estimating the immediate left interference effects, immediate right neighbour effects, left interference effects at distance 2, and right interference effects at distance 2 respectively from the neighbouring units is

$$\mathbf{C}_\delta = \mathbf{C}_\gamma = \mathbf{C}_\alpha = \mathbf{C}_\eta == \frac{\mathrm{v}(\text{v-5})}{(\text{v-4})}\left(\mathbf{I}_{\mathrm{v}} - \frac{\mathbf{11}'}{\mathrm{v}}\right), \ \mathrm{v} > 5. \qquad \ldots (2.1.26)$$

Hence the series of design obtained is *totally balanced* for estimating the contrasts pertaining to direct effects of treatments and interference effects arising from the neighbouring units up to distance 2.

Example 2.1.2.3: For v = 7, following is a complete block design balanced for both sided interference effects from the neighbouring units up to distance 2 with parameters v = 7 = k, b = 6 = r, μ_1= 1:

1	2	3	4	5	6	0
1	3	5	0	2	4	6
1	4	0	3	6	2	5

1	5	2	6	3	0	4
1	6	4	2	0	5	3
1	0	6	5	4	3	2

Remark 2.1.2.3: For the above class of design, when interference effects from only one side say, left neighbouring units are considered, the information matrices for estimating direct effects and information matrices for estimating interference effects from the neighbouring units up to distance 2 can be obtained as:

$$\mathbf{C}_\tau = \mathbf{C}_\delta = \mathbf{C}_\gamma = \frac{v(v\text{-}3)}{(v\text{-}2)}\left(\mathbf{I}_v - \frac{\mathbf{11}'}{v}\right), \ v > 3. \qquad \dots (2.1.27)$$

Remark 2.1.2.4: For the above class of designs, we can generalize the result by taking interference effects up to distance h ($1 \le h \le k\text{-}1$). The information matrix for estimating the direct effects and interference effects is thus obtained as follows:

$$\mathbf{C} = \frac{v\left[v\text{-}(2h+1)\right]}{(v\text{-}2h)}\left(\mathbf{I}_v - \frac{\mathbf{11}'}{v}\right), \ v > (2h+1). \qquad \dots (2.1.28)$$

Method 2.1.2.3: Tomar *et al.* (2005) obtained a series of block design balanced for adjacent left and right neighboring units for v = mt + 1 [prime or prime power (m > 3)], b = tv, r = mt, k = m and μ_1= 1 by developing following initial blocks modulo v and augmenting the whole set of blocks generated from each initial block one after another:

$$x^{w}, x^{w+t}, x^{w+2t}, \dots x^{w+(m-1)t} \ ; \quad \text{for } w = 0,1, \dots, t\text{-}1,$$

where x is the primitive element of GF (v). This class of design is also found to be balanced for interference effects from neighbouring units up to distance 2.

For this class of designs:

$$\mathbf{R}_\tau = \mathbf{R}_\delta = \mathbf{R}_\gamma = \mathbf{R}_\alpha = \mathbf{R}_\eta = (v\text{-}1)\mathbf{I}_v, \mathbf{K} = k\mathbf{I}_b,$$
$$\mathbf{M}_1 = \mathbf{M}_2 = \mathbf{M}_3 = \mathbf{M}_4 = \mathbf{M}_5 = \mathbf{M}_6 = \mathbf{M}_7 = \mathbf{M}_8 = \mathbf{M}_9 = \mathbf{M}_{10} = (\mathbf{11}' - \mathbf{I}_v),$$
$$\mathbf{N}_u \mathbf{N}'_{u'} = (v\text{-}k)\mathbf{I}_v + (k\text{-}1)\mathbf{11}', \ u, u' = 1, 2, \dots 5 \qquad \dots (2.1.29)$$

Hence,

$$\mathbf{R}_\tau - \mathbf{N}_1\mathbf{K}^{-1}\mathbf{N}'_1 = \mathbf{R}_\delta - \mathbf{N}_2\mathbf{K}^{-1}\mathbf{N}'_2 = \mathbf{R}_\gamma - \mathbf{N}_3\mathbf{K}^{-1}\mathbf{N}'_3$$
$$= \mathbf{R}_\alpha - \mathbf{N}_4\mathbf{K}^{-1}\mathbf{N}'_4 = \mathbf{R}_\eta - \mathbf{N}_5\mathbf{K}^{-1}\mathbf{N}'_5 = \frac{v(k\text{-}1)}{k}\left[\mathbf{I}_v - \frac{\mathbf{11}'}{v}\right] \qquad \dots (2.1.30)$$

and

$$\begin{aligned}&\mathbf{M}_1-\mathbf{N}_1\mathbf{K}^{-1}\mathbf{N}_2' = \mathbf{M}_2-\mathbf{N}_1\mathbf{K}^{-1}\mathbf{N}_3' = \mathbf{M}_3-\mathbf{N}_2\mathbf{K}^{-1}\mathbf{N}_3'\\ &= \mathbf{M}_4-\mathbf{N}_1\mathbf{K}^{-1}\mathbf{N}_4' = \mathbf{M}_5-\mathbf{N}_1\mathbf{K}^{-1}\mathbf{N}_5' = \mathbf{M}_6-\mathbf{N}_2\mathbf{K}^{-1}\mathbf{N}_4' =\\ &\mathbf{M}_7-\mathbf{N}_2\mathbf{K}^{-1}\mathbf{N}_5' = \mathbf{M}_8-\mathbf{N}_3\mathbf{K}^{-1}\mathbf{N}_4' = \mathbf{M}_9-\mathbf{N}_3\mathbf{K}^{-1}\mathbf{N}_5'\\ &= \mathbf{M}_{10}-\mathbf{N}_4\mathbf{K}^{-1}\mathbf{N}_5' = \frac{v}{k}\left[\frac{\mathbf{11}'}{v}-\mathbf{I}_v\right]\end{aligned} \quad \ldots (2.1.31)$$

Thus, the joint information matrix for estimating the direct as well as interference effects from the neighbouring units up to distance 2 can be obtained as:

$$\mathbf{C} = \begin{bmatrix} \frac{v(k\text{-}1)}{k}\left(\mathbf{I}_v-\frac{\mathbf{11}'}{v}\right) & \frac{v}{k}\left(\frac{\mathbf{11}'}{v}-\mathbf{I}_v\right) & \frac{v}{k}\left(\frac{\mathbf{11}'}{v}-\mathbf{I}_v\right) & \frac{v}{k}\left(\frac{\mathbf{11}'}{v}-\mathbf{I}_v\right) & \frac{v}{k}\left(\frac{\mathbf{11}'}{v}-\mathbf{I}_v\right) \\ \frac{v}{k}\left(\frac{\mathbf{11}'}{v}-\mathbf{I}_v\right) & \frac{v(k\text{-}1)}{k}\left(\mathbf{I}_v-\frac{\mathbf{11}'}{v}\right) & \frac{v}{k}\left(\frac{\mathbf{11}'}{v}-\mathbf{I}_v\right) & \frac{v}{k}\left(\frac{\mathbf{11}'}{v}-\mathbf{I}_v\right) & \frac{v}{k}\left(\frac{\mathbf{11}'}{v}-\mathbf{I}_v\right) \\ \frac{v}{k}\left(\frac{\mathbf{11}'}{v}-\mathbf{I}_v\right) & \frac{v}{k}\left(\frac{\mathbf{11}'}{v}-\mathbf{I}_v\right) & \frac{v(k\text{-}1)}{k}\left(\mathbf{I}_v-\frac{\mathbf{11}'}{v}\right) & \frac{v}{k}\left(\frac{\mathbf{11}'}{v}-\mathbf{I}_v\right) & \frac{v}{k}\left(\frac{\mathbf{11}'}{v}-\mathbf{I}_v\right) \\ \frac{v}{k}\left(\frac{\mathbf{11}'}{v}-\mathbf{I}_v\right) & \frac{v}{k}\left(\frac{\mathbf{11}'}{v}-\mathbf{I}_v\right) & \frac{v}{k}\left(\frac{\mathbf{11}'}{v}-\mathbf{I}_v\right) & \frac{v(k\text{-}1)}{k}\left(\mathbf{I}_v-\frac{\mathbf{11}'}{v}\right) & \frac{v}{k}\left(\frac{\mathbf{11}'}{v}-\mathbf{I}_v\right) \\ \frac{v}{k}\left(\frac{\mathbf{11}'}{v}-\mathbf{I}_v\right) & \frac{v}{k}\left(\frac{\mathbf{11}'}{v}-\mathbf{I}_v\right) & \frac{v}{k}\left(\frac{\mathbf{11}'}{v}-\mathbf{I}_v\right) & \frac{v}{k}\left(\frac{\mathbf{11}'}{v}-\mathbf{I}_v\right) & \frac{v(k\text{-}1)}{k}\left(\mathbf{I}_v-\frac{\mathbf{11}'}{v}\right) \end{bmatrix}$$

Thus, the information matrix for estimating the direct effects of treatments is

$$\mathbf{C}_\tau = \frac{v(k\text{-}5)}{(k\text{-}4)}\left(\mathbf{I}_v-\frac{\mathbf{11}'}{v}\right),\ k > 5. \quad \ldots (2.1.32)$$

Similarly,

$$\mathbf{C}_\delta = \mathbf{C}_\gamma = \mathbf{C}_\alpha = \mathbf{C}_\eta = \frac{v(k\text{-}5)}{(k\text{-}4)}\left(\mathbf{I}_v-\frac{\mathbf{11}'}{v}\right),\ k > 5. \quad \ldots (2.1.33)$$

The series is also *totally balanced* for estimating the contrast pertaining to direct effects of treatments and interference effects from the neighbouring units up to distance 2.

Example 2.1.2.4: Let m = 6, t = 2, then the following two initial blocks modulo 11 for w = 0 and w = 1 are obtained:

1 4 3 12 9 10 and 2 8 6 11 5 7

Developing these blocks, we obtain the following totally balanced circular incomplete block design with interference effects up to distance 2 from the neighbouring units and with v = 13, b = 26, r = 12, k = 6, μ_1= 1:

1	4	3	12	9	10
2	5	4	0	10	11
3	6	5	1	11	12

4	7	6	2	12	0
5	8	7	3	0	1
6	9	8	4	1	2
7	10	9	5	2	3
8	11	10	6	3	4
9	12	11	7	4	5
10	0	12	8	5	6
11	1	0	9	6	7
12	2	1	10	7	8
0	3	2	11	8	9
2	8	6	11	5	7
3	9	7	12	6	8
4	10	8	0	7	9
5	11	9	1	8	10
6	12	10	2	9	11
7	0	11	3	10	12
8	1	12	4	11	0
9	2	0	5	12	1
10	3	1	6	0	2
11	4	2	7	1	3
12	5	3	8	2	4
0	6	4	9	3	5
1	7	5	10	4	6

Remark 2.1.2.5: For the above class of design, when interference effects from only one side say, left neighbouring units are considered, the information matrices for estimating direct effects and information matrices for estimating interference effects from the neighbouring units up to distance 2 is obtained as:

$$\mathbf{C}_\tau = \mathbf{C}_\delta = \mathbf{C}_\gamma = \frac{v(k\text{-}3)}{(k\text{-}2)}\left(\mathbf{I}_v - \frac{\mathbf{11}'}{v}\right), \ k > 3. \qquad \ldots (2.1.34)$$

Remark 2.1.2.6: The above class is generalized by taking interference effects up to distance h ($1 \leq h \leq k\text{-}1$). The information matrix for estimating the direct effects and interference effects is thus obtained as follows:

$$\mathbf{C} = \frac{v\left[k\text{-}(2h+1)\right]}{(k\text{-}2h)}\left(\mathbf{I}_v - \frac{\mathbf{11}'}{v}\right), \ k > (2h+1) \qquad \ldots (2.1.35)$$

2.1.3 Analysis of Experimental Data of Block Design when Treatments Exhibit Neighbour Effects

In this section, we have given the analysis of block design with neighbour effects arising from adjacent units on both sides (left and right) at distance 2 as in field experiment it is generally assumed that neighbour effects beyond distance 2 are negligible. The experimental set up has been defined and the method of analysis has been illustrated for 11 genotypes under complete blocking situation using simulated data. A Multiple range comparison among genotype effects in the presence of all other effects using Tukey-Kramer test has also been performed using SAS 9.3.

We consider the fixed effects additive model as given in (2.1.1) for analyzing a block design with second order interference effects. Under this set up with second order neighbour effects, there are six different source of variability viz. block, treatment, left neighbour effects of treatments, right neighbour effects of treatments, second order left neighbour effects of treatments and second order right neighbour effects of treatments. The split up of sources of variation and degrees of freedom is shown in the ANOVA Table 2.1.1 with respect to block design involving treatments exhibiting neighbour effects up to distance 2.

Table 2.1.1: Sources of variation and degrees of freedom under block design set up involving treatments exhibiting second order neighbour effects

Sources of variation	**Degrees of freedom**
Block	b – 1
Treatments (Adjusted for all the neighbour effects)	v – 1
Immediate left neighbour (Adjusted for direct effect and other neighbour effects	v – 1
Immediate right neighbour (Adjusted for direct effect and other neighbour effects	v – 1
Second order left neighbour (Adjusted for direct effect and other neighbour effects	v – 1
Second order right neighbour (Adjusted for direct effect and other neighbour effects	v – 1
Error	By subtraction
Total	n – 1

Here, v is the number of treatments and n is the total number of observations. Since the direct effect of treatments and all its neighbour effects cannot be partitioned orthogonally, one must adjust each of the effects for all other effects. PROC GLM in SAS 9.3 facilitates this aportioning by mentioning SS2 (Sum of squares of type II) in the *model* statement. More importantly, one must use a variance balanced design to get comparison among each of the effects equally precisely.

Illustration: Consider an agricultural experiment for comparing 11 genotypes with respect to their yield. The genotypes tried in the experiment are highly competitive in nature. In order to see whether the response (i.e, yield) is affected significantly by the effects of neighbouring genotypes, the experiment was laid out in 10 complete blocks based on the designs obtained by Method 2.1.2.2. Following is the layout of the experiment along with simulated data set:

Block 1	1.90 (11)	8.00 (1)	2.50 (2)	1.11 (3)	6.90 (4)	2.42 (5)	6.45 (6)	14.52 (7)	29.03 (8)	10.00 (9)	7.60 (10)
Block 2	9.12 (2)	8.11 (4)	7.37 (6)	3.69 (8)	2.76 (10)	10.00 (1)	2.02 (3)	1.81 (5)	19.96 (7)	13.31 (9)	10.64 (11)
Block 3	5.13 (6)	1.87 (9)	8.00 (1)	7.26 (4)	1.61 (7)	8.06 (10)	16.13 (2)	25.34 (5)	7.60 (8)	22.81 (11)	41.06 (3)
Block 4	6.76 (8)	12.01 (1)	8.00 (5)	2.07 (9)	20.74 (2)	12.44 (6)	9.68 (10)	53.22 (3)	35.48 (7)	28.39 (11)	9.46 (4)
Block 5	25.92 (4)	41.47 (9)	8.29 (3)	14.52 (8)	1.61 (2)	3.23 (7)	11.01 (1)	11.09 (6)	15.84 (11)	63.36 (5)	95.04 (10)
Block 6	9.00 (1)	3.87 (7)	1.94 (2)	17.42 (8)	9.95 (3)	49.77 (9)	31.10 (4)	11.46 (10)	76.03 (5)	19.01 (11)	13.31 (6)
Block 7	93.14 (3)	16.93 (10)	21.77 (6)	36.29 (2)	3.63 (9)	4.84 (5)	46.01 (1)	11.83 (8)	16.56 (4)	49.67 (11)	62.09 (7)
Block 8	67.58 (5)	43.01 (2)	21.50 (10)	4.30 (7)	19.35 (4)	50.00 (1)	40.01 (9)	13.69 (6)	10.95 (3)	60.83 (11)	20.28 (8)
Block 9	12.44 (10)	16.59 (8)	33.18 (6)	36.50 (4)	41.06 (2)	47.90 (11)	59.88 (9)	89.81 (7)	8.16 (5)	9.07 (3)	14.01 (1)
Block 10	14.51 (7)	64.51 (6)	24.19 (5)	6.91 (4)	1.15 (3)	2.53 (2)	14.01 (1)	19.01 (11)	76.03 (10)	80.00 (9)	18.00 (8)

*Numbers in the parenthesis indicate genotypes

The above design is first analyzed as a complete block design with 11 treatments in 10 complete blocks without considering the interference effects from neighbouring units and the result is as shown below.

Output of SAS 9.3 after performing ANOVA

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	19	12785.066	672.898	1.41	0.1410
Error	90	42865.987	476.289		
Total	109	55651.053			

Source	DF	Type III SS	Mean Square	F Value	Pr > F
Block	9	9899.089	1099.899	2.31	0.0219
Treat	10	2885.977	288.598	0.61	0.8050

The same design is to be now analyzed considering the effects of neighbouring units. The SAS code and the output are now given.

SAS code for ANOVA (Type II) using proc glm considering neighbour effects from both left and right side upto distance 2

```
Data nbbd;
Input block treat left_ne_dist_one left_ne_dist_two right_ne_dist_one right_ne_dist_two
        yld;
```

Cards;

1	1	11	10	2	3	8.00
1	2	1	11	3	4	2.50
1	3	2	1	4	5	1.11
1	4	3	2	5	6	6.90
1	5	4	3	6	7	2.42
1	6	5	4	7	8	6.45
1	7	6	5	8	9	14.52
1	8	7	6	9	10	29.03
1	9	8	7	10	11	10.00
1	10	9	8	11	1	7.60
1	11	10	9	1	2	1.90
2	1	10	8	3	5	10.00
2	2	11	9	4	6	9.12
2	3	1	10	5	7	2.02
2	4	2	11	6	8	8.11
2	5	3	1	7	9	1.81
2	6	4	2	8	10	7.37
2	7	5	3	9	11	19.96
2	8	6	4	10	1	3.69
2	9	7	5	11	2	13.31
2	10	8	6	1	3	2.76
2	11	9	7	2	4	10.64
3	1	9	6	4	7	8.00
3	2	10	7	5	8	16.13
3	3	11	8	6	9	41.06
3	4	1	9	7	10	7.26
3	5	2	10	8	11	25.34
3	6	3	11	9	1	5.13
3	7	4	1	10	2	1.61
3	8	5	2	11	3	7.60
3	9	6	3	1	4	1.87
3	10	7	4	2	5	8.06
3	11	8	5	3	6	22.81
4	1	8	4	5	9	12.01
4	2	9	5	6	10	20.74
4	3	10	6	7	11	53.22
4	4	11	7	8	1	9.46
4	5	1	8	9	2	8.00
4	6	2	9	10	3	12.44
4	7	3	10	11	4	35.48
4	8	4	11	1	5	6.76
4	9	5	1	2	6	2.07
4	10	6	2	3	7	9.68

4	11	7	3	4	8	28.39
5	1	7	2	6	11	11.01
5	2	8	3	7	1	1.61
5	3	9	4	8	2	8.29
5	4	10	5	9	3	25.92
5	5	11	6	10	4	63.36
5	6	1	7	11	5	11.09
5	7	2	8	1	6	3.23
5	8	3	9	2	7	14.52
5	9	4	10	3	8	41.47
5	10	5	11	4	9	95.04
5	11	6	1	5	10	15.84
6	1	6	11	7	2	9.00
6	2	7	1	8	3	1.94
6	3	8	2	9	4	9.95
6	4	9	3	10	5	31.10
6	5	10	4	11	6	76.03
6	6	11	5	1	7	13.31
6	7	1	6	2	8	3.87
6	8	2	7	3	9	17.42
6	9	3	8	4	10	49.77
6	10	4	9	5	11	11.46
6	11	5	10	6	1	19.01
7	1	5	9	8	4	46.01
7	2	6	10	9	5	36.29
7	3	7	11	10	6	93.14
7	4	8	1	11	7	16.56
7	5	9	2	1	8	4.84
7	6	10	3	2	9	21.77
7	7	11	4	3	10	62.09
7	8	1	5	4	11	11.83
7	9	2	6	5	1	3.63
7	10	3	7	6	2	16.93
7	11	4	8	7	3	49.67
8	1	4	7	9	6	50.00
8	2	5	8	10	7	43.01
8	3	6	9	11	8	10.95
8	4	7	10	1	9	19.35
8	5	8	11	2	10	67.58
8	6	9	1	3	11	13.69
8	7	10	2	4	1	4.30
8	8	11	3	5	2	20.28
8	9	1	4	6	3	40.01
8	10	2	5	7	4	21.50

8	11	3	6	8	5	60.83
9	1	3	5	10	8	14.01
9	2	4	6	11	9	41.06
9	3	5	7	1	10	9.07
9	4	6	8	2	11	36.50
9	5	7	9	3	1	8.16
9	6	8	10	4	2	33.18
9	7	9	11	5	3	89.81
9	8	10	1	6	4	16.59
9	9	11	2	7	5	59.88
9	10	1	3	8	6	12.44
9	11	2	4	9	7	47.90
10	1	2	3	11	10	14.01
10	2	3	4	1	11	2.53
10	3	4	5	2	1	1.15
10	4	5	6	3	2	6.91
10	5	6	7	4	3	24.19
10	6	7	8	5	4	64.51
10	7	8	9	6	5	14.51
10	8	9	10	7	6	18.00
10	9	10	11	8	7	80.00
10	10	11	1	9	8	76.03
10	11	1	2	10	9	19.01

```
;
PROC glm;
Class  block treat left_ne_dist_one   left_ne_dist_two  right_ne_dist_one right_ne_dist_two;
Model yld  = block     treat     left_ne_dist_one         left_ne_dist_two     right_ne_dist_one
        right_ne_dist_two/ss2;
lsmeans treat/pdiff adjust=tukey lines;
Run;
quit;
```

SAS Output

The GLM Procedure

Dependent Variable: yld

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	59	45483.999	770.915	3.79	<.0001
Error	50	10167.053	203.341		
Total	109	55651.052			

Source	DF	Type II SS	Mean Square	F Value	Pr > F
Block	9	9899.088	1099.899	5.41	<.0001
Treat	10	5831.419	583.142	2.87	0.0067
left_ne_dist_one	10	9706.325	970.632	4.77	<.0001
left_ne_dist_two	10	13347.284	1334.729	6.56	<.0001
right_ne_dist_one	10	8688.005	868.800	4.27	0.0003
right_ne_dist_two	10	9202.507	920.251	4.53	0.0001

*ANOVA type II was performed to get the adjusted effects of genotypes and all other neighbour effects

Tukey-Kramer Comparison Lines for Least Squares Means of Genotype effects

Genotype	LS Means
1	10.38^{b}
2	12.56^{ab}
3	21.89^{ab}
4	18.33^{ab}
5	28.55^{ab}
6	19.06^{ab}
7	25.44^{ab}
8	15.40^{ab}
9	31.37^{ab}
10	29.77^{ab}
11	33.28^{a}

* Genotypes with same letter are not significantly different

Initially, the data was analysed by using the usual two-way Analysis of Variance technique (ANOVA) with two known sources of variance as genotypes and blocks. It is observed that the genotype effects are not significant i.e. the genotypes are not significantly different among each other. Thereafter, by considering border plots at both ends of every block up to distance 2 (In order to make the design balanced for neighbour effects), the same data was analysed once again based on the model as given in Equation (2.1.1), where apart from the two known sources of variations i.e. blocks and genotypes, four additional sources of variations i.e. the left and right neighbour effects of genotypes up to distance 2 have also been considered. Interestingly, it has been observed that the direct effects of genotypes came out to be significant at 5% level of significance. Further, we have also found that all the neighbour effects i.e. both left and right neighbour effects up to distance 2 came out to be significant. Thus, we can say that, neighbour effects from the adjacent units can play a significant role in the precision of the experiment. So, when there are evidences of neighbour effects, one has to consider these effects in to the model and analyse the data accordingly for drawing valid conclusions based on the experiment.

As the genotype effects came out to be significant when we have considered neighbour effects in to the model, thus a multiple comparison was also performed among genotype effects using Tukey-Kramer comparison test. It has been observed that genotype 11 was found to be the best and genotype 1 was giving the lowest yield and significantly different from the rest.

2.2 Optimal Block Designs with Spatial Indirect Effects from Neighbouring Experimental Units at Distance 2

Block designs balanced for interference effects from the neighbouring units have been found quite useful by experimenters under these situations and these designs also lead to simplified analysis. But these cannot be justified on statistical grounds unless they possess some optimal statistical properties as well. A lot of work which deals with optimality properties of block design with interference effects from the neighbouring units are available in literature [Bhaumik (1995), Azais and Druilhet (1997), Raghavarao and Zhou (1998), Druilhet (1999), Kunert *et al.* (2003), Jaggi *et al.* (2007), Pateria *et al.* (2011), etc.].

The purpose of this section is to establish the universal optimality of complete block designs with interference effects from the left neighbouring units up to distance 2. The blocks are circular in the sense that the first border treatments at the left end of each block is same as the treatment on the interior plot at the right end of the block and border treatment at distance 2 from the first plot of each block is same as the treatment at the second last plot of each block at the right end. The model considered is a four-way classified model consisting of direct effect of the treatment applied to a particular plot, effect of those treatments applied to the immediate left neighbouring units, effect of those treatments applied to the left neighbouring units at distance 2 and block effect. Conditions have been obtained for the block design to be universally optimal for estimating direct as well as interference effects from the left neighbouring units up to distance 2. Some classes of block designs have been identified to be universally optimal for the estimation of direct, immediate left and second order left neighbour effects.

2.2.1 Experimental Setup and Model

We consider a class of circular block designs with v treatments whose effects are to be studied in b blocks each of size k and here s^{th} (s = 1, 2, …, v) treatment is replicated r_s times. Let Y_{ij} be the response from the i^{th} plot in the j^{th} block (i = 1, 2,…, k; j = 1, 2,…, b). Following is the fixed effects additive model for a block design with neighbour effects from left neighbouring units:

$$\mathbf{Y} = \mu\mathbf{1} + \boldsymbol{\Delta}'\boldsymbol{\tau} + \boldsymbol{\Delta}_1'\boldsymbol{\delta} + \boldsymbol{\Delta}_2'\boldsymbol{\gamma} + \mathbf{D}'\boldsymbol{\beta} + \mathbf{e}, \qquad \dots (2.2.1)$$

where $\mathbf{Y}$ is a n × 1 vector of observations, μ is the general mean, $\mathbf{1}$ is a n × 1 vector of ones, $\boldsymbol{\Delta}'$ is a n × v matrix of observations versus direct treatments, $\boldsymbol{\tau}$ is a v × 1 vector of direct treatment effects, $\boldsymbol{\Delta}_1'$ is a n × v matrix of observations versus interference effect from the immediate left neighbour treatment, $\boldsymbol{\delta}$ is v × 1 vector of immediate left neighbour interference effects, $\boldsymbol{\Delta}_2'$ is a n × v matrix of observations versus interference effect from left neighbour treatment at distance 2 (leaving one plot), $\boldsymbol{\gamma}$ is v × 1 vector of left neighbor interference effects at distance 2, $\mathbf{D}'$ is a $n \times b$ incidence matrix of observations versus blocks, $\boldsymbol{\beta}$ is a $b \times 1$ vector of block effects and $\mathbf{e}$ is a $n \times 1$ vector of errors with $\mathbf{e} \sim N(\mathbf{0}, \sigma^2 \mathbf{I}_n)$.

Let,

$\boldsymbol{\Delta\Delta}_1' = \mathbf{M}_1$, $v \times v$ incidence matrix of direct treatments versus immediate left neighbour treatments.

$\boldsymbol{\Delta\Delta}_2' = \mathbf{M}_2$, $v \times v$ incidence matrix of direct treatments versus left neighbour treatments at distance 2.

$\boldsymbol{\Delta}_1\boldsymbol{\Delta}_2' = \mathbf{M}_3$, $v \times v$ incidence matrix of immediate left neighbour treatments versus left neighbour treatments at distance 2.

$\boldsymbol{\Delta}\mathbf{D}' = \mathbf{N}_1$, $v \times b$ incidence matrix of direct treatments versus blocks.

$\boldsymbol{\Delta}_1\mathbf{D}' = \mathbf{N}_2$, $v \times b$ incidence matrix of immediate left neighbour treatments versus blocks.

$\boldsymbol{\Delta}_2\mathbf{D}' = \mathbf{N}_3$, $v \times b$ incidence matrix of left neighbour treatments at distance 2 versus blocks.

$\mathbf{r} = (r_1, r_2, \ldots, r_v)'$ be the $v \times 1$ replication vector of the direct treatment with r_s as the number of times s^{th} $(s = 1,2,\ldots,v)$ treatment appears in the design.

$\mathbf{r}_1 = (r_{11}, r_{12}, \ldots, r_{1v})'$ be the $v \times 1$ replication vector of the immediate left neighbour treatment with r_{1s} as the number of times the treatments in the design has s^{th} treatment as immediate left neighbour.

$\mathbf{r}_2 = (r_{21}, r_{22}, \ldots, r_{2v})'$ be the $v \times 1$ replication vector of the left neighbour treatment at distance 2 with r_{2s} as the number of times the treatments in the design has s^{th} treatment as left neighbour at distance 2.

$\mathbf{R}_{\boldsymbol{\tau}} = \text{diag}(r_1, r_2, \ldots, r_v)$, $\mathbf{R}_{\boldsymbol{\delta}} = \text{diag}(r_{11}, r_{12}, \ldots, r_{1v})$, $\mathbf{R}_{\boldsymbol{\gamma}} = \text{diag}(r_{21}, r_{22}, \ldots, r_{2v})$.

The normal equation for estimating the direct effects of treatments as well as left interference effects of treatments up to distance 2 is

$$\mathbf{C}\,\hat{\boldsymbol{\theta}} = \mathbf{Q}, \qquad \ldots (2.2.2)$$

where $\mathbf{C}$ is the joint information matrix pertaining to direct effects, interference effects from the immediate left neighbouring units and the interference effects from the left neighbouring units at distance 2 which is obtained as

$$\mathbf{C} = \begin{bmatrix} \mathbf{R}_\tau - \frac{1}{k}\mathbf{N}_1\mathbf{N}_1' & \mathbf{M}_1 - \frac{1}{k}\mathbf{N}_1\mathbf{N}_2' & \mathbf{M}_2 - \frac{1}{k}\mathbf{N}_1\mathbf{N}_3' \\ \mathbf{M}_1' - \frac{1}{k}\mathbf{N}_2\mathbf{N}_1' & \mathbf{R}_\delta - \frac{1}{k}\mathbf{N}_2\mathbf{N}_2' & \mathbf{M}_3 - \frac{1}{k}\mathbf{N}_2\mathbf{N}_3' \\ \mathbf{M}_2' - \frac{1}{k}\mathbf{N}_3\mathbf{N}_1 & \mathbf{M}_3' - \frac{1}{k}\mathbf{N}_3\mathbf{N}_2' & \mathbf{R}_\gamma - \frac{1}{k}\mathbf{N}_3\mathbf{N}_3' \end{bmatrix} \quad \dots (2.2.3)$$

$\boldsymbol{\theta} = [\boldsymbol{\tau}' \quad \boldsymbol{\delta}' \quad \boldsymbol{\gamma}']'$, **Q** is the vector of adjusted totals pertaining to direct treatment, immediate left and left neighbour at distance 2 which is obtained as

$$\mathbf{Q} = \begin{bmatrix} \mathbf{T} - \frac{1}{k}\mathbf{N}_1\mathbf{B} \\ \mathbf{L}_1 - \frac{1}{k}\mathbf{N}_2\mathbf{B} \\ \mathbf{L}_2 - \frac{1}{k}\mathbf{N}_3\mathbf{B} \end{bmatrix}. \quad \dots (2.2.4)$$

Here, **T** is the v × 1 vector of direct treatment totals, $\mathbf{L}_1$ is the v × 1 vector of treatment totals corresponding to immediate left neighbour treatments, $\mathbf{L}_2$ is the v × 1 vector of treatment totals corresponding to left neighbour treatments at distance 2, **B** is the b × 1 vector of block totals.

The 3v × 3v matrix **C** is symmetric, non-negative definite and doubly centered. From (2.2.3), the information matrix for estimating direct treatment effects ($\mathbf{C}_\tau$) is

$$\mathbf{C}_\tau = \mathbf{C}_{11} - \mathbf{C}_{12}\mathbf{C}_{22}^{-}\mathbf{C}_{21} \quad \dots (2.2.5)$$

where

$$\mathbf{C}_{11} = \mathbf{R}_\tau - \frac{1}{k}\mathbf{N}_1\mathbf{N}_1'$$

$$\mathbf{C}_{12} = \begin{bmatrix} \mathbf{M}_1 - \frac{1}{k}\mathbf{N}_1\mathbf{N}_2' & \mathbf{M}_2 - \frac{1}{k}\mathbf{N}_1\mathbf{N}_3' \end{bmatrix}$$

$$\mathbf{C}_{22} = \begin{bmatrix} \mathbf{R}_\delta - \frac{1}{k}\mathbf{N}_2\mathbf{N}_2' & \mathbf{M}_3 - \frac{1}{k}\mathbf{N}_2\mathbf{N}_3' \\ \mathbf{M}_3' - \frac{1}{k}\mathbf{N}_3\mathbf{N}_2' & \mathbf{R}_\gamma - \frac{1}{k}\mathbf{N}_3\mathbf{N}_3' \end{bmatrix}$$

Likewise, the information matrices for estimating the first order left interference effects ($\mathbf{C}_\delta$) and second order left interference effects ($\mathbf{C}_\gamma$) can be obtained.

2.2.2 Universal Optimality of Block Designs under Interference Effect Model

A design $d^* \in D$ (the class of competing designs) is universally optimal (Kiefer, 1975), if its information matrix is such that

i) $\mathbf{C}_{d*}$ is completely symmetric i.e., $\mathbf{C}_{d*}$ is of the form $a_1\mathbf{I}_v + a_2\mathbf{11}'_v$ where $\mathbf{I}_v$ is the identity matrix of order v and $\mathbf{11}'$ is a square matrix of order v with all elements unity and a_1 and a_2 are scalars;

ii) trace ($\mathbf{C}_{d*}$) > trace ($\mathbf{C}_d$) for all d with $\mathbf{C}_d \in \mathbf{B}_{v,0}$, the class of all symmetric, non-negative definite matrices of order v with row sums equal to zero.

A universally optimal design is necessarily A-, D-, and E-optimal. Here, the search for a universally optimal block design with second order interference effects from neighbouring units is restricted in the class D (v, b, k) of left circular block designs having v treatments arranged in b blocks of size k each.

We consider a class of designs D_1 (v, b, k) in which each treatment appears in a given block an equal number of times, say λ times and for each ordered pair of treatments excluding identical pairs, there exists constant number (μ_1) of plots that have first chosen treatment as immediate left neighbour and the second one as left neighbour at distance 2.

We now prove the following theorem:

Theorem 2.2.2.1: A design $d^* \in D_1$ (v, b, k) is universally optimal for the estimation of direct effects under interference effects model if $\mathbf{M}_1 = \mathbf{M}_2 = \mu_1(\mathbf{11}' - \mathbf{I}_v)$, where μ_1 is a scalar.

Proof: Let d^* be a design in D_1 (v, b, k). We now show that the information matrix given in (2.2.5) pertaining to direct effects, $\mathbf{C}_{\tau d*}$**,** is completely symmetric and has maximum trace in the class of competing designs. Here,

$$\mathbf{R}_\tau = \mathbf{R}_\delta = \mathbf{R}_\gamma = r\,\mathbf{I}_v,\ \mathbf{DD}' = \mathbf{K} = k\,\mathbf{I}_b,$$
$$\mathbf{N_1} = \mathbf{N_2} = \mathbf{N_3} = \lambda\mathbf{11}' \text{ and } \mathbf{M}_3 = \mu_1(\mathbf{11}' - \mathbf{I}_v), \quad \dots (2.2.6)$$

where r = bλ, k = vλ, λ = k / v, μ_1 = bk /v (v-1).

The information matrix for estimating direct effects of treatments is obtained from (2.2.5) with

$$\mathbf{C}_{11} = \frac{bk}{v}\left(\mathbf{I}_v - \frac{\mathbf{11}'}{v}\right),$$
$$\mathbf{C}_{12} = \begin{bmatrix} \mathbf{M}_1 - \frac{bk}{v^2}\mathbf{11}' & \mathbf{M}_2 - \frac{bk}{v^2}\mathbf{11}' \end{bmatrix}$$

and

$$\mathbf{C}_{22} = \frac{bk}{v}\begin{bmatrix} \left(\mathbf{I}_v - \frac{\mathbf{11}'}{v}\right) & \frac{1}{(v\text{-}1)}\left(\frac{\mathbf{11}'}{v} - \mathbf{I}_v\right) \\ \frac{1}{(v\text{-}1)}\left(\frac{\mathbf{11}'}{v} - \mathbf{I}_v\right) & \left(\mathbf{I}_v - \frac{\mathbf{11}'}{v}\right) \end{bmatrix}, \quad \dots (2.2.7)$$

It can be seen that

$$\mathbf{C}_{22}^{-} = \frac{(v\text{-}1)}{bk(v\text{-}2)}\begin{bmatrix} (v\text{-}1)\mathbf{I}_v - \frac{\mathbf{11}'_v}{v} & \left(\mathbf{I}_v - \frac{\mathbf{11}'_v}{v}\right) \\ \left(\mathbf{I}_v - \frac{\mathbf{11}'_v}{v}\right) & (v\text{-}1)\mathbf{I}_v \end{bmatrix}. \quad \ldots (2.2.8)$$

The information matrix for estimating the direct effects of treatments is then obtained as

$$\mathbf{C}_{\tau d^*} = \frac{bk}{v}\left(\mathbf{I}_v - \frac{\mathbf{11}'}{v}\right) - \begin{bmatrix} \mathbf{M}_1 - \frac{bk}{v^2}\mathbf{11}' & \mathbf{M}_2 - \frac{bk}{v^2}\mathbf{11}' \end{bmatrix}$$

$$\frac{(v\text{-}1)}{bk(v\text{-}2)}\begin{bmatrix} (v\text{-}1)\mathbf{I}_v - \frac{\mathbf{11}'_v}{v} & \left(\mathbf{I}_v - \frac{\mathbf{11}'_v}{v}\right) \\ \left(\mathbf{I}_v - \frac{\mathbf{11}'_v}{v}\right) & (v\text{-}1)\mathbf{I}_v \end{bmatrix}\begin{bmatrix} \mathbf{M}_1 - \frac{bk}{v^2}\mathbf{11}' \\ \mathbf{M}_2 - \frac{bk}{v^2}\mathbf{11}' \end{bmatrix} \quad \ldots (2.2.9)$$

$$= \frac{bk}{v}\left(\mathbf{I}_v - \frac{\mathbf{11}'}{v}\right) - \frac{1}{bk(v\text{-}2)}\left[(v\text{-}1)^2\{\mathbf{M}_1\mathbf{M}'_1 + \mathbf{M}_2\mathbf{M}'_2\} + (v\text{-}1)\{\mathbf{M}_1\mathbf{M}'_2 + \mathbf{M}_2\mathbf{M}'_1\}\right]$$

$$+ \frac{2bk(v\text{-}1)}{v^2(v\text{-}2)}\mathbf{11}'$$

The trace of $\mathbf{C}_{\tau d*}$ is maximized iff $\text{trace}(\mathbf{M}_1\mathbf{M}'_1 + \mathbf{M}_2\mathbf{M}'_2)$ and $\text{trace}(\mathbf{M}_1\mathbf{M}'_2 + \mathbf{M}_2\mathbf{M}'_1)$ is minimum. These traces will be minimum when $\mathbf{M}_1 = \mathbf{M}_2 = \mu_1(\mathbf{11}' - \mathbf{I}_v)$ i.e. the design is balanced in the sense that each treatment has every other treatment as immediate left neighbour and left neighbour at distance 2, μ_1 number of times. Therefore, the information matrix for estimating direct effects of treatments is:

$$\mathbf{C}_{\tau d*} = \frac{bk(v\text{-}3)}{(v\text{-}1)(v\text{-}2)}\left(\mathbf{I}_v - \frac{\mathbf{11}'}{v}\right). \quad \ldots (2.2.10)$$

On the similar lines the following two theorems can also be proved:

Theorem 2.2.2.2: A design $d^* \in D_1$ (v, b, k) is universally optimal for the estimation of first order interference effects from the neighbouring units if $\mathbf{M}_1 = \mathbf{M}_3 = \mu_1(\mathbf{11}' - \mathbf{I}_v)$.

Theorem 2.2.2.3: A design $d^* \in D_1$ (v, b, k) is universally optimal for the estimation of second order interference effects from the neighbouring units if $\mathbf{M_2} = \mathbf{M_3} = \mu_1(\mathbf{11}' - \mathbf{I})$.

One can obtain a series of block design with v (> 3) prime treatments for the above class of universally optimal designs. These designs for v (prime) treatments are obtained by arranging the contents of the v-1 complete blocks of the design by writing the treatments in systematic order within a block with a difference of 1,2,…,v-1 between the treatments (modulo v) in the consecutive blocks. The first block is formed by taking the difference of one between treatments, the second block by taking the difference of two and so on, the $(v-1)^{th}$ block by taking the difference of (v-1). The series of complete block design is balanced for interference effect up to distance 2 with parameters v = k, b = (v-1) = r, $\mu_1 = 1$.

Example 2.2.2.1: The following left circular design with 5 treatments in 4 blocks of size 5 each is universally optimal for the estimation of direct effects, first order and second order interference effects from the neighbouring units ($\lambda = \mu_1 = 1$):

1	2	3	4	0
1	3	0	2	4
1	4	2	0	3
1	0	4	3	2

We consider another class of designs D_2 (v, b, k) in which each treatment appears in a given block an equal number of times, say λ (>1) times and for each ordered pair of treatments including identical pairs, there exists constant number (μ_1) of plots that have first chosen treatment as immediate left neighbour and the second one as left neighbour at distance 2.

We now prove the following theorem:

Theorem 2.2.2.4: A design $d^{**} \in D_2$ (v, b, k) is universally optimal for the estimation of direct effects if $\mathbf{M}_1 = \mathbf{M}_2 = \mu_1\mathbf{11}'$, where μ_1 is a scalar.

Proof: Let d^{**} be a design in D_2 (v, b, k). We now show that the information matrix pertaining to direct effects, $\mathbf{C}_{\tau d^{**}}$ is completely symmetric and has maximum trace in the class of competing designs.

Here,

$$\mathbf{R}_\tau = \mathbf{R}_\delta = \mathbf{R}_\gamma = r\,\mathbf{I}_v,\ \mathbf{DD}' = \mathbf{K} = k\,\mathbf{I}_b,$$
$$\mathbf{N}_1 = \mathbf{N}_2 = \mathbf{N}_3 = \lambda\mathbf{11}' \text{ and } \mathbf{M}_3 = \mu_1\mathbf{11}', \qquad \ldots (2.2.11)$$

where $r = b\lambda$, $k = v\lambda$, $\lambda = k / v$, $\mu_1 = bk/v^2$.

The information matrix for estimating direct effects of treatments is obtained as follows:

$$\mathbf{C}_\tau = \mathbf{C}_{11} - \mathbf{C}_{12}\mathbf{C}_{22}^{-}\mathbf{C}_{21}, \qquad \ldots (2.2.12)$$

where

$$\mathbf{C}_{11} = \frac{bk}{v}\left(\mathbf{I}_v - \frac{\mathbf{11}'}{v}\right),$$

$$\mathbf{C}_{12} = \begin{bmatrix} \mathbf{M}_1 - \frac{bk}{v^2}\mathbf{11}' & \mathbf{M}_2 - \frac{bk}{v^2}\mathbf{11}' \end{bmatrix}$$

and

$$\mathbf{C}_{22} = \begin{bmatrix} \frac{bk}{v}\left(\mathbf{I}_v - \frac{\mathbf{11}'}{v}\right) & \mathbf{0} \\ \mathbf{0} & \frac{bk}{v}\left(\mathbf{I}_v - \frac{\mathbf{11}'}{v}\right) \end{bmatrix}.$$

Clearly $\mathbf{C}_{11}$ is the information matrix for estimating direct effects of treatments when no interference effects are present in the model. Thus,

$$\text{Trace }(\mathbf{C}_{\tau d^{**}}) = \text{Trace }(\mathbf{C}_{11}) - \text{Trace }(\mathbf{C}_{12}\mathbf{C}_{22}^{-}\mathbf{C}_{21}) \leq \text{Trace }(\mathbf{C}_{11})$$

since $\mathbf{C}_{12}\mathbf{C}_{22}^{-}\mathbf{C}_{21}$ is a non-negative definite matrix. Hence, for the given class of designs D_2, trace of $\mathbf{C}_{\tau d^{**}}$ is maximized if $\mathbf{C}_{12}\mathbf{C}_{22}^{-}\mathbf{C}_{21} = \mathbf{0}$, i.e. when $\mathbf{C}_{12} = \left[\ \mathbf{M}_1 - \frac{bk}{v^2}\mathbf{11}' \quad \mathbf{M}_2 - \frac{bk}{v^2}\mathbf{11}'\ \right] = \mathbf{0}$ and $\mathbf{C}_{12}$ will be equal to zero when $\mathbf{M}_1 = \mathbf{M}_2 = \mu_1\mathbf{11}'$, i.e. the design is strongly balanced in the sense that every treatment has every other treatment including itself as immediate left neighbour and left neighbour at distance 2, μ_1 number of time. Therefore, the information matrix for estimating direct effects of treatments is:

$$\mathbf{C}_{\tau d^{**}} = \frac{bk}{v}\left(\mathbf{I}_v - \frac{\mathbf{11}'}{v}\right). \qquad \dots (2.2.13)$$

Thus, $\mathbf{C}_{\tau d^{**}}$ is completely symmetric and has the maximum trace in the class of competing designs.

On the similar lines the following two theorems can also be proved:

Theorem 2.2.2.5: A design $d^{**} \in D_2$ (v, b, k) is universally optimal for the estimation of first order interference effects from the neighbouring units if $\mathbf{M}_1 = \mathbf{M}_3 = \mu_1\mathbf{11}'$.

Theorem 2.2.2.6: A design $d^{**} \in D_2$ (v, b, k) is universally optimal for the estimation of second order interference effects from the neighbouring units if $\mathbf{M}_2 = \mathbf{M}_3 = \mu_1\mathbf{11}'$.

A series of block designs D_2 (v, b = v^2, k = 3v) for second order interference effects on one side are universally optimal for the estimation of direct effects and first order interference effects and second order interference effects. These designs with r = $3v^2$, λ = 3, μ_1 = 3v can be obtained using the method of construction of change over design for second order residual effects by Sharma (1977) considering the units as blocks and making the blocks left-circular up to distance 2.

Example 2.2.2.2: The following left circular design with 4 treatments in 16 blocks of size 12 each is universally optimal for the estimation of direct effects, first order and second order interference effects from the neighbouring units ($\lambda = 3$, $\mu_1 = 12$):

1	1	1	2	2	2	3	3	3	0	0	0
2	1	2	3	2	3	0	3	0	1	0	1
3	1	3	0	2	0	1	3	1	2	0	2
0	1	0	1	2	1	2	3	2	3	0	3
1	2	2	2	3	3	3	0	0	0	1	1
2	2	3	3	3	0	0	0	1	1	1	2
3	2	0	0	3	1	1	0	2	2	1	3
0	2	1	1	3	2	2	0	3	3	1	0
1	3	3	2	0	0	3	1	1	0	2	2
2	3	0	3	0	1	0	1	2	1	2	3
3	3	1	0	0	2	1	1	3	2	2	0
0	3	2	1	0	3	2	1	0	3	2	1
1	0	0	2	1	1	3	2	2	0	3	3
2	0	1	3	1	2	0	2	3	1	3	0
3	0	2	0	1	3	1	2	0	2	3	1
0	0	3	1	1	0	2	2	1	3	3	2

2.3 Block Designs with Spatial Indirect Effects under a Non-Additive Model

A distinctive feature of block design with interference effects from the neighbouring units is that the effect of a treatment applied to a plot is the sum of direct effect due to the treatment applied to the given plot and various interference effects of treatments applied to the neighbouring plots along with the block effects. However, under block design setup with interference effects from the neighbouring units, situations may arise where there could be interaction among direct and various interference effects from the neighbouring units. In such situations, effects are no longer additive in nature and thus instead of considering an additive model, we have to explore non-additivity in block models with interference effects from neighbouring units. Patterson (1970) considered direct × residual interaction under change over design setup and obtained designs that give the estimates of interaction effects. The robustness of some optimality results on change over designs when the underlying model is a non-additive incorporating an interaction due to direct and residual effects of treatments was investigated by Sen and Mukerjee (1987). Bose and Mukherjee (2000) studied cross over design in the presence of higher order carryover effects for non-additive models. Some small and efficient cross-over designs under a non-additive model have been obtained by Bose and Dey (2003). Varghese *et al.* (2009) studied optimal cross over design under treatment × unit interaction.

Here, a non-additive block model with block effects, direct effects of treatment, interference effects of treatment from the immediate left neighbouring units, interference effects of treatment from the immediate right neighbouring units, left interference effects × direct effects, direct effects × right interference effects, left interference effects × right interference effects and left interference effects × direct effects × right interference effects has been considered. Under this model, a class of complete, circular block designs balanced for both-sided interference effects from the neighbouring units has been shown to be universally optimal for the estimation of both direct effects and various interference effects from the neighbouring units among the class of all competing designs.

2.3.1 Non-Additive Block Model Under Two-Sided Interference Effects of Treatments

We consider a class of block designs with v treatments applied in b blocks each of size k and each treatment replicated r number of times. Let, d(i, j) denote the treatment applied to the i^{th} plot of the j^{th} block, i = 1, 2,…, k; j = 1, 2,…, b. The layout includes border plots at both end of every block making the design as circular. Let Y_{ij} be the response from the i^{th} plot in the j^{th} block. We define the following non-additive model:

$$Y_{ij} = \mu + \tau_{d(i,j)} + \delta_{d(i-1,j)} + \alpha_{d(i+1,j)} + \gamma_{d(i-1,j)d(i,j)} + \varphi_{d(i,j)d(i+1,j)} + \pi_{d(i-1,j)d(i+1,j)} + \rho_{d(i-1,j)\,d(i,j)d(i+1,j)} + \beta_j + e_{ij}; \quad \ldots (2.3.1)$$

for all i = 1, 2,…, k and j = 1, 2,…, b. Here, μ is the general mean, $\tau_{d(i,j)}$ is the direct effect of treatment d(i, j), $\delta_{d(i-1,j)}$ is the interference effect due to the left neighbour treatment d(i-1,j), $\alpha_{d(i+1,j)}$ is the interference effect due to the right neighbour treatment d(i+1, j), $\gamma_{d(i-1,j)d(i,j)}$ is the interaction effect between the direct treatment d(i, j) and it's left neighbour treatment d(i-1, j), $\varphi_{d(i,j)d(i+1,j)}$ is the interaction effect between the direct treatment d(i, j) and it's right neighbour treatment d(i+1, j), $\pi_{d(i-,j)d(i+1,j)}$ is the interaction effect between the left neighbour treatment d(i-1, j) and right neighbour treatment d(i+1, j) of d(i, j), $\rho_{d(i-,j)d(i,j)d(i+1,j)}$ is the interaction effect between the direct treatment d(i, j), it's left neighbour treatment d(i-1, j) and it's right neighbour treatment d(i+1, j), β_j is the j^{th} block effect, e_{ij} is the error term which are independently and identically distributed $N(0, \sigma^2)$.

This experimental setup can be looked upon as a v^3 factorial experiment with three factors F_1, F_2 and F_3. The direct effects correspond to the main effect of F_1, the interference effects from the immediate left neighbouring unit correspond to the main effect of F_2, the interference effects from the immediate right neighbouring unit correspond to the main effect of F_3, the direct effects versus interference effects from the left neighbouring units corresponds to the factorial interaction F_1F_2, the direct effects versus interference effects from the right neighbouring units corresponds to the usual factorial interaction F_1F_3, the interference effects from the left neighbouring units versus interference effects from the right neighbouring units corresponds to the usual factorial interaction F_2F_3 and direct effects versus interference effects from both left and right neighbouring units corresponds to the usual factorial interaction $F_1F_2F_3$. This setup can be used to analyze designs under the following model introduced by Kurkjian and Zelen (1962) by applying the calculus for factorial arrangements:

$$Y_{ij} = \mu + \beta_j + \boldsymbol{\omega}'_{ij}\boldsymbol{\xi} + e_{ij}; \quad i = 1, 2,\ldots, k;\ j = 1, 2,\ldots, b \quad \ldots (2.3.2)$$

where $v^3 \times 1$ vector $\boldsymbol{\xi}$ is the vector of v^3 factorial effects of treatment combinations and

$$\boldsymbol{\omega}_{ij} = \mathbf{e}_{d(i-1,j)} \otimes \mathbf{e}_{d(i,j)} \otimes \mathbf{e}_{d(i+1,j)}, \quad \ldots (2.3.3)$$

where $\mathbf{e}_{d(i,\, j)}$ is a v×1 vector with 1 in the position corresponding to the treatment d(i, j) and zero elsewhere and $\otimes$ denotes the kronecker product. For i = 1, $\mathbf{e}_{d(0,\, j)}$ indicate the border plot of the jth block at left side and for i = k, $\mathbf{e}_{d(k+1,\, j)}$ indicate the border plot of the jth block at right side. In matrix notation, Model (2.3.2) can be written as:

$$\mathbf{Y} = \mu\mathbf{1} + \mathbf{D}'\boldsymbol{\beta} + \boldsymbol{\omega}'\xi + \mathbf{e}, \qquad \ldots (2.3.4)$$

where **Y** is a n × 1 vector of observations (n is the total number of observations), **1** is a n × 1 vector of unity, **D**' is a n × b incidence matrix of observations versus blocks, $\boldsymbol{\beta}$ is a b × 1 vector of block effects, $\boldsymbol{\omega}'$ is n × v^3 incidence matrix of observation versus v^3 factorial treatment combinations with $\boldsymbol{\omega}'_{ij}$ as the rows of $\boldsymbol{\omega}'$ and **e** is a n × 1 vector of errors where errors are independently and identically normally distributed random variable with zero mean and constant variance. This model can be expressed as:

$$\mathbf{Y} = \mathbf{X}_d\boldsymbol{\theta} + \mathbf{e}, \qquad \ldots (2.3.5)$$

with $\mathbf{X}_d = \begin{bmatrix}\mathbf{1} & \mathbf{D}' & \boldsymbol{\omega}'\end{bmatrix}$ and $\boldsymbol{\theta} = \begin{bmatrix}\mu & \boldsymbol{\beta}' & \xi'\end{bmatrix}'$.

Thus,

$$\mathbf{X}_d'\mathbf{X}_d = \begin{bmatrix}\mathbf{1}'\mathbf{1} & \mathbf{1}'\mathbf{D}' & \mathbf{1}'\boldsymbol{\omega}' \\ \mathbf{D}\mathbf{1} & \mathbf{D}\mathbf{D}' & \mathbf{D}\boldsymbol{\omega}' \\ \boldsymbol{\omega}\mathbf{1} & \boldsymbol{\omega}\mathbf{D}' & \boldsymbol{\omega}\boldsymbol{\omega}'\end{bmatrix} = \begin{bmatrix} n & k\mathbf{1}_b' & \sum_{i=1}^{k}\sum_{j=1}^{b}\boldsymbol{\omega}'_{ij} \\ k\mathbf{1}_b & k\,\mathbf{I}_b & \boldsymbol{\Omega}_d' \\ \sum_{i=1}^{k}\sum_{j=1}^{b}\boldsymbol{\omega}_{ij} & \boldsymbol{\Omega}_d & \mathbf{R}_d \end{bmatrix} = \begin{bmatrix}\mathbf{X}_{2d}'\mathbf{X}_{2d} & \mathbf{X}_{2d}'\mathbf{X}_{1d} \\ \mathbf{X}_{1d}'\mathbf{X}_{2d} & \mathbf{X}_{1d}'\mathbf{X}_{1d}\end{bmatrix}$$

Where $\mathbf{R}_d = \boldsymbol{\omega}\boldsymbol{\omega}'$, $\boldsymbol{\Omega}_d = \boldsymbol{\omega}\mathbf{D}'$, $\mathbf{X}_{2d} = \begin{bmatrix}\mathbf{1} & \mathbf{D}'\end{bmatrix}$ and $\mathbf{X}_{1d} = \boldsymbol{\omega}'$.

The information matrix for estimating ξ i.e. for estimating direct effect of treatments, interference effects of treatment from the left neighbouring units, interference effects of treatment from the right neighbouring units, left neighbour interference × direct interaction effects, direct effects × right neighbour interference interaction effects, left interference × right interference interaction effects and left interference × direct × right interference interaction effects is given by

$$\mathbf{C}_d = \mathbf{X}_{1d}'\mathbf{X}_{1d} - \mathbf{X}_{1d}'\mathbf{X}_{2d}\left(\mathbf{X}_{2d}'\mathbf{X}_{2d}\right)^{-}\mathbf{X}_{2d}'\mathbf{X}_{1d}$$

$$= \mathbf{R}_d - \begin{bmatrix}\sum_{i=1}^{k}\sum_{j=1}^{b}\boldsymbol{\omega}_{ij} & \boldsymbol{\Omega}_d\end{bmatrix}\begin{bmatrix}0 & \mathbf{0} \\ \mathbf{0} & \frac{1}{k}\mathbf{I}_b\end{bmatrix}\begin{bmatrix}\sum_{i=1}^{k}\sum_{j=1}^{b}\boldsymbol{\omega}'_{ij} \\ \boldsymbol{\Omega}_d'\end{bmatrix} \qquad \ldots (2.3.6)$$

$$= \mathbf{R}_d - \frac{1}{k}\boldsymbol{\Omega}_d\boldsymbol{\Omega}_d'$$

The v^3 × v^3 matrix $\mathbf{C}_d$ is symmetric, non-negative definite with zero row and column sums.

2.3.2 Universal Optimality of Block Design Under Non-Additive Interference Effects Model

Consider a complete, circular block design d_1 with interference effects from the neighbouring units which satisfy the following conditions:

i) Each treatment appears in a given block an equal number of times

ii) For each ordered pair of treatments including identical pairs, there exists constant number (μ_1) of plots that have first chosen treatment as immediate left neighbour and the second one as immediate right neighbour.

iii) Every v^3 treatment combination of left neighbour × direct × right neighbour appears a constant number of times (say λ) in the design.

We now state the following theorem to establish the optimality of design d_1 in the competing class of designs, D.

Theorem 2.3.2.1: A complete, circular block design d_1 balanced for interference effects from the neighbouring units on both-sides, whenever exists, is universally optimal for the estimation of direct effects of treatments among all the competing designs under the non-additive model (2.3.1).

Proof: Let $\mathbf{P}_v$ be a (v-1) × v matrix such that $\left(\frac{1}{\sqrt{v}}\mathbf{1}_v, \mathbf{P}'_v\right)$ is orthogonal. We define

$$\mathbf{P}^{100} = \mathbf{P}_v \otimes \left(\frac{1}{\sqrt{v}}\mathbf{1}'_v\right) \otimes \left(\frac{1}{\sqrt{v}}\mathbf{1}'_v\right);\ \mathbf{P}^{010} = \left(\frac{1}{\sqrt{v}}\mathbf{1}'_v\right) \otimes \mathbf{P}_v \otimes \left(\frac{1}{\sqrt{v}}\mathbf{1}'_v\right);\ \mathbf{P}^{001} = \left(\frac{1}{\sqrt{v}}\mathbf{1}'_v\right) \otimes \left(\frac{1}{\sqrt{v}}\mathbf{1}'_v\right) \otimes \mathbf{P}_v;$$

$$\mathbf{P}^{110} = \mathbf{P}_v \otimes \mathbf{P}_v \otimes \left(\frac{1}{\sqrt{v}}\mathbf{1}'_v\right);\ \mathbf{P}^{011} = \left(\frac{1}{\sqrt{v}}\mathbf{1}'_v\right) \otimes \mathbf{P}_v \otimes \mathbf{P}_v;\ \mathbf{P}^{101} = \mathbf{P}_v \otimes \left(\frac{1}{\sqrt{v}}\mathbf{1}'_v\right) \otimes \mathbf{P}_v;\ \mathbf{P}^{111} = \mathbf{P}_v \otimes \mathbf{P}_v \otimes \mathbf{P}_v;$$

$\mathbf{P}^{100}\xi_1$, $\mathbf{P}^{010}\xi_1$, $\mathbf{P}^{001}\xi_1$, $\mathbf{P}^{110}\xi_1$, $\mathbf{P}^{011}\xi_1$, $\mathbf{P}^{101}\xi_1$, and $\mathbf{P}^{111}\xi_1$, together represent a complete set of orthonormal treatment contrasts.

Following Mukerjee (1980), the coefficient matrix of the reduced normal equations for estimating the direct effect of treatments is given by

$$\mathbf{C}_{(dir)} = \mathbf{C}_{11} - \mathbf{C}_{12}\mathbf{C}_{22}^{-}\mathbf{C}_{21} \qquad \dots (2.3.7)$$

where,

$$\mathbf{C}_{11} = \mathbf{P}^{010}\mathbf{C}_d\left(\mathbf{P}^{010}\right)'$$

$$\mathbf{C}_{12} = \left[\mathbf{P}^{010}\mathbf{C}_d\left(\mathbf{P}^{100}\right)' \quad \mathbf{P}^{010}\mathbf{C}_d\left(\mathbf{P}^{001}\right)' \quad \mathbf{P}^{010}\mathbf{C}_d\left(\mathbf{P}^{110}\right)' \quad \mathbf{P}^{010}\mathbf{C}_d\left(\mathbf{P}^{011}\right)' \quad \mathbf{P}^{010}\mathbf{C}_d\left(\mathbf{P}^{101}\right)' \quad \mathbf{P}^{010}\mathbf{C}_d\left(\mathbf{P}^{111}\right)'\right]$$

$$\mathbf{C}_{21} = \left[\mathbf{P}^{010}\mathbf{C}_d\left(\mathbf{P}^{100}\right)' \quad \mathbf{P}^{010}\mathbf{C}_d\left(\mathbf{P}^{001}\right)' \quad \mathbf{P}^{010}\mathbf{C}_d\left(\mathbf{P}^{110}\right)' \quad \mathbf{P}^{010}\mathbf{C}_d\left(\mathbf{P}^{011}\right)' \quad \mathbf{P}^{010}\mathbf{C}_d\left(\mathbf{P}^{101}\right)' \quad \mathbf{P}^{010}\mathbf{C}_d\left(\mathbf{P}^{111}\right)'\right]'$$

and

$$\mathbf{C}_{22}=\begin{bmatrix}\mathbf{P}^{100}\mathbf{C}_d(\mathbf{P}^{100})' & \mathbf{P}^{100}\mathbf{C}_d(\mathbf{P}^{001})' & \mathbf{P}^{100}\mathbf{C}_d(\mathbf{P}^{110})' & \mathbf{P}^{100}\mathbf{C}_d(\mathbf{P}^{011})' & \mathbf{P}^{100}\mathbf{C}_d(\mathbf{P}^{101})' & \mathbf{P}^{100}\mathbf{C}_d(\mathbf{P}^{111})' \\ \mathbf{P}^{001}\mathbf{C}_d(\mathbf{P}^{100})' & \mathbf{P}^{001}\mathbf{C}_d(\mathbf{P}^{001})' & \mathbf{P}^{001}\mathbf{C}_d(\mathbf{P}^{110})' & \mathbf{P}^{001}\mathbf{C}_d(\mathbf{P}^{011})' & \mathbf{P}^{001}\mathbf{C}_d(\mathbf{P}^{101})' & \mathbf{P}^{001}\mathbf{C}_d(\mathbf{P}^{111})' \\ \mathbf{P}^{110}\mathbf{C}_d(\mathbf{P}^{100})' & \mathbf{P}^{110}\mathbf{C}_d(\mathbf{P}^{001})' & \mathbf{P}^{110}\mathbf{C}_d(\mathbf{P}^{110})' & \mathbf{P}^{110}\mathbf{C}_d(\mathbf{P}^{011})' & \mathbf{P}^{110}\mathbf{C}_d(\mathbf{P}^{101})' & \mathbf{P}^{110}\mathbf{C}_d(\mathbf{P}^{111})' \\ \mathbf{P}^{011}\mathbf{C}_d(\mathbf{P}^{100})' & \mathbf{P}^{011}\mathbf{C}_d(\mathbf{P}^{001})' & \mathbf{P}^{011}\mathbf{C}_d(\mathbf{P}^{110})' & \mathbf{P}^{011}\mathbf{C}_d(\mathbf{P}^{011})' & \mathbf{P}^{011}\mathbf{C}_d(\mathbf{P}^{101})' & \mathbf{P}^{011}\mathbf{C}_d(\mathbf{P}^{111})' \\ \mathbf{P}^{101}\mathbf{C}_d(\mathbf{P}^{100})' & \mathbf{P}^{101}\mathbf{C}_d(\mathbf{P}^{001})' & \mathbf{P}^{101}\mathbf{C}_d(\mathbf{P}^{110})' & \mathbf{P}^{101}\mathbf{C}_d(\mathbf{P}^{011})' & \mathbf{P}^{101}\mathbf{C}_d(\mathbf{P}^{101})' & \mathbf{P}^{101}\mathbf{C}_d(\mathbf{P}^{111})' \\ \mathbf{P}^{111}\mathbf{C}_d(\mathbf{P}^{100})' & \mathbf{P}^{111}\mathbf{C}_d(\mathbf{P}^{001})' & \mathbf{P}^{111}\mathbf{C}_d(\mathbf{P}^{110})' & \mathbf{P}^{111}\mathbf{C}_d(\mathbf{P}^{011})' & \mathbf{P}^{111}\mathbf{C}_d(\mathbf{P}^{101})' & \mathbf{P}^{111}\mathbf{C}_d(\mathbf{P}^{111})'\end{bmatrix}$$

$\mathbf{C}_d$ is as in equation (2.3.6). For the design d_1, $\mathbf{R}_{d_1}=\lambda(\mathbf{I}_v\otimes\mathbf{I}_v\otimes\mathbf{I}_v)=bkv^{-3}(\mathbf{I}_v\otimes\mathbf{I}_v\otimes\mathbf{I}_v)$, where the symbols have their usual meaning as defined earlier. Then we can write:

$$\begin{aligned}&\mathbf{P}^{010}\mathbf{R}_{d_1}(\mathbf{P}^{010})'=\lambda\mathbf{I}_{v-1}=bkv^{-3}\mathbf{I}_{v-1};\\ &\mathbf{P}^{010}\mathbf{R}_{d_1}(\mathbf{P}^{100})'=\mathbf{0};\ \mathbf{P}^{010}\mathbf{R}_{d_1}(\mathbf{P}^{001})'=\mathbf{0};\ \mathbf{P}^{010}\mathbf{R}_{d_1}(\mathbf{P}^{110})'=\mathbf{0};\\ &\mathbf{P}^{010}\mathbf{R}_{d_1}(\mathbf{P}^{011})'=\mathbf{0};\ \mathbf{P}^{010}\mathbf{R}_{d_1}(\mathbf{P}^{101})'=\mathbf{0};\ \mathbf{P}^{010}\mathbf{R}_{d_1}(\mathbf{P}^{111})'=\mathbf{0};\end{aligned}\qquad \ldots (2.3.8)$$

and $\mathbf{\Omega}_{d_1}$ is such that $\mathbf{P}^{010}\mathbf{\Omega}_{d_1}\mathbf{\Omega}'_{d_1}=\mathbf{0}$.

Hence, for the design d_1, the coefficient matrix of the reduced normal equations for estimating the direct effect of treatments is given by:

$$\begin{aligned}\mathbf{C}_{d_1(dir)}&=\mathbf{C}_{11}^{1}-\mathbf{C}_{12}^{1}\mathbf{C}_{22}^{1-}\mathbf{C}_{21}^{1}\\ &=\lambda\mathbf{I}_{v-1}\\ &=bkv^{-3}\mathbf{I}_{v-1}\end{aligned}\qquad \ldots (2.3.9)$$

where

$$\mathbf{C}_{11}^{1}=\mathbf{P}^{010}\mathbf{C}_{d_1}(\mathbf{P}^{010})'$$

$$\mathbf{C}_{12}^{1}=\begin{bmatrix}\mathbf{P}^{010}\mathbf{C}_{d_1}(\mathbf{P}^{100})' & \mathbf{P}^{010}\mathbf{C}_{d_1}(\mathbf{P}^{001})' & \mathbf{P}^{010}\mathbf{C}_{d_1}(\mathbf{P}^{110})' & \mathbf{P}^{010}\mathbf{C}_{d_1}(\mathbf{P}^{011})' & \mathbf{P}^{010}\mathbf{C}_{d_1}(\mathbf{P}^{101})' & \mathbf{P}^{010}\mathbf{C}_{d_1}(\mathbf{P}^{111})'\end{bmatrix}$$

$$\mathbf{C}_{21}^{1}=\begin{bmatrix}\mathbf{P}^{010}\mathbf{C}_{d_1}(\mathbf{P}^{100})' & \mathbf{P}^{010}\mathbf{C}_{d_1}(\mathbf{P}^{001})' & \mathbf{P}^{010}\mathbf{C}_{d_1}(\mathbf{P}^{110})' & \mathbf{P}^{010}\mathbf{C}_{d_1}(\mathbf{P}^{011})' & \mathbf{P}^{010}\mathbf{C}_{d_1}(\mathbf{P}^{101})' & \mathbf{P}^{010}\mathbf{C}_{d_1}(\mathbf{P}^{111})'\end{bmatrix}'$$

$$\mathbf{C}_{22}^{1}=\begin{bmatrix}\mathbf{P}^{100}\mathbf{C}_{d_1}(\mathbf{P}^{100})' & \mathbf{P}^{100}\mathbf{C}_{d_1}(\mathbf{P}^{001})' & \mathbf{P}^{100}\mathbf{C}_{d_1}(\mathbf{P}^{110})' & \mathbf{P}^{100}\mathbf{C}_{d_1}(\mathbf{P}^{011})' & \mathbf{P}^{100}\mathbf{C}_{d_1}(\mathbf{P}^{101})' & \mathbf{P}^{100}\mathbf{C}_{d_1}(\mathbf{P}^{111})' \\ \mathbf{P}^{001}\mathbf{C}_{d_1}(\mathbf{P}^{100})' & \mathbf{P}^{001}\mathbf{C}_{d_1}(\mathbf{P}^{001})' & \mathbf{P}^{001}\mathbf{C}_{d_1}(\mathbf{P}^{110})' & \mathbf{P}^{001}\mathbf{C}_{d_1}(\mathbf{P}^{011})' & \mathbf{P}^{001}\mathbf{C}_{d_1}(\mathbf{P}^{101})' & \mathbf{P}^{001}\mathbf{C}_{d_1}(\mathbf{P}^{111})' \\ \mathbf{P}^{110}\mathbf{C}_{d_1}(\mathbf{P}^{100})' & \mathbf{P}^{110}\mathbf{C}_{d_1}(\mathbf{P}^{001})' & \mathbf{P}^{110}\mathbf{C}_{d_1}(\mathbf{P}^{110})' & \mathbf{P}^{110}\mathbf{C}_{d_1}(\mathbf{P}^{011})' & \mathbf{P}^{110}\mathbf{C}_{d_1}(\mathbf{P}^{101})' & \mathbf{P}^{110}\mathbf{C}_{d_1}(\mathbf{P}^{111})' \\ \mathbf{P}^{011}\mathbf{C}_{d_1}(\mathbf{P}^{100})' & \mathbf{P}^{011}\mathbf{C}_{d_1}(\mathbf{P}^{001})' & \mathbf{P}^{011}\mathbf{C}_{d_1}(\mathbf{P}^{110})' & \mathbf{P}^{011}\mathbf{C}_{d_1}(\mathbf{P}^{011})' & \mathbf{P}^{011}\mathbf{C}_{d_1}(\mathbf{P}^{101})' & \mathbf{P}^{011}\mathbf{C}_{d_1}(\mathbf{P}^{111})' \\ \mathbf{P}^{101}\mathbf{C}_{d_1}(\mathbf{P}^{100})' & \mathbf{P}^{101}\mathbf{C}_{d_1}(\mathbf{P}^{001})' & \mathbf{P}^{101}\mathbf{C}_{d_1}(\mathbf{P}^{110})' & \mathbf{P}^{101}\mathbf{C}_{d_1}(\mathbf{P}^{011})' & \mathbf{P}^{101}\mathbf{C}_{d_1}(\mathbf{P}^{101})' & \mathbf{P}^{101}\mathbf{C}_{d_1}(\mathbf{P}^{111})' \\ \mathbf{P}^{111}\mathbf{C}_{d_1}(\mathbf{P}^{100})' & \mathbf{P}^{111}\mathbf{C}_{d_1}(\mathbf{P}^{001})' & \mathbf{P}^{111}\mathbf{C}_{d_1}(\mathbf{P}^{110})' & \mathbf{P}^{111}\mathbf{C}_{d_1}(\mathbf{P}^{011})' & \mathbf{P}^{111}\mathbf{C}_{d_1}(\mathbf{P}^{101})' & \mathbf{P}^{111}\mathbf{C}_{d_1}(\mathbf{P}^{111})'\end{bmatrix}$$

Thus from equation (2.3.9), it can be seen that $\mathbf{C}_{d_1(dir)}$ is completely symmetric.

From equation (2.3.7), it is clear that $\mathbf{P}^{010}\mathbf{C}_d\left(\mathbf{P}^{010}\right)' \geq \mathbf{C}_{(dir)} \Rightarrow \mathbf{P}^{010}\mathbf{C}_d\left(\mathbf{P}^{010}\right)' - \mathbf{C}_{(dir)}$ is nonnegative definite. Again equation (2.3.6) implies that $\mathbf{R}_d \geq \mathbf{C}_d \Rightarrow \mathbf{R}_d - \mathbf{C}_d$ is a nonnegative definite. Hence $\mathbf{P}^{010}\mathbf{R}_d\left(\mathbf{P}^{010}\right)' - \mathbf{P}^{010}\mathbf{C}_d\left(\mathbf{P}^{010}\right)'$ is a nonnegative definite. Therefore,

$$\begin{aligned}\text{trace}\left[\mathbf{C}_{(dir)}\right] \leq \text{trace}\left[\mathbf{P}^{010}\mathbf{C}_d\left(\mathbf{P}^{010}\right)'\right] &\leq \text{trace}\left[\mathbf{P}^{010}\mathbf{R}_{d_1}\left(\mathbf{P}^{010}\right)'\right] \\ &= \text{trace}\left[\mathbf{P}^{010}\mathbf{C}_{d_1}\left(\mathbf{P}^{010}\right)'\right] \\ &= \text{trace}\left[\mathbf{C}_{d_1(dir)}\right]\end{aligned}$$

Hence,

$$\text{trace}\left[\mathbf{C}_{(dir)}\right] \leq \text{trace}\left[\mathbf{C}_{d_1(dir)}\right] \text{for all designs in D} \qquad \ldots (2.3.10)$$

Thus, on the lines of Bose and Dey (2003), the block design d_1 with interference effects from the neighbouring units is universally optimal for the estimation of direct effects of treatments among all the competing design under the non-additive model (2.3.1).

On the similar lines the following theorem can also be proved:

Theorem 2.3.2.2: A complete, circular block design d_1 balanced for two-sided interference effects from the left neighbouring and right neighbouring units, whenever exists, is universally optimal for the estimation of interference effects of treatments from the immediate left and right neighbouring units among all the competing designs under the non-additive model (2.3.1).

2.3.3 Series of Universally Optimal Block Designs

For given v, a series of complete circular block designs balanced for interference effects from both sides with parameters v, $b = v^2$, $r = 3v^2$, $k = 3v$, $\lambda = v - 1$ and $\mu_1 = 3v$ are obtained using the method of construction of change over design for second order residual effects by Sharma (1977) considering the units as blocks and making the blocks circular. The design so obtained will ensure that each v^3 treatment combinations of left neighbour × direct × right neighbour will appear a constant number ($\lambda = v - 1$) of times in the design. The resultant design will be universally optimal for the estimation of direct effects of treatment as well as both left and right interference effects of treatments under the non-additive model.

Example 2.3.3.1: The following complete circular block design with 4 treatments in 16 blocks (rows as blocks) of size 12 each is universally optimal for the estimation of direct effects, interference effects from the immediate left and right neighbouring units under the non-additive model ($\lambda = 3$, $\mu_1 = 12$):

1	1	1	2	2	2	3	3	3	4	4	4
2	1	2	3	2	3	4	3	4	1	4	1
3	1	3	4	2	4	1	3	1	2	4	2
4	1	4	1	2	1	2	3	2	3	4	3
1	2	2	2	3	3	3	4	4	4	1	1
2	2	3	3	3	4	4	4	1	1	1	2
3	2	4	4	3	1	1	4	2	2	1	3
4	2	1	1	3	2	2	4	3	3	1	4
1	3	3	2	4	4	3	1	1	4	2	2
2	3	4	3	4	1	4	1	2	1	2	3
3	3	1	4	4	2	1	1	3	2	2	4
4	3	2	1	4	3	2	1	4	3	2	1
1	4	4	2	1	1	3	2	2	4	3	3
2	4	1	3	1	2	4	2	3	1	3	4
3	4	2	4	1	3	1	2	4	2	3	1
4	4	3	1	1	4	2	2	1	3	3	2

2.3.4 Non-Additive Block Model Under One-Sided Interference Effects of Treatments

Here, a non-additive block model with block effects, direct effects of treatment, interference effects of treatment from the immediate left neighbouring units and left interference × direct effects has been considered. Druilhet and Tinsson (2009) studied optimal repeated measurement designs for a model with partial interactions. Park *et al.* (2011) studied efficient crossover designs in the presence of interaction between direct and residual effects of treatments.

Consider a class of block designs with v treatments applied in b blocks each of size k and each treatment replicated r number of times. Let d(i, j) denote the treatment applied to the i^{th} plot of the j^{th} block, i = 1, 2,… ,k; j = 1, 2,… ,b. Let Y_{ij} be the response from the i^{th} plot in the j^{th} block.

We define the following non-additive model:

$$Y_{ij} = \mu + \tau_{d(i,j)} + \delta_{d(i-1,j)} + \gamma_{d(i-1,j)d(i,j)} + \beta_j + e_{ij}\,;\ i = 1,2,\ldots,k;\ j = 1,2,\ldots,b \qquad \ldots (2.3.11)$$

where μ is the general mean, $\tau_{d(i,j)}$ is the direct effect of treatment d(i, j), $\delta_{d(i-1,\ j)}$ is the interference effect due to the left neighbour treatment d(i-1, j), $\gamma_{d(i-1,\ j)d(i,\ j)}$ is the interaction effect between the direct treatment d(i, j) and it's left neighbour treatment d(i-1, j), β_j is the j^{th} block effect, e_{ij} is the error term which are independently and identically distributed $N(0, \sigma^2)$.

This experimental setup can be considered as a v^2 factorial experiment with two factors F_1 and F_2. The direct effects correspond to the main effect F_1, the interference effects from the immediate left neighbouring unit correspond to the main effect F_2 and interference effects from the left neighbouring units versus the direct effects corresponds to the usual factorial interaction F_1F_2. These designs may be analyzed under the following model by applying the calculus for factorial arrangements:

$$Y_{ij} = \mu + \beta_j + \boldsymbol{\lambda}'_{ij}\boldsymbol{\xi}_1 + e_{ij}; \quad i = 1,2,\ldots,k;\ j = 1,2,\ldots,b, \qquad \ldots (2.3.12)$$

where $v^2 \times 1$ vector $\boldsymbol{\xi}_1 = [\xi_{11}, \xi_{12}, \ldots, \xi_{1v}, \xi_{21}, \ldots, \xi_{2v}, \ldots, \xi_{v1}, \ldots, \xi_{vv}]'$ is the vector of the effects of v^2 factorial treatment combinations and

$$\boldsymbol{\lambda}_{ij} = \mathbf{e}_{d(i-1,\, j)} \otimes \mathbf{e}_{d(i,\, j)}, \qquad \ldots (2.3.13)$$

where $\mathbf{e}_{d(i,\, j)}$ is a v×1 vector with 1 in the position corresponding to the treatment d(i, j) and zero elsewhere and $\otimes$ denotes the kronecker product. For i = 1, $\mathbf{e}_{d(0,\, j)}$ indicate the left border plot of the j^{th} block. In matrix notation, model (2.3.12) can be written as:

$$\mathbf{Y} = \mu\mathbf{1} + \mathbf{D}'\boldsymbol{\beta} + \boldsymbol{\lambda}'\boldsymbol{\xi}_1 + \mathbf{e}, \qquad \ldots (2.3.14)$$

where $\mathbf{Y}$ is a n × 1 vector of observations (n is the number of observations), $\mathbf{1}$ is a n × 1 vector of unity, $\mathbf{D}'$ is a n × b incidence matrix of observations versus blocks, $\boldsymbol{\beta}$ is a b × 1 vector of block effects, $\boldsymbol{\lambda}'$ is n × v^2 incidence matrix of observation versus v^2 factorial treatment combinations with $\boldsymbol{\lambda}'_{ij}$ as the rows of $\boldsymbol{\lambda}'$ and $\mathbf{e}$ is a n × 1 vector of errors where errors are independently and identically normally distributed random variable with zero mean and constant variance.

Model (2.3.14) can also be expressed as:

$$\mathbf{Y} = \mathbf{X}_d\boldsymbol{\theta} + \mathbf{e}, \qquad \ldots (2.3.15)$$

with $\mathbf{X}_d = \begin{bmatrix}\mathbf{1} & \mathbf{D}' & \boldsymbol{\lambda}'\end{bmatrix}$ and $\boldsymbol{\theta} = \begin{bmatrix}\mu & \boldsymbol{\beta}' & \boldsymbol{\xi}'\end{bmatrix}'$.

Thus,

$$\mathbf{X}'_d\mathbf{X}_d = \begin{bmatrix}\mathbf{1}'\mathbf{1} & \mathbf{1}'\mathbf{D}' & \mathbf{1}'\boldsymbol{\lambda}' \\ \mathbf{D}\mathbf{1} & \mathbf{D}\mathbf{D}' & \mathbf{D}\boldsymbol{\lambda}' \\ \boldsymbol{\lambda}\mathbf{1} & \boldsymbol{\lambda}\mathbf{D}' & \boldsymbol{\lambda}\boldsymbol{\lambda}'\end{bmatrix} = \begin{bmatrix} n & k\mathbf{1}'_b & \sum_{i=1}^{k}\sum_{j=1}^{b}\boldsymbol{\lambda}'_{ij} \\ k\mathbf{1}_b & k\mathbf{I}_b & \mathbf{M}'_d \\ \sum_{i=1}^{k}\sum_{j=1}^{b}\boldsymbol{\lambda}_{ij} & \mathbf{M}_d & \mathbf{V}_d \end{bmatrix} = \begin{bmatrix}\mathbf{X}'_{2d}\mathbf{X}_{2d} & \mathbf{X}'_{2d}\mathbf{X}_{1d} \\ \mathbf{X}'_{1d}\mathbf{X}_{2d} & \mathbf{X}'_{1d}\mathbf{X}_{1d}\end{bmatrix},$$

where $\mathbf{V}_d = \boldsymbol{\lambda}\boldsymbol{\lambda}'$, $\mathbf{M}_d = \boldsymbol{\lambda}\mathbf{D}'$, $\mathbf{X}_{2d} = \begin{bmatrix}\mathbf{1} & \mathbf{D}'\end{bmatrix}$ and $\mathbf{X}_{1d} = \boldsymbol{\lambda}'$.

The information matrix for estimating $\boldsymbol{\xi}_1$ i.e. for estimating direct effects of treatment, interference effects of treatment from the left neighbouring units and left neighbour interference × direct interaction effects is given by

$$\mathbf{C}_d = \mathbf{V}_d - \frac{1}{k}\mathbf{M}_d\mathbf{M}'_d \qquad \dots (2.3.16)$$

Consider a class of complete circular block designs with interference effects from the neighbouring units D^* which satisfy the following conditions:

i) Each treatment appears an equal number of times in each block of the design.
ii) Each treatment appears as left neighbour of every other treatment (including itself) equal number of times (say μ_1) in the design i.e. the design is strongly balanced.

We now state the following theorem to establish the optimality of above class of designs:

Theorem 2.3.4.1: A complete circular block design $d_1 \in D^*$ balanced for one-sided interference effects from the left neighbouring units, whenever exists, is universally optimal for the estimation of direct effects of treatments among all the competing designs under the non-additive model (2.5.1).

Proof: Let $\mathbf{P}_v$ be a (v-1) × v matrix such that $\left(\frac{1}{\sqrt{v}}\mathbf{1}_v, \mathbf{P}'_v\right)$ is orthogonal. We define

$$\mathbf{P}^{01} = \left(\frac{1}{\sqrt{v}}\mathbf{1}'_v\right)\otimes\mathbf{P}_v;\ \mathbf{P}^{10} = \mathbf{P}_v\otimes\left(\frac{1}{\sqrt{v}}\mathbf{1}'_v\right);\ \mathbf{P}^{11} = \mathbf{P}_v\otimes\mathbf{P}_v \qquad \dots (2.3.17)$$

$\mathbf{P}^{01}\xi_1$, $\mathbf{P}^{10}\xi_1$ and $\mathbf{P}^{11}\xi_1$ together represent a complete set of orthonormal treatment contrasts.

For any design in D^*, the coefficient matrix of the reduced normal equations for estimating the complete set of orthonormal contrasts pertaining to direct effect of treatments is given by:

$$\mathbf{C}_{(dir)} = \mathbf{P}^{01}\mathbf{C}_d\left(\mathbf{P}^{01}\right)' - \begin{bmatrix}\mathbf{P}^{01}\mathbf{C}_d\left(\mathbf{P}^{10}\right)' & \mathbf{P}^{01}\mathbf{C}_d\left(\mathbf{P}^{11}\right)'\end{bmatrix}\mathbf{G}^{-}\begin{bmatrix}\mathbf{P}^{10}\mathbf{C}_d\left(\mathbf{P}^{01}\right)' \\ \mathbf{P}^{11}\mathbf{C}_d\left(\mathbf{P}^{01}\right)'\end{bmatrix}, \qquad \dots (2.3.18)$$

where $\mathbf{C}_d$ is as in equation (2.3.16) and $\mathbf{G}^-$ is a generalized inverse of $\mathbf{G}$ given by

$$\mathbf{G} = \begin{bmatrix}\mathbf{P}^{10}\mathbf{C}_d\left(\mathbf{P}^{10}\right)' & \mathbf{P}^{10}\mathbf{C}_d\left(\mathbf{P}^{11}\right)' \\ \mathbf{P}^{11}\mathbf{C}_d\left(\mathbf{P}^{10}\right)' & \mathbf{P}^{11}\mathbf{C}_d\left(\mathbf{P}^{11}\right)'\end{bmatrix}.$$

For the design d_1, $\mathbf{V}_{d_1} = \left(\mathbf{I}_v\otimes\mathbf{I}_v\right)$ and as every v^2 treatment combinations of left neighbour × direct treatment appear equal number of times in the design, hence $\mathbf{M}_{d_1}$ is such that $\mathbf{P}^{01}\mathbf{M}_{d_1}\mathbf{M}'_{d_1} = \mathbf{0}$.

Here,

$$\mathbf{P}^{01}\mathbf{V}_{d_1}\left(\mathbf{P}^{01}\right)' = \mathbf{I}_{v-1};$$
$$\mathbf{P}^{01}\mathbf{V}_{d_1}\left(\mathbf{P}^{10}\right)' = \mathbf{0};\ \mathbf{P}^{01}\mathbf{V}_{d_1}\left(\mathbf{P}^{11}\right)' = \mathbf{0}. \qquad \ldots (2.3.19)$$

Hence, for the design d_1, the coefficient matrix of the reduced normal equations for estimating the complete set of orthonormal contrasts pertaining to direct effect of treatments is given by:

$$\mathbf{C}_{d_1(\text{dir})} = \mathbf{P}^{01}\mathbf{C}_{d_1}\left(\mathbf{P}^{01}\right)' - \begin{bmatrix} \mathbf{P}^{01}\mathbf{C}_{d_1}\left(\mathbf{P}^{10}\right)' & \mathbf{P}^{01}\mathbf{C}_{d_1}\left(\mathbf{P}^{11}\right)' \end{bmatrix} \begin{bmatrix} \mathbf{P}^{10}\mathbf{C}_{d_1}\left(\mathbf{P}^{10}\right)' & \mathbf{P}^{10}\mathbf{C}_{d_1}\left(\mathbf{P}^{11}\right)' \\ \mathbf{P}^{11}\mathbf{C}_{d_1}\left(\mathbf{P}^{10}\right)' & \mathbf{P}^{11}\mathbf{C}_{d_1}\left(\mathbf{P}^{11}\right)' \end{bmatrix}^{-} \begin{bmatrix} \mathbf{P}^{10}\mathbf{C}_{d_1}\left(\mathbf{P}^{01}\right)' \\ \mathbf{P}^{11}\mathbf{C}_{d_1}\left(\mathbf{P}^{01}\right)' \end{bmatrix} \qquad \ldots (2.3.20)$$

$= \mathbf{I}_{v-1}$, hence $\mathbf{C}_{d_1(\text{dir})}$ is completely symmetric.

Now from equation (2.3.18), it is clear that $\mathbf{P}^{01}\mathbf{C}_{d}\left(\mathbf{P}^{01}\right)' \geq \mathbf{C}_{(\text{dir})} \Rightarrow \mathbf{P}^{01}\mathbf{C}_{d}\left(\mathbf{P}^{01}\right)' - \mathbf{C}_{(\text{dir})}$ is nonnegative definite for all design in D^*. Again equation (2.3.16) implies that $\mathbf{V}_d \geq \mathbf{C}_d \Rightarrow \mathbf{V}_d - \mathbf{C}_d$ is a nonnegative definite for all design in D^*. Hence $\mathbf{P}^{01}\mathbf{V}_{d}\left(\mathbf{P}^{01}\right)' - \mathbf{P}^{01}\mathbf{C}_{d}\left(\mathbf{P}^{01}\right)'$ is a nonnegative definite for all such designs.

Therefore,

$$\begin{aligned} \text{trace}\left[\mathbf{C}_{(\text{dir})}\right] \leq \text{trace}\left[\mathbf{P}^{01}\mathbf{C}_{d}\left(\mathbf{P}^{01}\right)'\right] &\leq \text{trace}\left[\mathbf{P}^{01}\mathbf{V}_{d_1}\left(\mathbf{P}^{01}\right)'\right] \\ &= \text{trace}\left[\mathbf{P}^{01}\mathbf{C}_{d_1}\left(\mathbf{P}^{01}\right)'\right] \\ &= \text{trace}\left[\mathbf{C}_{d_1(\text{dir})}\right] \end{aligned}$$

Hence,

$$\text{trace}\left[\mathbf{C}_{(\text{dir})}\right] \leq \text{trace}\left[\mathbf{C}_{d_1(\text{dir})}\right] \text{ for all designs in } D^* \qquad \ldots (2.3.21)$$

On the lines of Bose and Dey (2003), the considered class of block designs with interference effects from the neighbouring units is universally optimal for the estimation of direct effects of treatments among all the competing designs under the given non-additive model based on the sufficient conditions of Kiefer (1975).

On the similar lines the following theorem is also proved:

Theorem 2.3.4.2: A complete circular block design d_1 balanced for one-sided interference effects from the left neighbouring units, whenever exists, is universally optimal for the separate estimation of interference effects of treatments among all the competing designs under the non-additive model (2.3.11).

For given v (where v + 1 should be prime), design with v/2 blocks each of size v can be obtained by applying treatment c at i^{th} plot of j^{th} block (i = 1, 2, …, v and j = 1, 2, …, v/2) i.e at $(i, j)^{th}$ position such that ij = c [mod (v + 1)]. Every block of this design is extended by taking its mirror image and augmenting it. The series of block design so obtained is balanced for one sided interference effect with parameters v, b = v/2, r = v, k = 2v, $\mu_1 = 1$. The design so obtained is universally optimal for the estimation of both direct effects of treatment and interference effects of treatments under the non-additive model.

Example 2.3.4.1: The following left circular block design for v = 4 is universally optimal for estimating both direct and interference effects of treatment under interference × direct non-additive model with b = 2, r = 4, k = 8 and $\mu_1 = 1$:

1	2	3	4	4	3	2	1
2	4	1	3	3	1	4	2

The complete circular block design $d_1 \in D^*$ balanced for one-sided interference effects from the left neighbouring units is universally optimal for the separate estimation of both direct effects and interference effects of treatments under the non-additive model. But the design d_1 so obtained is not optimal for the estimation of interference × direct interaction effects. If the interest is in interference × direct interaction effects also, a complete circular block design $d_2 \in D^*$ with each treatment appearing as left neighbour of every treatment (including itself) equal number of times in each block as well can be considered.

We now state the following theorem:

Theorem 2.3.4.3: A complete circular block design $d_2 \in D^*$ balanced for one-sided interference effects from the left neighbouring units, whenever exists, is universally optimal for the separate estimation of direct effects of treatments among all the competing designs under the non-additive model (2.3.11).

Proof: To prove this, we define $\mathbf{P}_v$, $\mathbf{P}^{01}$, $\mathbf{P}^{10}$, $\mathbf{P}^{11}$ as earlier. For the design d_2,

$$\mathbf{V}_{d_2} = b\,k\,v^{-2}\left(\mathbf{I}_v \otimes \mathbf{I}_v\right)$$

and as every v^2 treatment combinations of left neighbour × direct treatment also appears equal number of times in each block $\mathbf{M}_{d_2} = k\,v^{-2}\left(\mathbf{1}_{v^2} \otimes \mathbf{1}'_b\right)$, where the symbols have their usual meaning as defined earlier. Then we can write:

$$\mathbf{P}^{01}\mathbf{V}_{d_2}\left(\mathbf{P}^{01}\right)' = b\,k\,v^{-2}\mathbf{I}_{v-1};$$
$$\mathbf{P}^{01}\mathbf{V}_{d_2}\left(\mathbf{P}^{10}\right)' = \mathbf{0};\ \mathbf{P}^{01}\mathbf{V}_{d_2}\left(\mathbf{P}^{11}\right)' = \mathbf{0} \qquad \ldots (2.3.22)$$
$$\mathbf{P}^{01}\mathbf{M}_{d_1}\mathbf{M}'_{d_1} = \mathbf{0}.$$

Hence, for the design d_2, the coefficient matrix of the reduced normal equations for estimating the complete set of orthonormal contrasts pertaining to direct effect of treatments is given by:

$$\mathbf{C}_{d_2(\mathrm{dir})} = \mathbf{P}^{01}\mathbf{C}_{d_2}\left(\mathbf{P}^{01}\right)' - \begin{bmatrix}\mathbf{P}^{01}\mathbf{C}_{d_2}\left(\mathbf{P}^{10}\right)' & \mathbf{P}^{01}\mathbf{C}_{d_2}\left(\mathbf{P}^{11}\right)'\end{bmatrix} \begin{bmatrix}\mathbf{P}^{10}\mathbf{C}_{d_2}\left(\mathbf{P}^{10}\right)' & \mathbf{P}^{10}\mathbf{C}_{d_2}\left(\mathbf{P}^{11}\right)' \\ \mathbf{P}^{11}\mathbf{C}_{d_2}\left(\mathbf{P}^{10}\right)' & \mathbf{P}^{11}\mathbf{C}_{d_2}\left(\mathbf{P}^{11}\right)'\end{bmatrix}^{-} \begin{bmatrix}\mathbf{P}^{10}\mathbf{C}_{d_2}\left(\mathbf{P}^{01}\right)' \\ \mathbf{P}^{11}\mathbf{C}_{d_2}\left(\mathbf{P}^{01}\right)'\end{bmatrix} \qquad \ldots (2.3.23)$$

$= b\,k\,v^{-2}\mathbf{I}_{v-1}$, hence $\mathbf{C}_{d_2(\mathrm{dir})}$ is completely symmetric.

Now proceeding in the similar way as in Theorem 2.3.4.1, it can be easily proved that

$$\mathrm{trace}\left[\mathbf{C}_{(\mathrm{dir})}\right] \le \mathrm{trace}\left[\mathbf{C}_{d_2(\mathrm{dir})}\right] \text{ for all design in } D^* \qquad \ldots (2.3.24)$$

Thus, the considered class of block designs with interference effects from the neighbouring units is proved to be universally optimal for the estimation of direct effects of treatments among all the competing design under the non-additive model (2.3.11).

On the similar lines it can also be proved that the complete circular block design d_2 balanced for one-sided interference effects from the left neighbouring units, whenever exists, is universally optimal for the estimation of interference effects of treatments and interference × direct interaction effects separately among all the competing designs under the non-additive model (2.3.11).

Remark: Since the design d_2 is universally optimal for the estimation of direct effects, interference effects and interference × direct interaction effects of treatments, so more number of experimental units is required.

For given v, a series of block designs balanced for interference effect with parameters v = b, r = k = $2v^2$, μ_1= 2v, can be obtained by writing the j^{th} block (modulo v) of the design as follows and then considering border plots at left end of each block:

j, 1, j, 2, j, 3, …, j, v, j + 1, 1, j + 1, 2, …, j + 1, v, …, j + v, 1, j + v, 2, …, j + v, v,

where j = 1, 2,…,v.

The design so obtained will ensure that each v^2 treatment combinations of direct × left neighbour will appear in each block equal number of times. The resultant design is universally optimal for the estimation of direct effects of treatment, interference effects of treatment and interference × direct interaction effects under the non-additive model.

Example 2.3.4.2: The following left circular block design for v = 3 is universally optimal for estimating direct effects, interference effects and interference × direct interaction effects of treatment under the non-additive model with b = 3, r = 18, k = 18 and μ_1= 6:

1	1	1	2	1	3	2	1	2	2	2	3	3	1	3	2	3	3
2	1	2	2	2	3	3	1	3	2	3	3	1	1	1	2	1	3
3	1	3	2	3	3	1	1	1	2	1	3	2	1	2	2	2	3

2.4 Experimental Designs with Spatial Indirect Effects for Open-Pollination in Polycross Trials

The general aim in the breeding of cross-pollinated crops is to develop improved synthetic population. The characters of interest are generally quantitative and therefore exhibit continuous variation. Polycross is a simple method of rapid screening of genetic stocks for their breeding value by natural hybridization of a group of selected genotypes grown in isolation from other compatible genotypes to promote random open-pollination. Polycross method is commonly used in the breeding of cross-fertilizing, wind-pollinated, perennial species. Polycross trials are to be designed in such a way that each genotype has an equal chance of pollinating, or being pollinated by, any of the others. This implies that genotypes must be flowering at the same time. A particular practical application of the use of polycross method occurs in the production of a synthetic variety, produced by combination of selected lines or plants and subsequently maintained by open-pollination. Since polycross seeds are produced under natural field conditions without manual operation, sufficient quantity of seeds can be produced which is of great significance for synthetic breeding. Polycross nursery is a specific type of field design to ensure random mating among test genotypes. Polycross seed nurseries are commonly used in breeding programs for sweet potato, forage (eg: grass) and fodder (eg: alfalfa) crops, seed orchards for forest trees, *etc*.

Neighbour restricted designs restrict randomization of entries in such a way that certain groups of entries do not occur together. They can be advantageously used in situations where some genotypes interfere in the growth or production of other genotypes due to different maturity or plant height. Generally, for easy pollination, male (female) genotypes are not to be kept as neighbours to other male (female) genotypes. Some designs are obtained for these situations, where a set of genotypes will not appear in the neighbouring positions of an identified genotype.

Further, for seed orchards having well known, prevailing wind directions, directional polycross designs are to be obtained. These would require fewer replications of genotypes as pollination can happen in a certain direction and hence balancing has to be done in that direction only. Some designs balanced for neighbour effects of genotypes in the direction of prevailing wind system have been constructed here.

Polycross designs balanced for neighbours in all directions are not available for all situations. In these designs, it is required that every genotype should have every other genotype as a nearest neighbour equally frequently. Here, two series of designs have been constructed ensuring the reduction of self occurrence in the nearest neighbourhood in any of the eight directions (North, South, East, West, North-East, North-West, South-East, South-West) surrounding it.

2.4.1 Neighbour Restricted Designs for Polycross Nurseries

When one source of heterogeneity is present in a particular direction in the experimental area, block designs are advisable. Neighbour restricted block designs, balanced for neighbour effects of genotypes appearing on both sides in a block, can be recommended for such situations. When more number of genotypes is to be grown, a larger experimental area is required for the same and hence chances are more for having heterogeneity in two cross-classified directions in the nursery. Neighbour restricted row-column designs are advisable for such situations.

2.4.1.1 Neighbour Restricted Block Designs

Let there be $v = 2m$ genotypes belonging to two groups each of size m. Consider any two orthogonal Latin squares in m symbols. Retain the first Latin square as such and renumber the symbols in the second Latin square by (m+1), (m+2),…, 2m. Now, interlace the columns of the second Latin square within those of the first. The resultant arrangement has m rows considered as blocks each of size 2m and each genotype replicated m times. Considering borders on both ends in a circular manner, i.e. left (right) border is the treatment in the right (left) most inner plot, the final arrangement obtained is such that the genotypes are neighbour balanced, on both sides, for genotypes from the other group.

Example 2.4.1.1: Consider v = 6 genotypes (m = 3). Two orthogonal Latin squares for m = 3 are:

1	2	3
2	3	1
3	1	2

1	2	3
3	1	2
2	3	1

Interlace the columns of the second Latin square within those of the first Latin square after renumbering 1, 2, 3 of the second Latin square by 4, 5, 6, respectively. Considering circular borders on both sides, the following neighbour restricted block design for polycross trials in 3 blocks of size 6 each is obtained:

Block	Left Border	Block Contents	Right Border
I	**6**	1 4 2 5 3 6	**1**

II	**5**	2	6	3	4	1	5	**2**
III	**4**	3	5	1	6	2	4	**3**

Here it can be seen that every genotype from group I (1, 2, 3) has all genotypes of group II (4, 5, 6) as left and right neighbour once and vice-versa.

2.4.1.2 Neighbour Restricted Row-Column Designs

Let $v = 4m$, with two groups having $2m$ genotypes each. m arrays each of size $v \times 2$, where each genotype is replicated $\frac{v}{2}$ times, can be obtained as follows:

	Array I		Array II		Array III			Array m	
	Col 1	Col 2	Col 1	Col 2	Col 1	Col 2		Col 1	Col 2
Row 1	1	v	1	v-2	1	v-4	…	1	$\frac{v}{2}$+2
Row 2	V	2	v-2	2	v-4	2	…	$\frac{v}{2}$+2	2
Row 3	2	v-1	2	v-3	2	v-5	…	2	$\frac{v}{2}$+1
Row 4	v-1	3	v-3	3	v-5	3	…	$\frac{v}{2}$+1	3
.	.	.	.	.	.	.	…	.	.
.	.	.	.	.	.	.	…	.	.
.	.	.	.	.	.	.	…	.	.
Row v	$\frac{v}{2}$+1	1	v-1	1	v-3	1	…	$\frac{v}{2}$+3	1

Two border rows are added to the resultant arrays, in a circular manner, which results in a balanced neighbour restricted row-column design in v rows and $\frac{v}{2}$ columns. Each genotype belonging to any one group has every genotype from the other group as neighbours 3 times.

Example: 2.4.1.2: Let $m = 2$ giving rise to $v = 8$. The following design in 2 arrays each of size 8×2 is a neighbour restricted row-column design where each genotype is replicated 4 times:

	Array I		Array II	
	Col 1	Col 2	Col 1	Col 2
Border Row	**5**	**1**	**7**	**1**
Row 1	1	8	1	6
Row 2	8	2	6	2
Row 3	2	7	2	5
Row 4	7	3	5	3
Row 5	3	6	3	8

Row 6	6	4	8	4
Row 7	4	5	4	7
Row 8	5	1	7	1
Border Row	**1**	**8**	**1**	**6**

Here, in each array, every genotype has two genotypes of other group as neighbour in rows or columns 3 times. In the entire design, all genotypes of one group will have all genotypes of other group occurring as neighbour thrice.

2.4.2 Polycross Designs for Directional Wind System

Let *v* be a prime number with (*v* - 1) a multiple of 3. Consider $\frac{v-1}{3}$ initial columns each with elements 1, 3, 5. Develop these initial columns by adding 3, 4, 9, 10, 15, 16, … (mod *v*) respectively, to first, second, third,…, $\left(\frac{v-1}{3}\right)^{th}$ initial columns. A border column is appended to the rightmost (the leftmost) column if the wind direction is from right to left (left to right) in the field. The border column is to be taken in a circular manner. The middle row in each array is the seed row while the other two rows act as border rows. Here, each genotype in the seed row has a chance to get pollinated by the 3 genotypes in its right side and if the wind is flowing from left to right, then each genotype in the seed row has a chance to get pollinated by the 3 genotypes in its left side. If the wind is flowing in any other direction, then the direction of blocks are to be taken accordingly.

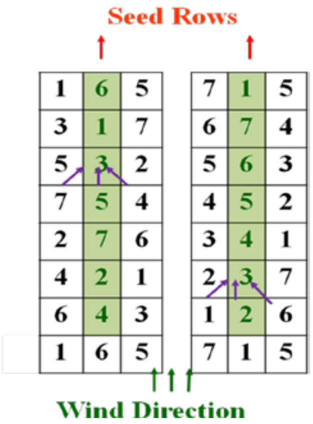
Example 2.4.2.1: Let *v* = 7. Assuming wind is prevailing from right to left in the field, the following design is obtained by developing 2 initial columns (1, 3, 5) by adding 3 to the first initial column and 4 to the second. Top and bottom rows and right most column, given in bold, in both arrays represent border plants that are used for pollination only and seeds are not collected from them.

	1	**4**	**7**	**3**	**6**	**2**	**5**	**1**	←	
Array I	3	6	2	5	1	4	7	**3**	←	Wind Direction
	5	**1**	**4**	**7**	**3**	**6**	**2**	**5**	←	

	1	**5**	**2**	**6**	**3**	**7**	**4**	**1**	←	
Array II	3	7	4	1	5	2	6	**3**	←	Wind Direction
	5	**2**	**6**	**3**	**7**	**4**	**1**	**5**	←	

Some more polycross designs suitable for directional seed orchards have been obtained using trial and error method that are suitable for situations where regular prevailing wind is predicted.

Example 2.4.2.2: For *v* = 7.



Example 2.4.2.3: For *v* = 10.

2	5	8	1	4	7	10	3	6	9	2	
3	6	9	2	5	8	1	4	7	10	3	⇇ Wind Direction
4	7	10	3	6	9	2	5	8	1	4	
9	6	3	10	7	4	1	8	5	2	9	
10	7	4	1	8	5	2	9	6	3	10	⇇ Wind Direction
1	8	5	2	9	6	3	10	7	4	1	
2	3	4	5	6	7	8	9	10	1	2	
4	5	6	7	8	9	10	1	2	3	4	⇇ Wind Direction
8	9	10	1	2	3	4	5	6	7	8	

Example 2.4.2.4: *v* = 13.

2	5	8	11	1	4	7	10	13	3	6	9	12	2	
3	6	9	12	2	5	8	11	1	4	7	10	13	3	⇇ Wind Direction
4	7	10	13	3	6	9	12	2	5	8	11	1	4	
12	9	6	3	13	10	7	4	1	11	8	5	2	12	
13	10	7	4	1	11	8	5	2	12	9	6	3	13	⇇ Wind Direction
1	11	8	5	2	12	9	6	3	13	10	7	4	1	
2	3	4	5	6	7	8	9	10	11	12	13	1	2	
4	5	6	7	8	9	10	11	12	13	1	2	3	4	⇇ Wind Direction
8	9	10	11	12	13	1	2	3	4	5	6	7	8	
3	10	4	11	5	12	6	13	7	1	8	2	9	3	
4	11	5	12	6	13	7	1	8	2	9	3	10	4	⇇ Wind Direction
5	12	6	13	7	1	8	2	9	3	10	4	11	5	

2.4.3 Octa Neighbour Balanced Polycross Designs

Modifying the methods given by Olesen (1976) and Morgan (1988), the following two series of polycross designs balanced for neighbour effects in 8 directions having less number of replications can be obtained.

2.4.3.1 Octa Neighbour Balanced Polycross Designs for *v* Genotypes (*v*+1 prime)

Olesen (1976) developed a method for constructing polycross designs for *v* genotypes balanced for neighbours (including self) in 8 directions, where each genotype is replicated v^2 times for even *v* (with *v*+1 as prime). A series of polycross designs for *v* genotypes (where *v*+1 is a prime number) is obtained here in $\frac{v}{2}$ squares of size *v* and each genotype replicated $\frac{v^2}{2}$ times, balanced for neighbours in eight directions as follows:

The genotype number *c* at (*i*, *j*, *k*), that is in row *i* and column *j* of square *k*, is defined as

$$ijk \equiv c \quad (i, j, c = 1, \ldots, v \text{ and } k = 1, 3, \ldots, v\text{-}1 \text{ or } k = 2, 4, \ldots, v) \quad (\text{mod } v + 1)$$

where *v* +1 is prime.

Example 2.4.3.1: Let *v* = 6. A polycross design as given below for 6 genotypes in 3 squares of size 6 and each genotype replicated 18 times, balanced for neighbours in eight directions is obtained by taking *k* = 1, 3, 5.

Square I							Square II							Square III					
1	2	3	4	5	6		3	6	2	5	1	4		5	3	1	6	4	2
2	4	6	1	3	5		6	5	4	3	2	1		3	6	2	5	1	4
3	6	2	5	1	4		2	4	6	1	3	5		1	2	3	4	5	6
4	1	5	2	6	3		5	3	1	6	4	2		6	5	4	3	2	1
5	3	1	6	4	2		1	2	3	4	5	6		4	1	5	2	6	3
6	5	4	3	2	1		4	1	5	2	6	3		2	4	6	1	3	5

2.4.3.2 Octa Neighbour Balanced Polycross Designs for *v* Genotypes (*v* odd)

Morgan (1988) developed a method for constructing polycross designs for *v* genotypes balanced for neighbours (including self) in 8 directions, where each genotype is replicated v^2 times for odd *v*. A series of polycross designs for *v* genotypes (*v* being an odd number) in *v* arrays of size $(\frac{v+1}{2}) \times v$ and each genotype replicated $\frac{v(v+1)}{2}$ times, balanced for neighbours in eight directions is obtained as follows:

The genotype number *c* at (*i*, *j*, *k*), that is in row *i* and column *j* of array *k*, is defined by the equation

$$c \equiv [g(i, i) + g(j, j) + k] \quad (\text{mod } v),\ i = 1,2, \ldots, (\tfrac{v+1}{2});\ j, k = 1,2, \ldots, v$$

where

$$\begin{aligned} g(i, j) &= (-1)^i \operatorname{int}(j/2), && \text{if } j = 1, 2, \ldots, m \\ &= (-1)^i \operatorname{int}((j + 1)/2), && \text{if } j = m + 1, m + 2, \ldots, v - 1 \\ &= v && \text{if } j = v \end{aligned}$$

where $m = [(-1)^{(v+1)/2}]\ (\frac{v+1}{2})\ (\text{mod } v)$.

Example 2.4.3.2: Let v = 5. The following polycross design for 5 genotypes in 5 arrays of size 3×5 and each genotype replicated 15 times, balanced for neighbours in eight directions, is obtained as below:

Array I						**Array II**						**Array III**						**Array IV**						**Array V**				
1	2	4	3	1		2	3	5	4	2		3	4	1	5	3		4	5	2	1	4		5	1	3	2	5
2	3	5	4	2		3	4	1	5	3		4	5	2	1	4		5	1	3	2	5		1	2	4	3	1
4	5	2	1	4		5	1	3	2	5		1	2	4	3	1		2	3	5	4	2		3	4	1	5	3

Chapter III

BLOCK DESIGNS WITH SPATIAL INDIRECT EFFECTS IN THE PRESENCE OF SYSTEMATIC TREND

3.1 Introduction

In block design set up, spatial trend in the experimental material may affect the plots within the blocks. In such situations, the response may also depend on the spatial position of the experimental unit within a block. For example, field plots in block similarly oriented may have similar fertility gradient. In field experiments, if the land is irrigated the nutrients supplied by the fertilizers may be equally distributed but when there is slope or while dealing with undulating land in hilly areas, this may not be the case as slope may cause a trend in experimental units. One way to overcome such situations is the application of suitable arrangement of treatments over plots within a block such that the arranged design is capable of completely eliminating the effects of defined components of a common trend. Such designs have been called as *Trend Free Block* (TFB) designs (Bradley and Yeh, 1980). These designs are constructed in such a manner that treatment effects and trend effects are orthogonal. Bradley and Yeh (1980) introduced the concept of a TFB design along with the necessary and sufficient condition for the existence of such designs. Research on various aspects of TFB designs continued in work of Yeh and Bradley (1983), Dhall (1986), Lal *et al.* (2005) and a number of others.

Here, we have considered block model with interference effect arising from the immediate left and right neighbouring experimental units incorporating trend component. Block model with interference effects from the neighbouring units at distance 2 (second order) and incorporating trend component have also been discussed. The case of one-sided interference effects have been considered as a particular case. The experimental setup has been defined and the information matrices for estimating direct as well as interference effects incorporating trend component have been derived. Further, the conditions for a block design with interference effects to be trend free have been obtained. Methods of constructing complete/ incomplete trend free block designs balanced for interference effects have been discussed and their characterization properties have been investigated.

3.2 Experimental Setup

Consider a class of proper block designs with v treatments and n = bk units that form b blocks each containing k units. Let Y_{ij} be the response from the i^{th} plot in the j^{th} block (i = 1, 2,…, k; j = 1, 2,…, b). It is assumed that the experiment is conducted in small plots in well separated linear blocks with no guard areas between the plots in a block. Further, the design is *circular*. It is also assumed that trend effects also affect the plots within blocks and the within-block trend effects can be represented by orthogonal polynomial of p^{th} degree ($p < k$).

3.2.1 Block Model with Interference Effects Incorporating Trend Component

Based on the above experimental setup, following fixed effects additive model is considered for analyzing a block design with interference effects from the immediate neighbouring units and incorporating trend component:

$$\mathbf{Y} = \mu\mathbf{1} + \mathbf{\Delta}'\boldsymbol{\tau} + \mathbf{\Delta}_1'\boldsymbol{\delta} + \mathbf{\Delta}_2'\boldsymbol{\gamma} + \mathbf{D}'\boldsymbol{\beta} + \mathbf{Z}\boldsymbol{\rho} + \mathbf{e}, \qquad \ldots (3.2.1)$$

where $\mathbf{Y}$ is a $n \times 1$ vector of observations, μ is the general mean, $\mathbf{1}$ is a $n \times 1$ vector of unity, $\mathbf{\Delta}'$ is a $n \times v$ matrix of observations versus direct treatments, $\boldsymbol{\tau}$ is a $v \times 1$ vector of direct treatment effects, $\mathbf{\Delta}_1'$ is a $n \times v$ matrix of observations versus interference effect from the immediate left neighbour treatment, $\boldsymbol{\delta}$ is $v \times 1$ vector of left neighbour interference effects, $\mathbf{\Delta}_2'$ is a $n \times v$ matrix of observations versus interference effect from the immediate right neighbour treatment, $\boldsymbol{\gamma}$ is $v \times 1$ vector of right neighbour interference effects, $\mathbf{D}'$ is a $n \times b$ incidence matrix of observations versus blocks, $\boldsymbol{\beta}$ is a $b \times 1$ vector of block effects, $\boldsymbol{\rho}$ is a $p \times 1$ vector representing the trend effects. The matrix $\mathbf{Z}$, of order $n \times p$, is the matrix of coefficients which is given by $\mathbf{Z} = \mathbf{1}_b \otimes \mathbf{F}$ where $\mathbf{F}$ is a $k \times p$ matrix with columns representing the (normalized) orthogonal polynomials and $\mathbf{e}$ is a $n \times 1$ vector of errors with $\mathbf{e} \sim N(\mathbf{0}, \sigma^2 \mathbf{I}_n)$. Without loss of generality, it can be assumed that the first k observations pertain to the first block, the next k observations pertain to the next block, and so on. Under this ordering, $\mathbf{D}' = \mathbf{I}_b \otimes \mathbf{1}_k$. Further, $\mathbf{1}'\mathbf{F} = \mathbf{0}$, $\mathbf{F}'\mathbf{F} = \mathbf{I_p}$ and hence $\mathbf{Z}'\mathbf{Z} = b\mathbf{I}_p$.

Rewriting the model as follows by writing parameter of interest first:

$$\mathbf{Y} = \mathbf{\Delta}'\boldsymbol{\tau} + \mathbf{\Delta}_1'\boldsymbol{\delta} + \mathbf{\Delta}_2'\boldsymbol{\gamma} + \mu\mathbf{1} + \mathbf{D}'\boldsymbol{\beta} + \mathbf{Z}\boldsymbol{\rho} + \mathbf{e}, \qquad \ldots (3.2.2)$$

Equation (3.2.2) can also be written as:

$$\mathbf{Y} = \mathbf{X}_1\boldsymbol{\theta}_1 + \mathbf{X}_2\boldsymbol{\theta}_2 + \mathbf{e}, \qquad \ldots (3.2.3)$$

where,

$$\mathbf{X}_1 = \left[\mathbf{\Delta}' \quad \mathbf{\Delta}_1' \quad \mathbf{\Delta}_2'\right],\ \mathbf{X}_2 = \left[\mathbf{1} \quad \mathbf{D}' \quad \mathbf{Z}\right],\ \boldsymbol{\theta}_1 = \left[\boldsymbol{\tau}' \quad \boldsymbol{\delta}' \quad \boldsymbol{\gamma}'\right]' \text{ and } \boldsymbol{\theta}_2 = \left[\mu \quad \boldsymbol{\beta}' \quad \boldsymbol{\rho}'\right]'.$$

Let,

$\mathbf{r} = (r_1, r_2, \ldots, r_v)'$ be the $v \times 1$ replication vector of the direct treatment with r_s as the number of times s^{th} ($s = 1, 2, \ldots, v$) treatment appears in the design.

$\mathbf{r}_1 = (r_{11}, r_{12}, \ldots, r_{1v})'$ be the $v \times 1$ replication vector of the immediate left neighbour treatment with r_{1s} as the number of times the treatments in the design has s^{th} treatment as immediate left neighbour.

$\mathbf{r}_2 = (r_{21}, r_{22}, ..., r_{2v})'$ be the v × 1 replication vector of the immediate right neighbour treatment with r_{2s} as the number of times the treatments in the design has s[th] treatment as immediate right neighbour.

$\mathbf{R}_\tau = \mathbf{\Delta\Delta}'$, $\mathbf{R}_\delta = \mathbf{\Delta_1\Delta_1'}$, $\mathbf{R}_\gamma = \mathbf{\Delta_2\Delta_2'}$, $\mathbf{M}_1 = \mathbf{\Delta\Delta_1'}$, $\mathbf{M}_2 = \mathbf{\Delta\Delta_2'}$, $\mathbf{M}_3 = \mathbf{\Delta_1\Delta_2'}$, $\mathbf{K = DD'}$.

$\mathbf{M}_1$ be a v × v incidence matrix of direct treatments versus immediate left neighbour treatments, $\mathbf{M}_2$ be a v × v incidence matrix of direct treatments versus immediate right neighbour treatments, $\mathbf{M}_3$ is a v × v incidence matrix of immediate left neighbor treatments versus immediate right neighbour treatments, $\mathbf{N}_1$ be a v × b incidence matrix of direct treatments versus blocks, $\mathbf{N}_2$ be a v × b incidence matrix of immediate left neighbour treatments versus blocks and $\mathbf{N}_3$ is a v × b incidence matrix of immediate right neighbour treatments versus blocks.

Therefore,

$$\mathbf{X}_1'\mathbf{X}_1 = \begin{bmatrix} \mathbf{\Delta\Delta}' & \mathbf{\Delta\Delta}_1' & \mathbf{\Delta\Delta}_2' \\ \mathbf{\Delta}_1\mathbf{\Delta}' & \mathbf{\Delta}_1\mathbf{\Delta}_1' & \mathbf{\Delta}_1\mathbf{\Delta}_2' \\ \mathbf{\Delta}_2\mathbf{\Delta}' & \mathbf{\Delta}_2\mathbf{\Delta}_1' & \mathbf{\Delta}_2\mathbf{\Delta}_2' \end{bmatrix} = \begin{bmatrix} \mathbf{R}_\tau & \mathbf{M}_1 & \mathbf{M}_2 \\ \mathbf{M}_1' & \mathbf{R}_\delta & \mathbf{M}_3 \\ \mathbf{M}_2' & \mathbf{M}_3' & \mathbf{R}_\gamma \end{bmatrix},$$

$$\mathbf{X}_1'\mathbf{X}_2 = \begin{bmatrix} \mathbf{\Delta 1} & \mathbf{\Delta D}' & \mathbf{\Delta Z} \\ \mathbf{\Delta}_1\mathbf{1} & \mathbf{\Delta}_1\mathbf{D}' & \mathbf{\Delta}_1\mathbf{Z} \\ \mathbf{\Delta}_2\mathbf{1} & \mathbf{\Delta}_2\mathbf{D}' & \mathbf{\Delta}_2\mathbf{Z} \end{bmatrix} = \begin{bmatrix} \mathbf{r} & \mathbf{N}_1 & \mathbf{\Delta Z} \\ \mathbf{r}_1 & \mathbf{N}_2 & \mathbf{\Delta}_1\mathbf{Z} \\ \mathbf{r}_2 & \mathbf{N}_3 & \mathbf{\Delta}_2\mathbf{Z} \end{bmatrix}$$

and

$$\mathbf{X}_2'\mathbf{X}_2 = \begin{bmatrix} \mathbf{1'1} & \mathbf{1'D}' & \mathbf{1'Z} \\ \mathbf{D1} & \mathbf{DD}' & \mathbf{DZ} \\ \mathbf{Z'1} & \mathbf{Z'D}' & \mathbf{Z'Z} \end{bmatrix} = \begin{bmatrix} n & k\mathbf{1}' & \mathbf{0} \\ k\mathbf{1} & k\mathbf{I}_b & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & b\mathbf{I}_p \end{bmatrix}.$$

The 3v × 3v symmetric, nonnegative definite, information matrix for estimating the direct effects, interference effects from the left neighbouring units and right neighbouring units is obtained as:

$$\mathbf{C} = \begin{bmatrix} \mathbf{R}_\tau - \mathbf{N}_1\mathbf{K}^{-1}\mathbf{N}_1' - \frac{1}{b}\mathbf{\Delta ZZ'\Delta}' & \mathbf{M}_1 - \mathbf{N}_1\mathbf{K}^{-1}\mathbf{N}_2' - \frac{1}{b}\mathbf{\Delta ZZ'\Delta}_1' & \mathbf{M}_2 - \mathbf{N}_1\mathbf{K}^{-1}\mathbf{N}_3' - \frac{1}{b}\mathbf{\Delta ZZ'\Delta}_2' \\ \mathbf{M}_1' - \mathbf{N}_2\mathbf{K}^{-1}\mathbf{N}_1' - \frac{1}{b}\mathbf{\Delta}_1\mathbf{ZZ'\Delta}' & \mathbf{R}_\delta - \mathbf{N}_2\mathbf{K}^{-1}\mathbf{N}_2' - \frac{1}{b}\mathbf{\Delta}_1\mathbf{ZZ'\Delta}_1' & \mathbf{M}_3 - \mathbf{N}_2\mathbf{K}^{-1}\mathbf{N}_3' - \frac{1}{b}\mathbf{\Delta}_1\mathbf{ZZ'\Delta}_2' \\ \mathbf{M}_2' - \mathbf{N}_3\mathbf{K}^{-1}\mathbf{N}_1' - \frac{1}{b}\mathbf{\Delta}_2\mathbf{ZZ'\Delta}' & \mathbf{M}_3' - \mathbf{N}_3\mathbf{K}^{-1}\mathbf{N}_2' - \frac{1}{b}\mathbf{\Delta}_2\mathbf{ZZ'\Delta}_1' & \mathbf{R}_\gamma - \mathbf{N}_3\mathbf{K}^{-1}\mathbf{N}_3' - \frac{1}{b}\mathbf{\Delta}_2\mathbf{ZZ'\Delta}_2' \end{bmatrix}$$

The information matrices for estimating the direct effects, interference effects from left and right neighbouring treatments can be obtained.

3.2.2 Block Model with Second Order Interference Effects Incorporating Trend Component

Following model is considered for analyzing a block design with second order interference effects i.e. interference effects from the neighbouring units at distance 2 and incorporating trend component under the above experimental setup:

$$\mathbf{Y} = \mu\mathbf{1} + \mathbf{\Delta}'\boldsymbol{\tau} + \mathbf{\Delta}_1'\boldsymbol{\delta} + \mathbf{\Delta}_2'\boldsymbol{\gamma} + \mathbf{\Delta}_3'\boldsymbol{\alpha} + \mathbf{\Delta}_4'\boldsymbol{\eta} + \mathbf{D}'\boldsymbol{\beta} + \mathbf{Z}\boldsymbol{\rho} + \mathbf{e}, \qquad \ldots (3.2.4)$$

where $\mathbf{\Delta}_3'$ is a n × v incidence matrix of observations versus second order left interference effects, $\boldsymbol{\alpha}$ is v × 1 vector of second order left neighbor interference effects, $\mathbf{\Delta}_4'$ is a n × v incidence matrix of observations versus second order right interference effects, $\boldsymbol{\eta}$ is v × 1 vector of second order right neighbour interference effects and all others symbols have their same meaning as defined earlier. Let,

$\mathbf{r}_3 = (r_{31}, r_{32}, \ldots, r_{3v})'$ be the v × 1 replication vector of the second order left neighbour treatments with r_{3s} being the number of times the treatments in the design has s^{th} treatment as left neighbour at distance 2.

$\mathbf{r}_4 = (r_{41}, r_{42}, \ldots, r_{4v})'$ be the v × 1 replication vector of the second order right neighbour treatments with r_{4s} being the number of times the treatments in the design has s^{th} treatment as right neighbour at distance 2,

Further let,

$$\mathbf{\Delta}_3\mathbf{\Delta}_3' = \mathbf{R}_\alpha = \text{diag}(r_{31}, r_{32}, \ldots, r_{3v}), \mathbf{\Delta}_4\mathbf{\Delta}_4' = \mathbf{R}_\eta = \text{diag}(r_{41}, r_{42}, \ldots, r_{4v}),$$
$$\mathbf{\Delta}\mathbf{\Delta}_3' = \mathbf{M}_4,\ \mathbf{\Delta}\mathbf{\Delta}_4' = \mathbf{M}_5, \mathbf{\Delta}_1\mathbf{\Delta}_3' = \mathbf{M}_6,\ \mathbf{\Delta}_1\mathbf{\Delta}_4' = \mathbf{M}_7,$$
$$\mathbf{\Delta}_2\mathbf{\Delta}_3' = \mathbf{M}_8,\ \mathbf{\Delta}_2\mathbf{\Delta}_4' = \mathbf{M}_9,\ \mathbf{\Delta}_3\mathbf{\Delta}_4' = \mathbf{M}_{10}$$
$$\mathbf{\Delta}_3\mathbf{D}' = \mathbf{N}_4,\ \mathbf{\Delta}_4\mathbf{D}' = \mathbf{N}_5,$$

here $\mathbf{M}_4$ is a v × v incidence matrix of direct treatments versus second order left neighbour treatments, $\mathbf{M}_5$ is a v × v incidence matrix of direct treatments versus second order right neighbour treatments, $\mathbf{M}_6$ is a v × v incidence matrix of immediate left neighbour treatments versus second order left neighbour treatments, $\mathbf{M}_7$ is a v × v incidence matrix of immediate left neighbour treatments versus second order right neighbour treatments, $\mathbf{M}_8$ is a v × v incidence matrix of immediate right neighbour treatments versus second order left neighbour treatments, $\mathbf{M}_9$ is a v × v incidence matrix of immediate right neighbour treatments versus second order right neighbour treatments, $\mathbf{M}_{10}$ is a v × v incidence matrix of second order left neighbour treatments versus second order right neighbour treatments. $\mathbf{N}_4$ is a v × b incidence matrix of second order left neighbour treatments versus blocks and $\mathbf{N}_5$ is a v × b incidence matrix of second order right neighbour treatments versus blocks.

The 5v × 5v symmetric, nonnegative definite joint information matrix for estimating the direct effects of treatment and neighbor effects up to distance 2 is obtained as:

$$\mathbf{C}=\begin{bmatrix}
\mathbf{R}_\tau-\mathbf{N}_1\mathbf{K}^{-1}\mathbf{N}'-\frac{1}{b}\boldsymbol{\Delta}\mathbf{Z}\mathbf{Z}'\boldsymbol{\Delta}' & \mathbf{M}_1-\mathbf{N}_1\mathbf{K}^{-1}\mathbf{N}_2'-\frac{1}{b}\boldsymbol{\Delta}\mathbf{Z}\mathbf{Z}'\boldsymbol{\Delta}_1' & \mathbf{M}_2-\mathbf{N}_1\mathbf{K}^{-1}\mathbf{N}_3'-\frac{1}{b}\boldsymbol{\Delta}\mathbf{Z}\mathbf{Z}'\boldsymbol{\Delta}_2' & \mathbf{M}_4-\mathbf{N}_1\mathbf{K}^{-1}\mathbf{N}_4'-\frac{1}{b}\boldsymbol{\Delta}\mathbf{Z}\mathbf{Z}'\boldsymbol{\Delta}_3' & \mathbf{M}_5-\mathbf{N}_1\mathbf{K}^{-1}\mathbf{N}_5'-\frac{1}{b}\boldsymbol{\Delta}\mathbf{Z}\mathbf{Z}'\boldsymbol{\Delta}_4' \\
\mathbf{M}_1'-\mathbf{N}_2\mathbf{K}^{-1}\mathbf{N}_1'-\frac{1}{b}\boldsymbol{\Delta}_1\mathbf{Z}\mathbf{Z}'\boldsymbol{\Delta}' & \mathbf{R}_\delta-\mathbf{N}_2\mathbf{K}^{-1}\mathbf{N}_2'-\frac{1}{b}\boldsymbol{\Delta}_1\mathbf{Z}\mathbf{Z}'\boldsymbol{\Delta}_1' & \mathbf{M}_3-\mathbf{N}_2\mathbf{K}^{-1}\mathbf{N}_3'-\frac{1}{b}\boldsymbol{\Delta}_1\mathbf{Z}\mathbf{Z}'\boldsymbol{\Delta}_2' & \mathbf{M}_6-\mathbf{N}_2\mathbf{K}^{-1}\mathbf{N}_4'-\frac{1}{b}\boldsymbol{\Delta}_1\mathbf{Z}\mathbf{Z}'\boldsymbol{\Delta}_3' & \mathbf{M}_7-\mathbf{N}_2\mathbf{K}^{-1}\mathbf{N}_5'-\frac{1}{b}\boldsymbol{\Delta}_1\mathbf{Z}\mathbf{Z}'\boldsymbol{\Delta}_4' \\
\mathbf{M}_2'-\mathbf{N}_3\mathbf{K}^{-1}\mathbf{N}_1'-\frac{1}{b}\boldsymbol{\Delta}_2\mathbf{Z}\mathbf{Z}'\boldsymbol{\Delta}' & \mathbf{M}_3'-\mathbf{N}_3\mathbf{K}^{-1}\mathbf{N}_2'-\frac{1}{b}\boldsymbol{\Delta}_2\mathbf{Z}\mathbf{Z}'\boldsymbol{\Delta}_1' & \mathbf{R}_\gamma-\mathbf{N}_3\mathbf{K}^{-1}\mathbf{N}_3'-\frac{1}{b}\boldsymbol{\Delta}_2\mathbf{Z}\mathbf{Z}'\boldsymbol{\Delta}_2' & \mathbf{M}_8-\mathbf{N}_3\mathbf{K}^{-1}\mathbf{N}_4'-\frac{1}{b}\boldsymbol{\Delta}_2\mathbf{Z}\mathbf{Z}'\boldsymbol{\Delta}_3' & \mathbf{M}_9-\mathbf{N}_3\mathbf{K}^{-1}\mathbf{N}_5'-\frac{1}{b}\boldsymbol{\Delta}_2\mathbf{Z}\mathbf{Z}'\boldsymbol{\Delta}_4' \\
\mathbf{M}_4'-\mathbf{N}_4\mathbf{K}^{-1}\mathbf{N}_1'-\frac{1}{b}\boldsymbol{\Delta}_3\mathbf{Z}\mathbf{Z}'\boldsymbol{\Delta}' & \mathbf{M}_6'-\mathbf{N}_4\mathbf{K}^{-1}\mathbf{N}_2'-\frac{1}{b}\boldsymbol{\Delta}_3\mathbf{Z}\mathbf{Z}'\boldsymbol{\Delta}_1' & \mathbf{M}_8'-\mathbf{N}_4\mathbf{K}^{-1}\mathbf{N}_3'-\frac{1}{b}\boldsymbol{\Delta}_3\mathbf{Z}\mathbf{Z}'\boldsymbol{\Delta}_2' & \mathbf{R}_\alpha-\mathbf{N}_4\mathbf{K}^{-1}\mathbf{N}_4'-\frac{1}{b}\boldsymbol{\Delta}_3\mathbf{Z}\mathbf{Z}'\boldsymbol{\Delta}_3' & \mathbf{M}_{10}-\mathbf{N}_4\mathbf{K}^{-1}\mathbf{N}_5'-\frac{1}{b}\boldsymbol{\Delta}_3\mathbf{Z}\mathbf{Z}'\boldsymbol{\Delta}_4' \\
\mathbf{M}_5'-\mathbf{N}_5\mathbf{K}^{-1}\mathbf{N}_1'-\frac{1}{b}\boldsymbol{\Delta}_4\mathbf{Z}\mathbf{Z}'\boldsymbol{\Delta}' & \mathbf{M}_7'-\mathbf{N}_5\mathbf{K}^{-1}\mathbf{N}_2'-\frac{1}{b}\boldsymbol{\Delta}_4\mathbf{Z}\mathbf{Z}'\boldsymbol{\Delta}_1' & \mathbf{M}_9'-\mathbf{N}_5\mathbf{K}^{-1}\mathbf{N}_3'-\frac{1}{b}\boldsymbol{\Delta}_4\mathbf{Z}\mathbf{Z}'\boldsymbol{\Delta}_2' & \mathbf{M}_{10}'-\mathbf{N}_5\mathbf{K}^{-1}\mathbf{N}_4'-\frac{1}{b}\boldsymbol{\Delta}_4\mathbf{Z}\mathbf{Z}'\boldsymbol{\Delta}_3' & \mathbf{R}_\eta-\mathbf{N}_5\mathbf{K}^{-1}\mathbf{N}_5'-\frac{1}{b}\boldsymbol{\Delta}_4\mathbf{Z}\mathbf{Z}'\boldsymbol{\Delta}_4'
\end{bmatrix}$$

The information matrices for estimating the direct effects and interference effects up to second order can be obtained from the joint information matrix.

3.3 Definitions

Following are some general definitions associated with the block design with interference effects incorporating trend component (here the definitions are given in respect of second order interference effects):

Definition 3.3.1: A block design is said to be balanced for second order interference effects from the neighbouring units if every treatment has every other treatment appearing as both left and right neighbour up to distance 2 constant number of times (say μ_1).

Further, a block design with both sided interference effects is *strongly balanced* if each treatment has every treatment, including itself, appearing as both left and right neighbours up to second order a constant number of times (say μ_2).

Definition 3.3.2: A block design with second order interference effects incorporating trend component, is called a *trend-free* design if the adjusted treatment sum of squares arising from direct effects of treatments and interference effects of treatments up to distance 2 under the corresponding model is same as the adjusted treatment sum of squares under the usual block model with second order interference effects without trend component.

Definition 3.3.3: A trend-free block design with second order interference effects is called *variance balanced* if the variance of any estimated elementary contrast among the direct effects is constant (say V_1), the variance of any estimated elementary contrast among the immediate left neighbour effects is constant (say V_2), the variance of any estimated elementary contrast among the immediate right neighbour effects is constant (say V_3), the variance of any estimated elementary contrast among the second order left neighbour effects is constant (say V_4) and the variance of any estimated elementary contrast among the second order right neighbour effects is constant (say V_5). A block design is *totally balanced* if $V_1 = V_2 = V_3 = V_4 = V_5$.

3.4 Conditions for the Block Design with Interference Effects to be Trend Free

The conditions for the block design with interference effects have been obtained here so that the treatment (direct, interference from left and right neighbouring units) effects and trend effects are orthogonal and the analysis of the design could then be done in the usual manner, as if no trend effects was present. Such designs are known as trend free designs. We now derive a necessary and sufficient condition for a block design with interference effects from the immediate neighbouring units to be trend free. The conditions for a block design with second order interference effects to be trend free have also been discussed subsequently.

Theorem 3.4.1: A block design with interference effects from immediate left and right neighbouring units and incorporating trend component is said to be trend free iff $\mathbf{\Delta Z} = \mathbf{0}$, $\mathbf{\Delta}_1 \mathbf{Z} = \mathbf{0}$ and $\mathbf{\Delta}_2 \mathbf{Z} = \mathbf{0}$, where the symbols have their usual meaning as defined earlier.

Proof: As defined in (3.2.3), $\mathbf{X}_2 = [\mathbf{1} \quad \mathbf{D}' \quad \mathbf{Z}]$. Let $\mathbf{X}_3 = [\mathbf{1} \quad \mathbf{D}']$.

We define,

$$\begin{aligned} \mathbf{A}_u &= \mathbf{I_n} - \mathbf{X}_u (\mathbf{X}_u'\mathbf{X}_u)^- \mathbf{X}_u' \quad (u = 2,3) \\ \mathbf{Q}_{u\tau} &= \mathbf{\Delta A}_u \mathbf{\Delta}' \\ \mathbf{Q}_{u\delta} &= \mathbf{\Delta}_1 \mathbf{A}_u \mathbf{\Delta}_1' \\ \mathbf{Q}_{u\gamma} &= \mathbf{\Delta}_2 \mathbf{A}_u \mathbf{\Delta}_2' \end{aligned} \qquad \ldots (3.4.1)$$

Thus, $$\mathbf{X}_2'\mathbf{X}_2 = \begin{bmatrix} n & k\mathbf{1}' & \mathbf{0} \\ k\mathbf{1} & k\mathbf{I}_b & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & b\mathbf{I}_p \end{bmatrix} \text{ and } \mathbf{X}_3'\mathbf{X}_3 = \begin{bmatrix} n & k\mathbf{1}' \\ k\mathbf{1} & k\mathbf{I}_b \end{bmatrix}. \qquad \ldots (3.4.2)$$

A g-inverse of $\mathbf{X}_2'\mathbf{X}_2$ and $\mathbf{X}_3'\mathbf{X}_3$ is given, respectively, by

$$(\mathbf{X}_2'\mathbf{X}_2)^- = \begin{bmatrix} 0 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{1}{k}\mathbf{I}_b & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \frac{1}{b}\mathbf{I}_p \end{bmatrix} \text{and } (\mathbf{X}_3'\mathbf{X}_3)^- = \begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & \frac{1}{k}\mathbf{I}_b \end{bmatrix}. \qquad \ldots (3.4.3)$$

Hence, $$\mathbf{A}_2 = \mathbf{I}_n - \mathbf{X}_2 (\mathbf{X}_2'\mathbf{X}_2)^- \mathbf{X}_2' = \mathbf{I}_n - \frac{1}{k}\mathbf{D}'\mathbf{D} - \frac{1}{b}\mathbf{ZZ}' \qquad \ldots (3.4.4)$$

and

$$\mathbf{A}_3 = \mathbf{I}_n - \mathbf{X}_3 (\mathbf{X}_3'\mathbf{X}_3)^- \mathbf{X}_3' = \mathbf{I}_n - \frac{1}{k}\mathbf{D}'\mathbf{D}. \qquad \ldots (3.4.5)$$

Now we first prove the necessary part. Let the block design with interference effects incorporating trend component be a trend free design. Thus, we have to prove $\mathbf{\Delta Z} = \mathbf{0}$, $\mathbf{\Delta}_1 \mathbf{Z} = \mathbf{0}$ and $\mathbf{\Delta}_2 \mathbf{Z} = \mathbf{0}$. Let T_z and T_0 be the adjusted treatment sum of squares arising from the direct effect of treatments under Model (3.2.1) and under the usual block model with interference effects without trend effect respectively. Further let T_{zL} and T_{0L} be the adjusted treatment sum of squares arising from interference effects of treatments in the left neighbouring units under Model (3.2.1) and under the usual block model with interference effects without trend effect respectively and T_{zR} and T_{0R} be the adjusted treatment sum of squares arising from interference effects of treatments from the right neighbouring units under Model (3.2.1) and under the usual block model with two-sided interference effects without trend effect respectively. Since the design is assumed to be trend free, we can write $T_z = T_0$, $T_{zL} = T_{0L}$ and $T_{zR} = T_{0R}$ i.e.

$$\mathbf{Y}'\mathbf{A}_2\boldsymbol{\Delta}'\mathbf{Q}^{-}_{2\tau}\boldsymbol{\Delta}\mathbf{A}_2\mathbf{Y} = \mathbf{Y}'\mathbf{A}_3\boldsymbol{\Delta}'\mathbf{Q}^{-}_{3\tau}\boldsymbol{\Delta}\mathbf{A}_3\mathbf{Y}\,, \qquad \ldots (3.4.6)$$

$$\mathbf{Y}'\mathbf{A}_2\boldsymbol{\Delta}_1'\mathbf{Q}^{-}_{2\delta}\boldsymbol{\Delta}_1\mathbf{A}_2\mathbf{Y} = \mathbf{Y}'\mathbf{A}_3\boldsymbol{\Delta}_1'\mathbf{Q}^{-}_{3\delta}\boldsymbol{\Delta}_1\mathbf{A}_3\mathbf{Y} \qquad \ldots (3.4.7)$$

and

$$\mathbf{Y}'\mathbf{A}_2\boldsymbol{\Delta}_2'\mathbf{Q}^{-}_{2\gamma}\boldsymbol{\Delta}_2\mathbf{A}_2\mathbf{Y} = \mathbf{Y}'\mathbf{A}_3\boldsymbol{\Delta}_2'\mathbf{Q}^{-}_{3\gamma}\boldsymbol{\Delta}_2\mathbf{A}_3\mathbf{Y}\,. \qquad \ldots (3.4.8)$$

Thus, from Equation (3.4.7),

$$\begin{aligned} &\mathbf{A}_2\boldsymbol{\Delta}'\mathbf{Q}^{-}_{2\tau}\boldsymbol{\Delta}\mathbf{A}_2 = \mathbf{A}_3\boldsymbol{\Delta}'\mathbf{Q}^{-}_{3\tau}\boldsymbol{\Delta}\mathbf{A}_3 \\ &\Rightarrow \boldsymbol{\Delta}\mathbf{A}_2\boldsymbol{\Delta}'\mathbf{Q}^{-}_{2\tau}\boldsymbol{\Delta}\mathbf{A}_2\boldsymbol{\Delta}' = \boldsymbol{\Delta}\mathbf{A}_3\boldsymbol{\Delta}'\mathbf{Q}^{-}_{3\tau}\boldsymbol{\Delta}\mathbf{A}_3\boldsymbol{\Delta}' \qquad \ldots (3.4.9) \\ &\Rightarrow \boldsymbol{\Delta}\left(\mathbf{A}_2 - \mathbf{A}_3\right)\boldsymbol{\Delta}' = \mathbf{0}. \end{aligned}$$

Similarly using Equation (3.4.7) and Equation (3.4.8),

$$\begin{aligned} &\mathbf{A}_2\boldsymbol{\Delta}_1'\mathbf{Q}^{-}_{2\delta}\boldsymbol{\Delta}_1\mathbf{A}_2 = \mathbf{A}_3\boldsymbol{\Delta}_1'\mathbf{Q}^{-}_{3\delta}\boldsymbol{\Delta}_1\mathbf{A}_3 \\ &\Rightarrow \boldsymbol{\Delta}_1\mathbf{A}_2\boldsymbol{\Delta}_1'\mathbf{Q}^{-}_{2\delta}\boldsymbol{\Delta}_1\mathbf{A}_2\boldsymbol{\Delta}_1' = \boldsymbol{\Delta}_1\mathbf{A}_3\boldsymbol{\Delta}_1'\mathbf{Q}^{-}_{3\delta}\boldsymbol{\Delta}_1\mathbf{A}_3\boldsymbol{\Delta}_1' \\ &\Rightarrow \boldsymbol{\Delta}_1\left(\mathbf{A}_2 - \mathbf{A}_3\right)\boldsymbol{\Delta}_1' = \mathbf{0}, \end{aligned}$$

and

$$\begin{aligned} &\mathbf{A}_2\boldsymbol{\Delta}_2'\mathbf{Q}^{-}_{2\gamma}\boldsymbol{\Delta}_2\mathbf{A}_2 = \mathbf{A}_3\boldsymbol{\Delta}_2'\mathbf{Q}^{-}_{3\gamma}\boldsymbol{\Delta}_2\mathbf{A}_3 \\ &\Rightarrow \boldsymbol{\Delta}_2\mathbf{A}_2\boldsymbol{\Delta}_2'\mathbf{Q}^{-}_{2\gamma}\boldsymbol{\Delta}_2\mathbf{A}_2\boldsymbol{\Delta}_2' = \boldsymbol{\Delta}_2\mathbf{A}_3\boldsymbol{\Delta}_2'\mathbf{Q}^{-}_{3\gamma}\boldsymbol{\Delta}_2\mathbf{A}_3\boldsymbol{\Delta}_2' \\ &\Rightarrow \boldsymbol{\Delta}_2\left(\mathbf{A}_2 - \mathbf{A}_2\right)\boldsymbol{\Delta}_2' = \mathbf{0}. \qquad \ldots (3.4.10) \end{aligned}$$

Substituting the value of $\mathbf{A}_2$ and $\mathbf{A}_3$ from Equation (3.4.4) and (3.4.5) into Equation (3.4.9) and (3.4.10) respectively and then solving the corresponding equations we get

$$\boldsymbol{\Delta}\mathbf{Z} = \mathbf{0},\ \boldsymbol{\Delta}_1\mathbf{Z} = \mathbf{0} \text{ and } \boldsymbol{\Delta}_2\mathbf{Z} = \mathbf{0}. \qquad \ldots (3.4.11)$$

To prove the sufficiency, we assume that the condition given in the above theorem is true i.e. $\boldsymbol{\Delta}\mathbf{Z} = \mathbf{0}$, $\boldsymbol{\Delta}_1\mathbf{Z} = \mathbf{0}$ and $\boldsymbol{\Delta}_2\mathbf{Z} = \mathbf{0}$. Pre-multiplying and post-multiplying both sides of Equation (3.4.4) and (3.4.5) by $\boldsymbol{\Delta}$ and $\boldsymbol{\Delta}'$ respectively and using (3.4.1) we get:

$$\mathbf{Q}_{2\tau} = \mathbf{\Delta A}_2\mathbf{\Delta}' = \mathbf{\Delta}\left[\mathbf{I}_n - \frac{1}{k}\mathbf{D}'\mathbf{D} - \frac{1}{b}\mathbf{ZZ}'\right]\mathbf{\Delta}' = \mathbf{R}_\tau - \frac{1}{k}\mathbf{N}_1\mathbf{N}_1' \ \left[\because \mathbf{\Delta Z} = \mathbf{0}\right] \qquad \ldots (3.4.12)$$

and

$$\mathbf{Q}_{3\tau} = \mathbf{\Delta A}_3\mathbf{\Delta}' = \mathbf{\Delta}\left[\mathbf{I}_n - \frac{1}{k}\mathbf{D}'\mathbf{D}\right]\mathbf{\Delta}' = \mathbf{R}_\tau - \frac{1}{k}\mathbf{N}_1\mathbf{N}_1'. \qquad \ldots (3.4.13)$$

As, $\mathbf{Q}_{2\tau} = \mathbf{Q}_{3\tau}$**,** thus it is obvious that $T_z = T_0$. Similarly, we can prove $T_{zL} = T_{0L}$ and $T_{zR} = T_{0R}$. Hence the condition given in the above theorem is both necessary and sufficient.

On the similar lines, we can obtain the conditions for a block design with second order interference effects to be trend free in the form of following theorem:

Theorem 3.4.2: A block design with second order interference effects from left and right neighbouring units and incorporating trend component is said to be trend free iff $\mathbf{\Delta Z} = \mathbf{0},\ \mathbf{\Delta}_1\mathbf{Z} = \mathbf{0}, \mathbf{\Delta}_2\mathbf{Z} = \mathbf{0}, \mathbf{\Delta}_3\mathbf{Z} = \mathbf{0}$ and $\mathbf{\Delta}_4\mathbf{Z} = \mathbf{0}$, where the symbols have their usual meaning as defined earlier.

Corollary 3.4.2.1: For a trend free block design with interference effects, the information matrix for estimating the direct effects as well as the information matrix for estimating the interference effects from left and right neighbouring units with trend is same as the information matrix for estimating the direct effects as well as the information matrix for estimating the interference effects from the immediate left and right neighbouring units without trend component.

3.5 Trend Free Designs

In this section, methods for construction of trend free block designs balanced for interference effects from the immediate neighbouring units have been described. In all the cases, it is assumed that the designs are circular. We choose **F** as a $k \times 1$ vector with columns representing the (normalized) orthogonal polynomials and **Z** can be obtained based on **F** as defined earlier in such a way that $\mathbf{\Delta Z} = \mathbf{0},\ \mathbf{\Delta}_1\mathbf{Z} = \mathbf{0}$ and $\mathbf{\Delta}_2\mathbf{Z} = \mathbf{0}$.

3.5.1 Trend Free Complete Block Designs

For v prime, the contents of the v-1 complete blocks of the design balanced for interference effects from neighbouring units are obtained by writing the treatments in systematic order within a block with a difference of 1,2,…,v-1 between the treatments (modulo v) in the consecutive blocks. The first block is formed by taking the difference of one between treatments, the second block by taking the difference of two and so on the $(v\text{-}1)^{th}$ block by taking the difference of (v-1). Considering these (v-1) blocks as initial blocks and developing them modulo v will result in a series of trend free totally balanced complete block design with parameters v, b = v(v-1) = r, μ_1= v and every treatment appears in every position in the design same number of times i.e. v-1.

For this class of designs,

$$\mathbf{R}_\tau = \mathbf{R}_\delta = \mathbf{R}_\gamma = \mathrm{v}(\mathrm{v}\text{-}1)\mathbf{I}_\mathrm{v},\ \mathbf{DD}' = \mathrm{v}\,\mathbf{I}_\mathrm{b},$$
$$\mathbf{N}_\mathrm{u}\mathbf{N}'_{\mathrm{u}'} = \mathrm{v}(\mathrm{v}\text{-}1)\mathbf{11}',\ \ [\mathrm{u},\mathrm{u}'=1,2,3],$$
$$\mathbf{M}_1 = \mathbf{M}_2 = \mathbf{M}_3 = \mathrm{v}\left(\mathbf{11}' - \mathbf{I}_\mathrm{v}\right). \qquad \ldots (3.5.1)$$

The joint information matrix for estimating the direct as well as interference effects from the neighbouring units is

$$\mathbf{C} = \begin{bmatrix} \mathrm{v}(\mathrm{v}\text{-}1)\left(\mathbf{I}_\mathrm{v} - \frac{\mathbf{11}'}{\mathrm{v}}\right) & \mathrm{v}\left(\frac{\mathbf{11}'}{\mathrm{v}} - \mathbf{I}_\mathrm{v}\right) & \mathrm{v}\left(\frac{\mathbf{11}'}{\mathrm{v}} - \mathbf{I}_\mathrm{v}\right) \\ \mathrm{v}\left(\frac{\mathbf{11}'}{\mathrm{v}} - \mathbf{I}_\mathrm{v}\right) & \mathrm{v}(\mathrm{v}\text{-}1)\left(\mathbf{I}_\mathrm{v} - \frac{\mathbf{11}'}{\mathrm{v}}\right) & \mathrm{v}\left(\frac{\mathbf{11}'}{\mathrm{v}} - \mathbf{I}_\mathrm{v}\right) \\ \mathrm{v}\left(\frac{\mathbf{11}'}{\mathrm{v}} - \mathbf{I}_\mathrm{v}\right) & \mathrm{v}\left(\frac{\mathbf{11}'}{\mathrm{v}} - \mathbf{I}_\mathrm{v}\right) & \mathrm{v}(\mathrm{v}\text{-}1)\left(\mathbf{I}_\mathrm{v} - \frac{\mathbf{11}'}{\mathrm{v}}\right) \end{bmatrix}$$

The information matrix for estimating the direct effects of treatment is

$$\mathbf{C}_\tau = \frac{\mathrm{v}^2(\mathrm{v}\text{-}3)}{(\mathrm{v}\text{-}2)}\left[\mathbf{I}_\mathrm{v} - \frac{\mathbf{11}'}{\mathrm{v}}\right],\ \mathrm{v} > 3. \qquad \ldots (3.5.2)$$

Similarly, the information matrices for estimating the interference effects from immediate left and right neighbouring units are obtained as

$$\mathbf{C}_\delta = \mathbf{C}_\gamma = \frac{\mathrm{v}^2(\mathrm{v}\text{-}3)}{(\mathrm{v}\text{-}2)}\left[\mathbf{I}_\mathrm{v} - \frac{\mathbf{11}'}{\mathrm{v}}\right],\ \mathrm{v} > 3. \qquad \ldots (3.5.3)$$

Thus, the designs so obtained are *totally balanced* for estimating the contrasts pertaining to direct effects of treatments and interference effects arising from the immediate left and right neighbouring units.

Remark 3.5.1: For the above class of designs, when interference effects from only one side, i.e. left neighbouring units is considered, the information matrices for estimating direct effects and the interference effects from the left neighbouring units are obtained as

$$\mathbf{C}_\tau = \mathbf{C}_\delta = \frac{\mathrm{v}^2(\mathrm{v}\text{-}2)}{(\mathrm{v}\text{-}1)}\left[\mathbf{I}_\mathrm{v} - \frac{\mathbf{11}'}{\mathrm{v}}\right],\ \ \mathrm{v} > 2. \qquad \ldots (3.5.4)$$

Example 3.5.1: For v = 5, the block design balanced for interference effects from the immediate left and right neighbouring units is

1	2	3	4	0
1	3	0	2	4
1	4	2	0	3
1	0	4	3	2

Considering these four blocks as initial blocks and developing them modulo 5 will result in a trend free totally balanced complete block design with parameters v = 5, b = 20 = r, μ_1= 5 and every treatment appears in every position in the design four times.

-2	-1	0	1	2
1	2	3	4	0
2	3	4	0	1
3	4	0	1	2
4	0	1	2	3
0	1	2	3	4
1	3	0	2	4
2	4	1	3	0
3	0	2	4	1
4	1	3	0	2
0	2	4	1	3
1	4	2	0	3
2	0	3	1	4
3	1	4	2	0
4	2	0	3	1
0	3	1	4	2
1	0	4	3	2
2	1	0	4	3
3	2	1	0	4
4	3	2	1	0
0	4	3	2	1

Orthogonal trend component of degree one without normalization [Fisher and Yates (1957)] is given in the upper row and

$$\mathbf{F} = \begin{bmatrix} \frac{-2}{\sqrt{10}} & \frac{-1}{\sqrt{10}} & 0 & \frac{1}{\sqrt{10}} & \frac{2}{\sqrt{10}} \end{bmatrix}' = \begin{bmatrix} -0.63 & -0.31 & 0 & 0.31 & 0.63 \end{bmatrix}'$$

3.5.2 Trend Free Incomplete Block Designs

Tomar *et al.* (2005) obtained a series of incomplete block design balanced for interference effects from the neighbouring units for v = mt + 1 [prime or prime power (m > 3) by developing following initial blocks modulo v and augmenting the whole set of blocks generated from each initial block one after another:

$$x^{w}, x^{w+t}, x^{w+2t}, \dots x^{w+(m-1)t}; \quad \text{for } w = 0,1, \dots , t-1,$$

where x is the primitive element of GF (v). The design so obtained is a trend free block design balanced for interference effect with parameters v = mt + 1, b = tv, r = tm, k = m, μ_1= 1 and every treatment appears in every position in the design t number of times.

For this class of designs,

$$\mathbf{R}_\tau = \mathbf{R}_\delta = \mathbf{R}_\gamma = r\,\mathbf{I}_v,\ \mathbf{DD}' = k\,\mathbf{I}_v,$$
$$\mathbf{N}_u\mathbf{N}'_{u'} = (v\text{-}k)\mathbf{I}_v + (k\text{ - }1)\mathbf{11}'\ \ [u,u'=1,2,3],$$
$$\mathbf{M}_1 = \mathbf{M}_2 = \mathbf{M}_3 = (\mathbf{11}' - \mathbf{I}_v). \qquad \dots (3.5.5)$$

The joint information matrix for estimating the direct as well as interference effect from the neighbouring units is

$$\mathbf{C} = \begin{bmatrix} \frac{v(k\text{-}1)}{k}\left(\mathbf{I}_v - \frac{\mathbf{11}'}{v}\right) & \frac{v}{k}\left(\frac{\mathbf{11}'}{v} - \mathbf{I}_v\right) & \frac{v}{k}\left(\frac{\mathbf{11}'}{v} - \mathbf{I}_v\right) \\ \frac{v}{k}\left(\frac{\mathbf{11}'}{v} - \mathbf{I}_v\right) & \frac{v(k\text{-}1)}{k}\left(\mathbf{I}_v - \frac{\mathbf{11}'}{v}\right) & \frac{v}{k}\left(\frac{\mathbf{11}'}{v} - \mathbf{I}_v\right) \\ \frac{v}{k}\left(\frac{\mathbf{11}'}{v} - \mathbf{I}_v\right) & \frac{v}{k}\left(\frac{\mathbf{11}'}{v} - \mathbf{I}_v\right) & \frac{v(k\text{-}1)}{k}\left(\mathbf{I}_v - \frac{\mathbf{11}'}{v}\right) \end{bmatrix}$$

The information matrix for estimating the direct effects of treatment is

$$\mathbf{C}_\tau = \frac{v(k\text{-}3)}{(k\text{-}2)}\left[\mathbf{I}_v - \frac{\mathbf{11}'}{v}\right],\ k > 3. \qquad \dots (3.5.6)$$

The information matrices for estimating the interference effects from left and right neighbouring units are obtained as

$$\mathbf{C}_\delta = \mathbf{C}_\gamma = \frac{v(k\text{-}3)}{(k\text{-}2)}\left[\mathbf{I}_v - \frac{\mathbf{11}'}{v}\right],\ k > 3. \qquad \dots (3.5.7)$$

The series of designs so obtained are trend free *totally balanced* for estimating the contrasts pertaining to direct effects of treatments and interference effects arising from the immediate left and right neighbouring units.

Remark 3.5.2: For the above class of designs, when interference effects from only left neighbouring units is considered, the information matrices for estimating direct effects and the interference effects are obtained as

$$\mathbf{C}_\tau = \mathbf{C}_\delta = \frac{v(k\text{-}2)}{(k\text{-}1)}\left[\mathbf{I}_v - \frac{\mathbf{11}'}{v}\right],\ k > 2. \qquad \dots (3.5.8)$$

Example 3.5.2: Let m = 5, t = 2, then we get the following two initial blocks modulo 11 for w = 0 and w = 1:

1 4 5 9 3 and 2 8 10 7 6

Developing these blocks, we obtain the following trend free totally balanced incomplete block design with v = 11, b = 22, r = 10, k = 5, μ_1= 1 and every treatment appears in every position in the design two times. **F** is same as in Example 3.5.1.

-2	-1	0	1	2
1	4	5	9	3
2	5	6	10	4
3	6	7	0	5
4	7	8	1	6
5	8	9	2	7
6	9	10	3	8
7	10	0	4	9
8	0	1	5	10
9	1	2	6	0
10	2	3	7	1
0	3	4	8	2
2	8	10	7	6
3	9	0	8	7
4	10	1	9	8
5	0	2	10	9
6	1	3	0	10
7	2	4	1	0
8	3	5	2	1
9	4	6	3	2
10	5	7	4	3
0	6	8	5	4
1	7	9	6	5

Remark 3.5.3: It has also been seen that the both the above class of designs so obtained are trend free up to p^{th} degree (p < k).

3.5.3 Trend Free Block Designs Balanced for Second Order Interference Effects

In this section, methods for construction of trend free block designs balanced for interference effects from the neighbouring units up to distance 2 have been described. In all the cases, it is assumed that the designs are circular. We choose **F** as a k × 1 vector with column representing the (normalized) orthogonal polynomial and **Z** can be obtained based on **F** as defined earlier in such a way that $\mathbf{\Delta Z = 0}$, $\mathbf{\Delta_1 Z = 0}, \mathbf{\Delta_2 Z = 0}, \mathbf{\Delta_3 Z = 0}$ and $\mathbf{\Delta_4 Z = 0}$.

Let there be v (prime) treatments labeled as 0, 1, 2, …, v-1. A series of trend free complete block designs strongly balanced for interference effects up to distance 2 are obtained by developing the blocks of the design as follows for all q = 0, 1, …, (v-1) and p = 1, 2, …, (v-1)/2:

q, q + p, q + 2p,…, q + (v-2)p, q + (v-1)p, q + (v-2)p,…, q + 2p, q + p, q   (modulo v)

The parameters of the design so obtained are v, b = v(v-1)/2, r = (v-1)(2v-1)/2, k = 2v-1, μ_1 = v-1 and μ_2 = (v-1)/2. Here, every treatment appears in every position in the design same number of times i.e.(v-1)/2.

For this class of designs,

$$\mathbf{R}_\tau = \mathbf{R}_\delta = \mathbf{R}_\gamma = \mathbf{R}_\alpha = \mathbf{R}_\eta = \frac{(\text{v-1})(2\text{v}-1)}{2}\mathbf{I}_\text{v}, \mathbf{K} = k\mathbf{I}_b = (2\text{v}-1)\mathbf{I}_b,$$

$$\mathbf{M}_1 = \mathbf{M}_2 = \mathbf{M}_3 = \mathbf{M}_4 = \mathbf{M}_5 = \mathbf{M}_6 = \mathbf{M}_7 = \mathbf{M}_8 = \mathbf{M}_9 = \mathbf{M}_{10} = \frac{(\text{v-1})}{2}(2\mathbf{11}' - \mathbf{I}_\text{v}),$$

$$\mathbf{N}_u \mathbf{N}'_{u'} = \frac{(\text{v-1})}{2}\left[\mathbf{I}_\text{v} + 4(\text{v-1})\mathbf{11}'\right], \text{ u, u}' = 1, 2, \ldots 5. \qquad \ldots (3.5.9)$$

The information matrix for estimating the direct effects of treatment is

$$\mathbf{C}_\tau = \frac{2\text{v}(\text{v-1})(\text{v-3})}{(2\text{v}-5)}\left(\mathbf{I}_\text{v} - \frac{\mathbf{11}'}{\text{v}}\right), \text{ v} > 3. \qquad \ldots (3.5.10)$$

The information matrices for estimating the immediate left neighbour effects, immediate right neighbour effects, second order left neighbour effects and second order right neighbour effects are obtained as

$$\mathbf{C}_\delta = \mathbf{C}_\gamma = \mathbf{C}_\alpha = \mathbf{C}_\eta = \frac{2\text{v}(\text{v-1})(\text{v-3})}{(2\text{v}-5)}\left(\mathbf{I}_\text{v} - \frac{\mathbf{11}'}{\text{v}}\right), \text{ v} > 3. \qquad \ldots (3.5.11)$$

Remark 3.5.4: For the above class of designs, when interference effects from only left neighbouring units is considered, the information matrices for estimating direct effects and the interference effects from the left neighbouring units up to distance 2 are obtained as

$$\mathbf{C}_\tau = \mathbf{C}_\delta = \mathbf{C}_\gamma = \frac{2\text{v}(\text{v-1})(\text{v-2})}{(2\text{v}-3)}\left(\mathbf{I}_\text{v} - \frac{\mathbf{11}'}{\text{v}}\right), \text{ v} > 2. \qquad \ldots (3.5.12)$$

Example 3.5.3: Let v = 5. The following is a trend free complete block design strongly balanced for second order interference effects with v = 5, b = 10, r = 18, k = 9, μ_1= 4, μ_2 = 2 and every treatment appears in every position in the design two times:

-4	-3	-2	-1	0	1	2	3	4
0	1	2	3	4	3	2	1	0
1	2	3	4	0	4	3	2	1
2	3	4	0	1	0	4	3	2
3	4	0	1	2	1	0	4	3
4	0	1	2	3	2	1	0	4
0	2	4	1	3	1	4	2	0

1	3	0	2	4	2	0	3	1
2	4	1	3	0	3	1	4	2
3	0	2	4	1	4	2	0	3
4	1	3	0	2	0	3	1	4

Orthogonal trend component of degree one without normalization [Fisher and Yates (1957)] is given in the upper row and **F** can be obtained accordingly.

For v prime, a series of trend free incomplete block design strongly balanced for interference effects up to distance 2 can also be obtained by developing the blocks of the design as follows for all q = 0, 1, …, (v-1) and p = 1, 2, …, (v-1)/2:

q, q + p, q + 2p, … , q + (v-3)p, q + (v-2)p, q + (v-3)p,…, q + 2p, q + p, q (modulo v)

The parameters of this class of designs are v, b = v(v-1)/2, r = (v-1)(2v-3)/2, k = 2v-3, μ_1 = v-2 and μ_2 = (v-1)/2. Here, every treatment appears in every position in the design same number of times i.e.(v-1)/2. For this class of designs,

$$\mathbf{R}_\tau = \mathbf{R}_\delta = \mathbf{R}_\gamma = \mathbf{R}_\alpha = \mathbf{R}_\eta = \frac{(\text{v-1})(2\text{v}-3)}{2}\mathbf{I}_\text{v}, \mathbf{K} = \text{k}\,\mathbf{I}_\text{b} = (2\text{v}-3)\mathbf{I}_\text{b},$$

$$\mathbf{M}_1 = \mathbf{M}_2 = \mathbf{M}_3 = \mathbf{M}_4 = \mathbf{M}_5 = \mathbf{M}_6 = \mathbf{M}_7 = \mathbf{M}_8 = \mathbf{M}_9 = \mathbf{M}_{10} = (\text{v-2})\mathbf{11'} - \frac{(\text{v-3})}{2}\mathbf{I}_\text{v},$$

$$\mathbf{N}_\text{u}\,\mathbf{N}'_{\text{u}'} = 2(\text{v-2})^2\,\mathbf{11'} + \frac{(5\text{v}-9)}{2}\mathbf{I}_\text{v}, \text{ u, u}' = 1, 2, \ldots 5 \qquad \ldots (3.5.13)$$

The information matrix for estimating the direct effect of treatments is

$$\mathbf{C}_\tau = \frac{2\text{v}(\text{v-2})(\text{v-4})}{(2\text{v}-7)}\left(\mathbf{I}_\text{v} - \frac{\mathbf{11'}}{\text{v}}\right), \text{ v} > 4. \qquad \ldots (3.5.14)$$

The information matrices for estimating the immediate left neighbour effects, immediate right neighbour effects, second order left neighbour effects and second order right neighbour effects are obtained as

$$\mathbf{C}_\delta = \mathbf{C}_\gamma = \mathbf{C}_\alpha = \mathbf{C}_\eta = \frac{2\text{v}(\text{v-2})(\text{v-4})}{(2\text{v}-7)}\left(\mathbf{I}_\text{v} - \frac{\mathbf{11'}}{\text{v}}\right), \text{ v} > 4. \qquad \ldots (3.5.15)$$

Remark 3.5.5: For the above class of designs, when interference effects from only left neighbouring units is considered, the information matrices for estimating direct effects and the interference effects from the left neighbouring units up to distance 2 are obtained as

$$\mathbf{C}_\tau = \mathbf{C}_\delta = \mathbf{C}_\gamma = \frac{2\text{v}(\text{v-2})(\text{v}-3)}{(2\text{v}-5)}\left(\mathbf{I}_\text{v} - \frac{\mathbf{11'}}{\text{v}}\right), \text{ v} > 3. \qquad \ldots (3.5.16)$$

Example 3.5.4: For v = 5, following is a strongly balanced trend free second order neighbour balanced incomplete block design with v = 5, b = 10, r = 14, k = 7, $\mu_1 = 3$, $\mu_2 = 2$ and every treatment appears in every position in the design two times.

-3	-2	-1	0	1	2	3
0	1	2	3	2	1	0
1	2	3	4	3	2	1
2	3	4	0	4	3	2
3	4	0	1	0	4	3
4	0	1	2	1	0	4
0	2	4	1	4	2	0
1	3	0	2	0	3	1
2	4	1	3	1	4	2
3	0	2	4	2	0	3
4	1	3	0	3	1	4

Remark 3.5.6: It has also been seen that the both the series of designs so obtained are trend free up to p^{th} degree ($p < k$).

Chapter IV

ROW-COLUMN DESIGNS BALANCED FOR SPATIAL INDIRECT EFFECTS

4.1 Introduction

It is seen that most of the work on designs with spatial indirect effects is concentrated on block design. In agricultural field experiments, row-column designs are very useful when the heterogeneity present in the experimental material is in two directions. Freeman (1979) has given some row-column designs balanced for neighbours with and without border plots. Federer and Basford (1991) have given three methods of constructing balanced nearest neighbour row-column designs. Chan and Eccleston (2003) have given an algorithm which generates neighbour balanced row–column Designs. However, the designs obtained are found to be only combinatorially balanced.

We have considered here three different situations under row-column setup incorporating neighbour effects viz., a row-column design with one-sided neighbour effects, a row-column design with two-sided neighbour effects and a row-column design with four-sided neighbour effects. The information matrices for all the situations for estimating the direct and neighbour effects of treatments have been derived. Methods of constructing neighbour balanced row-column designs have been developed and its characterization properties have been studied.

4.2 Model and Definitions

Let v be the number of treatments and $y_{ij(m)}$ be response from the experimental unit occurring in the i^{th} row and the j^{th} column to which the m^{th} treatment is applied. It is assumed that border plots are added on the four sides of the design. Assuming that units are affected only by the adjacent neighbouring units and the neighbour effects are directional from four sides, $y_{ij(m)}$ can be represented by a model.

$$y_{ij(m)} = \mu + \tau_{[i,\, j](m)} + \delta_{[i,\, j-1](m)} + \gamma_{[i,\, j+1](m)} + \eta_{[i-1,\, j](m)} + \chi_{[i+1,\, j](m)} + \alpha_i + \beta_j + e_{ij(m)} \qquad \ldots (4.2.1)$$

$$m = 1,2,\ldots,v;\; i = 1,2,\ldots,p;\; j = 1,2,\ldots,q;$$

where, $\tau_{[i,\, j](m)}$ is the direct effect of m^{th} treatment, $\delta_{[i,\, j-1](m)}$ is the neighbour effect due to the treatment applied in the adjacent left plot, $\gamma_{[i,\, j+1](m)}$ is the neighbour effect due to the treatment applied in the adjacent right plot, $\eta_{[i-1,\, j](m)}$ is the neighbour effect due to the treatment applied in the adjacent top (upper) plot, $\chi_{[i+1,\, j](m)}$ is the neighbour effect due to the treatment applied in the adjacent bottom (lower) plot, α_i is the i^{th} row effect, β_j is the j^{th} column effect and $e_{ij(m)}$ is the random errors assumed to be independent with $E(e_{ij(m)}) = 0$ and constant variance σ^2. We now give some definitions associated with the row-column design with neighbour effects.

Definition 4.2.1: A row-column design with one sided, say left, neighbour effects is said to be balanced if every treatment has every other treatment appearing as a left neighbour a constant number of times (say μ_1 times) and strongly balanced if the same treatment also appears as a left neighbour a constant number of times (say μ_2 times). μ_1 may be equal to μ_2.

Definition 4.2.2: A row-column design with two sided neighbour effects, say left and right, is said to be balanced if every treatment has every other treatment appearing as a left and right neighbours a constant number of times (say μ_1 times) and strongly balanced if the same treatment also appears as a left and right neighbours a constant number of times (say μ_2 times). μ_1 may be equal to μ_2.

Definition 4.2.3: A row-column design with four sided neighbour effects is said to be balanced if every treatment has every other treatment appearing as a neighbour a constant number of times (say μ_1 times) and strongly balanced if the same treatment also appears as a neighbour a constant number of times (say μ_2 times) on each of the four sides (left, right, top and bottom). μ_1 may be equal to μ_2.

Definition 4.2.4: A neighbour balanced row-column (NBRC) design with four-sided neighbour effects is said to be circular if the treatment in the left border is the same as the treatment in the right-end inner plot, the treatment in the right border is the same as the treatment in the left-end inner plot, the treatment in the top border is the same as the treatment in the bottom-end inner plot and the treatment in the bottom border is the same as the treatment in the top-end inner plot.

Definition 4.2.5: A row-column design with four sided neighbour effects is said to be variance balanced for direct effects, if all the pair-wise contrasts pertaining to the direct effects of treatments are estimated with the same variance.

Three different forms of the model given in Equation (4.2.1) have now been discussed.

4.2.1 Row-Column Design with One-sided Neighbour Effects

Here, row-column design with one-sided neighbour effect, say left, is considered. Hence the model in (4.2.1) can be rewritten as

$$\mathbf{Y} = \mu\mathbf{1} + \boldsymbol{\Delta}'\boldsymbol{\tau} + \boldsymbol{\Delta}_1'\boldsymbol{\delta} + \mathbf{D}_1'\boldsymbol{\alpha} + \mathbf{D}_2'\boldsymbol{\beta} + \mathbf{e}, \qquad \dots (4.2.2)$$

where $\mathbf{Y}$ is a $n \times 1$ vector of observations, $\mathbf{1}$ is a $n \times 1$ vector of ones, $\boldsymbol{\Delta}'$ is a $n \times v$ incidence matrix of observations versus direct treatments, $\boldsymbol{\tau}$ is a $v \times 1$ vector of direct treatment effects, $\boldsymbol{\Delta}_1'$ is a $n \times v$ matrix of observations versus left neighbour treatment, $\boldsymbol{\delta}$ is a $v \times 1$ vector of left neighbour effects, $\mathbf{D}_1'$ is a $n \times p$ incidence matrix of observations versus rows, $\boldsymbol{\alpha}$ is a $p \times 1$ vector of row effects, $\mathbf{D}_2'$ is a $n \times q$ incidence matrix of observations versus columns, $\boldsymbol{\beta}$ is a $q \times 1$ vector of column effects and $\mathbf{e}$ is a $n \times 1$ vector of errors. Now, the design matrix $\mathbf{X}_{n \times (2v+p+q+1)}$ can be partitioned into parameters of interest (X_1) and nuisance parameters (X_2).

$$\mathbf{X}_1 = [\mathbf{\Delta}' \quad \mathbf{\Delta}_1'],\ \mathbf{X}_2 = [\mathbf{1} \quad \mathbf{D}_1' \quad \mathbf{D}_2'],$$

$$\mathbf{X}_1'\mathbf{X}_1 = \begin{bmatrix} \mathbf{\Delta\Delta}' & \mathbf{\Delta\Delta}_1' \\ \mathbf{\Delta}_1\mathbf{\Delta}' & \mathbf{\Delta}_1\mathbf{\Delta}_1' \end{bmatrix} = \begin{bmatrix} \mathbf{R}_\tau & \mathbf{N}_5 \\ \mathbf{N}_5' & \mathbf{R}_\delta \end{bmatrix},$$

$$\mathbf{X}_1'\mathbf{X}_2 = \begin{bmatrix} \mathbf{\Delta 1} & \mathbf{\Delta D}_1' & \mathbf{\Delta D}_2' \\ \mathbf{\Delta}_1\mathbf{1} & \mathbf{\Delta}_1\mathbf{D}_1' & \mathbf{\Delta}_1\mathbf{D}_2' \end{bmatrix} = \begin{bmatrix} \mathbf{r}_\tau & \mathbf{N}_1 & \mathbf{N}_2 \\ \mathbf{r}_\delta & \mathbf{N}_3 & \mathbf{N}_4 \end{bmatrix}$$

and $$\mathbf{X}_2'\mathbf{X}_2 = \begin{bmatrix} \mathbf{1}'\mathbf{1} & \mathbf{1}'\mathbf{D}_1' & \mathbf{1}'\mathbf{D}_2' \\ \mathbf{D}_1\mathbf{1} & \mathbf{D}_1\mathbf{D}_1' & \mathbf{D}_1\mathbf{D}_2' \\ \mathbf{D}_2\mathbf{1} & \mathbf{D}_2\mathbf{D}_1' & \mathbf{D}_2\mathbf{D}_2' \end{bmatrix} = \begin{bmatrix} \mathbf{n} & \mathbf{k}' & \mathbf{h}' \\ \mathbf{k} & \mathbf{K} & \mathbf{M} \\ \mathbf{h} & \mathbf{M}' & \mathbf{H} \end{bmatrix}.$$

Here, $\mathbf{N}_1$ is an incidence matrix of order v × p of direct treatments Vs rows; $\mathbf{N}_2$ is an incidence matrix of order v × p of left neighbour treatments Vs rows; $\mathbf{N}_3$ is an incidence matrix of order v × q of direct treatments Vs columns; $\mathbf{N}_4$ is an incidence matrix of order v × q of left neighbour treatments Vs columns; $\mathbf{N}_5$ is an incidence matrix of order v × v of direct treatments Vs left neighbour treatments; $\mathbf{M}$ is an incidence matrix of order p × q of rows Vs columns; $\mathbf{r}_\tau = (r_1, r_2, \ldots, r_v)$ is the v × 1 replication vector of direct treatments with r_m (m =1, 2 ,…, v) being the number of times the m^th^ treatment appears in the design; $\mathbf{r}_\delta = (r_{\delta 1}, r_{\delta 2}, \ldots, r_{\delta v})$ is the v × 1 replication vector of the left neighbour treatments with $r_{\delta m}$ being the number of times the treatments in the design has m^th^ treatment as left neighbour; $\mathbf{R}_\tau = \text{diag}(r_1, r_2, \ldots, r_v)$ is the diagonal matrix of replications of treatments as direct effect; $\mathbf{R}_\delta = \text{diag.}(r_{\delta 1}, r_{\delta 2}, \ldots, r_{\delta v})$ is the diagonal matrix of replications of treatments as left neighbour; $\mathbf{k} = (k_1,k_2,\ldots,k_p)$ is the p × 1 vector of row sizes; $\mathbf{h} = (h_1,h_2,\ldots,h_q)$ is the q × 1 vector of column sizes; $\mathbf{K} = \text{diag}(k_1,k_2,\ldots,k_p)$ is the diagonal matrix of row sizes; $\mathbf{H} = \text{diag}(h_1,h_2,\ldots,h_q)$ is the diagonal matrix of column sizes.

The joint information matrix for estimating all the effects (direct and neighbour) can be obtained as $\mathbf{C} = \mathbf{X}_1'\mathbf{X}_1 - \mathbf{X}_1'\mathbf{X}_2(\mathbf{X}_2'\mathbf{X}_2)^{-}\mathbf{X}_2'\mathbf{X}_1$, where $(\mathbf{X}_2'\mathbf{X}_2)^{-}$ is the generalised inverse of $\mathbf{X}_2'\mathbf{X}_2$ and is obtained using the following result:

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}' & \mathbf{H} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{A}^{-1} + \mathbf{FE}^{-1}\mathbf{F}' & -\mathbf{FE}^{-1} \\ -\mathbf{E}^{-1}\mathbf{F}' & \mathbf{E}^{-1} \end{bmatrix}, \text{ where } \mathbf{E} = \mathbf{H} - \mathbf{B}'\mathbf{A}^{-1}\mathbf{B} \text{ and } \mathbf{F} = \mathbf{A}^{-1}\mathbf{B}.$$

Hence,

$$(\mathbf{X}_2'\mathbf{X}_2)^{-} = \begin{bmatrix} 0 & \mathbf{0}' & \mathbf{0}' \\ \mathbf{0} & \mathbf{K}^{-1} + \mathbf{K}^{-1}\mathbf{ME}^{-}\mathbf{M}'\mathbf{K}^{-1} & -\mathbf{K}^{-1}\mathbf{ME}^{-} \\ \mathbf{0} & -\mathbf{E}^{-}\mathbf{M}'\mathbf{K}^{-1} & \mathbf{E}^{-} \end{bmatrix} \text{ with } \mathbf{E} = \mathbf{H} - \mathbf{M}'\mathbf{K}^{-1}\mathbf{M}.$$

Thus, $$\mathbf{C} = \begin{bmatrix} \mathbf{C}_{11} & \mathbf{C}_{12} \\ \mathbf{C}_{21} & \mathbf{C}_{22} \end{bmatrix}$$

$$\mathbf{C}_{11} = \mathbf{R}_\tau - (\mathbf{N}_1\mathbf{K}^{-1}\mathbf{N}_1' + \mathbf{N}_1\mathbf{K}^{-1}\mathbf{M}\mathbf{E}^-\mathbf{M}'\mathbf{K}^{-1}\mathbf{N}_1' - \mathbf{N}_2\mathbf{E}^-\mathbf{M}'\mathbf{K}^{-1}\mathbf{N}_1' - \mathbf{N}_1\mathbf{K}^{-1}\mathbf{M}\mathbf{E}^-\mathbf{N}_3' + \mathbf{N}_2\mathbf{E}^-\mathbf{N}_3');$$
$$\mathbf{C}_{12} = \mathbf{N}_5 - (\mathbf{N}_1\mathbf{K}^{-1}\mathbf{N}_2' + \mathbf{N}_1\mathbf{K}^{-1}\mathbf{M}\mathbf{E}^-\mathbf{M}'\mathbf{K}^{-1}\mathbf{N}_2' - \mathbf{N}_2\mathbf{E}^-\mathbf{M}'\mathbf{K}^{-1}\mathbf{N}_2' - \mathbf{N}_1\mathbf{K}^{-1}\mathbf{M}\mathbf{E}^-\mathbf{N}_4' + \mathbf{N}_2\mathbf{E}^-\mathbf{N}_4');$$
$$\mathbf{C}_{21} = \mathbf{N}_5' - (\mathbf{N}_3\mathbf{K}^{-1}\mathbf{N}_1' + \mathbf{N}_3\mathbf{K}^{-1}\mathbf{M}\mathbf{E}^-\mathbf{M}'\mathbf{K}^{-1}\mathbf{N}_1' - \mathbf{N}_4\mathbf{E}^-\mathbf{M}'\mathbf{K}^{-1}\mathbf{N}_1' - \mathbf{N}_3\mathbf{K}^{-1}\mathbf{M}\mathbf{E}^-\mathbf{N}_3' + \mathbf{N}_4\mathbf{E}^-\mathbf{N}_3');$$
$$\mathbf{C}_{22} = \mathbf{R}_\delta - (\mathbf{N}_3\mathbf{K}^{-1}\mathbf{N}_2' + \mathbf{N}_3\mathbf{K}^{-1}\mathbf{M}\mathbf{E}^-\mathbf{M}'\mathbf{K}^{-1}\mathbf{N}_2' - \mathbf{N}_4\mathbf{E}^-\mathbf{M}'\mathbf{K}^{-1}\mathbf{N}_2' - \mathbf{N}_3\mathbf{K}^{-1}\mathbf{M}\mathbf{E}^-\mathbf{N}_4' + \mathbf{N}_4\mathbf{E}^-\mathbf{N}_4').$$

The 2v × 2v matrix **C** is symmetric, non-negative definite with zero row and column sums. From the above, the information matrices for estimating the direct effects ($\mathbf{C}_\tau$) and neighbour effects ($\mathbf{C}_\delta$) can be estimated using $\mathbf{C}_\tau = \mathbf{C}_{11} - \mathbf{C}_{12}\mathbf{C}_{22}^-\mathbf{C}_{21}$ and $\mathbf{C}_\delta = \mathbf{C}_{22} - \mathbf{C}_{21}\mathbf{C}_{11}^-\mathbf{C}_{12}$ respectively.

4.2.2 Row-Column Design with Two-sided Neighbour Effects

Here, row-column design with two-sided neighbours, say left and right (or top and bottom), is considered. The model for this situation can be written as

$$\mathbf{Y} = \mu\mathbf{1} + \Delta'\boldsymbol{\tau} + \Delta_1'\boldsymbol{\delta} + \Delta_2'\boldsymbol{\gamma} + \mathbf{D}_1'\boldsymbol{\alpha} + \mathbf{D}_2'\boldsymbol{\beta} + \mathbf{e}, \quad \ldots (4.2.3)$$

where Δ_1' is a n × v matrix of observations versus left neighbour treatment, $\boldsymbol{\delta}$ is v × 1 vector of left neighbour effects, Δ_2' is a n × v matrix of observations versus right neighbour treatment, $\boldsymbol{\gamma}$ is v × 1 vector of right neighbour effects. Here,

$$\mathbf{X}_1 = [\Delta' \quad \Delta_1' \quad \Delta_2'],\ \mathbf{X}_2 = [\mathbf{1} \quad \mathbf{D}_1' \quad \mathbf{D}_2']$$

$$\mathbf{X}_1'\mathbf{X}_1 = \begin{bmatrix} \Delta\Delta' & \Delta\Delta_1' & \Delta\Delta_2' \\ \Delta_1\Delta' & \Delta_1\Delta_1' & \Delta_1\Delta_2' \\ \Delta_2\Delta' & \Delta_2\Delta_1' & \Delta_2\Delta_2' \end{bmatrix} = \begin{bmatrix} \mathbf{R}_\tau & \mathbf{N}_1 & \mathbf{N}_2 \\ \mathbf{N}_1' & \mathbf{R}_\delta & \mathbf{N}_3 \\ \mathbf{N}_2' & \mathbf{N}_3' & \mathbf{R}_\gamma \end{bmatrix},$$

$$\mathbf{X}_1'\mathbf{X}_2 = \begin{bmatrix} \Delta\mathbf{1} & \Delta\mathbf{D}_1' & \Delta\mathbf{D}_2' \\ \Delta_1\mathbf{1} & \Delta_1\mathbf{D}_1' & \Delta_1\mathbf{D}_2' \\ \Delta_2\mathbf{1} & \Delta_2\mathbf{D}_1' & \Delta_2\mathbf{D}_2' \end{bmatrix} = \begin{bmatrix} \mathbf{r}_\tau & \mathbf{N}_4 & \mathbf{N}_7 \\ \mathbf{r}_\delta & \mathbf{N}_5 & \mathbf{N}_8 \\ \mathbf{r}_\gamma & \mathbf{N}_6 & \mathbf{N}_9 \end{bmatrix},$$

and $$\mathbf{X}_2'\mathbf{X}_2 = \begin{bmatrix} \mathbf{1}'\mathbf{1} & \mathbf{1}'\mathbf{D}_1' & \mathbf{1}'\mathbf{D}_2' \\ \mathbf{D}_1\mathbf{1} & \mathbf{D}_1\mathbf{D}_1' & \mathbf{D}_1\mathbf{D}_2' \\ \mathbf{D}_2\mathbf{1} & \mathbf{D}_2\mathbf{D}_1' & \mathbf{D}_2\mathbf{D}_2' \end{bmatrix} = \begin{bmatrix} \mathbf{n} & \mathbf{k}' & \mathbf{h}' \\ \mathbf{k} & \mathbf{K} & \mathbf{M} \\ \mathbf{h} & \mathbf{M}' & \mathbf{H} \end{bmatrix}.$$

Here, $\mathbf{N}_1$ is an incidence matrix of order v × v of direct treatments Vs left neighbour treatments; $\mathbf{N}_2$ is an incidence matrix of order v × v of direct treatments Vs right neighbour treatments; $\mathbf{N}_3$ is an incidence matrix of order v × v of left neighbour treatments Vs right neighbour treatments; $\mathbf{N}_4$ is an incidence matrix of order v × p of direct treatments Vs rows; $\mathbf{N}_5$ is an incidence matrix of order v × p of left neighbour treatments Vs rows; $\mathbf{N}_6$ is an incidence matrix of order v × p of right neighbour treatments Vs rows; $\mathbf{N}_7$ is an incidence matrix of order v × q of direct treatments Vs columns; $\mathbf{N}_8$ is an incidence matrix of order v × q of left neighbour treatments Vs columns; $\mathbf{N}_9$ is an incidence matrix of order v × q of right neighbour treatments Vs columns; $\mathbf{r}_\gamma = (r_{\gamma 1}, r_{\gamma 2}, \ldots, r_{\gamma v})$ is the v × 1 replication vector of right neighbour treatments with $r_{\gamma m}$ being the number of times the treatments in the design has m[th] treatment as a right neighbour; $\mathbf{R}_\gamma$ = diag. $(r_{\gamma 1}, r_{\gamma 2}, \ldots, r_{\gamma v})$ is the diagonal matrix of replications of treatments as a right neighbour.

The joint information matrix for estimating the direct effects, left-neighbour effects and right-neighbour effects of treatments is obtained as follows:

$$\mathbf{C} = \begin{bmatrix} \mathbf{C}_{11} & \mathbf{C}_{12} & \mathbf{C}_{13} \\ \mathbf{C}_{21} & \mathbf{C}_{22} & \mathbf{C}_{23} \\ \mathbf{C}_{31} & \mathbf{C}_{32} & \mathbf{C}_{33} \end{bmatrix},$$

where $\mathbf{C}_{11} = \mathbf{R}_\tau - (\mathbf{N}_4\mathbf{K}^{-1}\mathbf{N}_4' + \mathbf{N}_4\mathbf{K}^{-1}\mathbf{M}\mathbf{E}^{-}\mathbf{M}'\mathbf{K}^{-1}\mathbf{N}_4' - \mathbf{N}_7\mathbf{E}^{-}\mathbf{M}'\mathbf{K}^{-1}\mathbf{N}_4' - \mathbf{N}_4\mathbf{K}^{-1}\mathbf{M}\mathbf{E}^{-}\mathbf{N}_7' + \mathbf{N}_7\mathbf{E}^{-}\mathbf{N}_7')$;

$\mathbf{C}_{12} = \mathbf{N}_1 - (\mathbf{N}_4\mathbf{K}^{-1}\mathbf{N}_5' + \mathbf{N}_4\mathbf{K}^{-1}\mathbf{M}\mathbf{E}^{-}\mathbf{M}'\mathbf{K}^{-1}\mathbf{N}_5' - \mathbf{N}_7\mathbf{E}^{-}\mathbf{M}'\mathbf{K}^{-1}\mathbf{N}_5' - \mathbf{N}_4\mathbf{K}^{-1}\mathbf{M}\mathbf{E}^{-}\mathbf{N}_8' + \mathbf{N}_7\mathbf{E}^{-}\mathbf{N}_8')$;

$\mathbf{C}_{13} = \mathbf{N}_1 - (\mathbf{N}_4\mathbf{K}^{-1}\mathbf{N}_6' + \mathbf{N}_4\mathbf{K}^{-1}\mathbf{M}\mathbf{E}^{-}\mathbf{M}'\mathbf{K}^{-1}\mathbf{N}_6' - \mathbf{N}_7\mathbf{E}^{-}\mathbf{M}'\mathbf{K}^{-1}\mathbf{N}_6' - \mathbf{N}_4\mathbf{K}^{-1}\mathbf{M}\mathbf{E}^{-}\mathbf{N}_9' + \mathbf{N}_7\mathbf{E}^{-}\mathbf{N}_9')$;

$\mathbf{C}_{22} = \mathbf{R}_\delta - (\mathbf{N}_5\mathbf{K}^{-1}\mathbf{N}_5' + \mathbf{N}_5\mathbf{K}^{-1}\mathbf{M}\mathbf{E}^{-}\mathbf{M}'\mathbf{K}^{-1}\mathbf{N}_5' - \mathbf{N}_8\mathbf{E}^{-}\mathbf{M}'\mathbf{K}^{-1}\mathbf{N}_5' - \mathbf{N}_5\mathbf{K}^{-1}\mathbf{M}\mathbf{E}^{-}\mathbf{N}_8' + \mathbf{N}_8\mathbf{E}^{-}\mathbf{N}_8')$;

$\mathbf{C}_{23} = \mathbf{N}_3 - (\mathbf{N}_5\mathbf{K}^{-1}\mathbf{N}_6' + \mathbf{N}_5\mathbf{K}^{-1}\mathbf{M}\mathbf{E}^{-}\mathbf{M}'\mathbf{K}^{-1}\mathbf{N}_6' - \mathbf{N}_8\mathbf{E}^{-}\mathbf{M}'\mathbf{K}^{-1}\mathbf{N}_6' - \mathbf{N}_5\mathbf{K}^{-1}\mathbf{M}\mathbf{E}^{-}\mathbf{N}_9' + \mathbf{N}_8\mathbf{E}^{-}\mathbf{N}_9')$;

$\mathbf{C}_{33} = \mathbf{R}_\gamma - (\mathbf{N}_6\mathbf{K}^{-1}\mathbf{N}_6' + \mathbf{N}_6\mathbf{K}^{-1}\mathbf{M}\mathbf{E}^{-}\mathbf{M}'\mathbf{K}^{-1}\mathbf{N}_6' - \mathbf{N}_9\mathbf{E}^{-}\mathbf{M}'\mathbf{K}^{-1}\mathbf{N}_6' - \mathbf{N}_6\mathbf{K}^{-1}\mathbf{M}\mathbf{E}^{-}\mathbf{N}_9' + \mathbf{N}_9\mathbf{E}^{-}\mathbf{N}_9')$.

Therefore the information matrix for estimating the direct effect of treatments can be obtained as

$$\mathbf{C}_\tau = \mathbf{C}_{11} - \begin{bmatrix} \mathbf{C}_{12} & \mathbf{C}_{13} \end{bmatrix} \begin{bmatrix} \mathbf{C}_{22} & \mathbf{C}_{23} \\ \mathbf{C}_{32} & \mathbf{C}_{33} \end{bmatrix}^{-} \begin{bmatrix} \mathbf{C}_{21} \\ \mathbf{C}_{31} \end{bmatrix}.$$

4.2.3 Row-Column Design with Neighbour Effects from Four Sides

Extending the model (4.2.3) for neighbour effects from four sides, the following model is defined:

$$\mathbf{Y} = \mu\mathbf{1} + \mathbf{\Delta}'\boldsymbol{\tau} + \mathbf{\Delta}_1'\boldsymbol{\delta} + \mathbf{\Delta}_2'\boldsymbol{\gamma} + \mathbf{\Delta}_3'\boldsymbol{\eta} + \mathbf{\Delta}_4'\boldsymbol{\chi} + \mathbf{D}_1'\boldsymbol{\alpha} + \mathbf{D}_2'\boldsymbol{\beta} + \mathbf{e}, \qquad \ldots (4.2.4)$$

$\mathbf{\Delta}_3'$ is a n × v matrix of observations versus top neighbour treatment, $\boldsymbol{\eta}$ is v × 1 vector of top neighbour effects, $\mathbf{\Delta}_4'$ is a n × v matrix of observations versus bottom neighbour treatment, $\boldsymbol{\chi}$ is v × 1 vector of bottom neighbour effects. Here,

$$\mathbf{X}_1 = \begin{bmatrix} \mathbf{\Delta}' & \mathbf{\Delta}_1' & \mathbf{\Delta}_2' & \mathbf{\Delta}_3' & \mathbf{\Delta}_4' \end{bmatrix},\ \mathbf{X}_2 = \begin{bmatrix} \mathbf{1} & \mathbf{D}_1' & \mathbf{D}_2' \end{bmatrix},$$

$$\mathbf{X}_1'\mathbf{X}_1 = \begin{bmatrix} \mathbf{\Delta}\mathbf{\Delta}' & \mathbf{\Delta}\mathbf{\Delta}_1' & \mathbf{\Delta}\mathbf{\Delta}_2' & \mathbf{\Delta}\mathbf{\Delta}_3' & \mathbf{\Delta}\mathbf{\Delta}_4' \\ \mathbf{\Delta}_1\mathbf{\Delta}' & \mathbf{\Delta}_1\mathbf{\Delta}_1' & \mathbf{\Delta}_1\mathbf{\Delta}_2' & \mathbf{\Delta}_1\mathbf{\Delta}_3' & \mathbf{\Delta}_1\mathbf{\Delta}_4' \\ \mathbf{\Delta}_2\mathbf{\Delta}' & \mathbf{\Delta}_2\mathbf{\Delta}_1' & \mathbf{\Delta}_2\mathbf{\Delta}_2' & \mathbf{\Delta}_2\mathbf{\Delta}_3' & \mathbf{\Delta}_2\mathbf{\Delta}_4' \\ \mathbf{\Delta}_3\mathbf{\Delta}' & \mathbf{\Delta}_3\mathbf{\Delta}_1' & \mathbf{\Delta}_3\mathbf{\Delta}_2' & \mathbf{\Delta}_3\mathbf{\Delta}_3' & \mathbf{\Delta}_3\mathbf{\Delta}_4' \\ \mathbf{\Delta}_4\mathbf{\Delta}' & \mathbf{\Delta}_4\mathbf{\Delta}_1' & \mathbf{\Delta}_4\mathbf{\Delta}_2' & \mathbf{\Delta}_4\mathbf{\Delta}_3' & \mathbf{\Delta}_4\mathbf{\Delta}_4' \end{bmatrix} = \begin{bmatrix} \mathbf{R}_\tau & \mathbf{N}_1 & \mathbf{N}_2 & \mathbf{N}_3 & \mathbf{N}_4 \\ \mathbf{N}_1' & \mathbf{R}_\delta & \mathbf{N}_5 & \mathbf{N}_6 & \mathbf{N}_7 \\ \mathbf{N}_2' & \mathbf{N}_5' & \mathbf{R}_\gamma & \mathbf{N}_8 & \mathbf{N}_9 \\ \mathbf{N}_3' & \mathbf{N}_6' & \mathbf{N}_8' & \mathbf{R}_\eta & \mathbf{N}_{10} \\ \mathbf{N}_4' & \mathbf{N}_7' & \mathbf{N}_9' & \mathbf{N}_{10}' & \mathbf{R}_\chi \end{bmatrix},$$

$$\mathbf{X}_1'\mathbf{X}_2 = \begin{bmatrix} \mathbf{\Delta 1} & \mathbf{\Delta D}_1' & \mathbf{\Delta D}_2' \\ \mathbf{\Delta}_1\mathbf{1} & \mathbf{\Delta}_1\mathbf{D}_1' & \mathbf{\Delta}_1\mathbf{D}_2' \\ \mathbf{\Delta}_2\mathbf{1} & \mathbf{\Delta}_2\mathbf{D}_1' & \mathbf{\Delta}_2\mathbf{D}_2' \\ \mathbf{\Delta}_3\mathbf{1} & \mathbf{\Delta}_3\mathbf{D}_1' & \mathbf{\Delta}_3\mathbf{D}_2' \\ \mathbf{\Delta}_4\mathbf{1} & \mathbf{\Delta}_4\mathbf{D}_1' & \mathbf{\Delta}_4\mathbf{D}_2' \end{bmatrix} = \begin{bmatrix} \mathbf{r}_\tau & \mathbf{N}_{11} & \mathbf{N}_{16} \\ \mathbf{r}_\delta & \mathbf{N}_{12} & \mathbf{N}_{17} \\ \mathbf{r}_\gamma & \mathbf{N}_{13} & \mathbf{N}_{18} \\ \mathbf{r}_\eta & \mathbf{N}_{14} & \mathbf{N}_{19} \\ \mathbf{r}_\chi & \mathbf{N}_{15} & \mathbf{N}_{20} \end{bmatrix},$$

$$\mathbf{X}_2'\mathbf{X}_2 = \begin{bmatrix} \mathbf{1}'\mathbf{1} & \mathbf{1}'\mathbf{D}_1' & \mathbf{1}'\mathbf{D}_2' \\ \mathbf{D}_1\mathbf{1} & \mathbf{D}_1\mathbf{D}_1' & \mathbf{D}_1\mathbf{D}_2' \\ \mathbf{D}_2\mathbf{1} & \mathbf{D}_2\mathbf{D}_1' & \mathbf{D}_2\mathbf{D}_2' \end{bmatrix} = \begin{bmatrix} \mathbf{n} & \mathbf{k}' & \mathbf{h}' \\ \mathbf{k} & \mathbf{K} & \mathbf{M} \\ \mathbf{h} & \mathbf{M} & \mathbf{H} \end{bmatrix}.$$

$\mathbf{N}_1$ is an incidence matrix of order v × v of direct treatments Vs left neighbour treatments; $\mathbf{N}_2$ is an incidence matrix of order v × v of direct treatments Vs right neighbour treatments; $\mathbf{N}_3$ is an incidence matrix of order v × v of direct treatments Vs top neighbour treatments; $\mathbf{N}_4$ is an incidence matrix of order v × v of direct treatments Vs bottom neighbour treatments; $\mathbf{N}_5$ is an incidence matrix of order v × v of left neighbour treatments Vs right neighbour treatments; $\mathbf{N}_6$ is an incidence matrix of order v × v of left neighbour treatments Vs top neighbour treatments; $\mathbf{N}_7$ is an incidence matrix of order v × v of left neighbour treatments Vs bottom neighbour treatments; $\mathbf{N}_8$ is an incidence matrix of order v × v of right neighbour treatments Vs top neighbour treatments; $\mathbf{N}_9$ is an incidence matrix of order v × v of right neighbour treatments Vs bottom neighbour treatments; $\mathbf{N}_{10}$ is an incidence matrix of order v × v of top neighbour treatments Vs bottom neighbour treatments; $\mathbf{N}_{11}$ is an incidence matrix of order v × p direct treatments Vs rows; $\mathbf{N}_{12}$ is an incidence matrix of order v × p of left neighbour treatments Vs rows; $\mathbf{N}_{13}$ is an incidence matrix of order v × p of right neighbour treatments Vs rows; $\mathbf{N}_{14}$ is an incidence matrix of order v × p of top neighbour treatments Vs rows; $\mathbf{N}_{15}$ is an incidence matrix of order v × p of bottom neighbour treatments Vs rows; $\mathbf{N}_{16}$ is an incidence matrix of order v × q of direct treatments Vs columns; $\mathbf{N}_{17}$ is an incidence matrix of order v × q of left neighbour treatments Vs columns; $\mathbf{N}_{18}$ is an incidence matrix of order v × q of right neighbour treatments Vs columns; $\mathbf{N}_{19}$ is an incidence matrix of order v × q of top neighbour treatments Vs columnss; $\mathbf{N}_{20}$ is an incidence matrix of order v × q of bottom neighbour treatments Vs columns; $\mathbf{r}_\eta = (r_{\eta 1}, r_{\eta 2}, \ldots, r_{\eta v})$ is the v × 1 replication vector of right neighbour treatments with $r_{\eta m}$ being the number of times the treatments in the design has m^{th} treatment as top neighbour; $\mathbf{r}_\chi = (r_{\chi 1}, r_{\chi 2}, \ldots, r_{\chi v})$ is the v × 1 replication vector of right neighbour treatments with $r_{\chi m}$ being the number of times the treatments in the design has m^{th} treatment as a bottom neighbour; $\mathbf{R}_\gamma = \text{diag.}(r_{\gamma 1}, r_{\gamma 2}, \ldots, r_{\gamma v})$ is the diagonal matrix of replications of treatments as a right neighbour; $\mathbf{R}_\eta = \text{diag.}(r_{\eta 1}, r_{\eta 2}, \ldots, r_{\eta v})$ is the diagonal matrix of replications of treatments as a top neighbour; $\mathbf{R}_\chi = \text{diag.}(r_{\chi 1}, r_{\chi 2}, \ldots, r_{\chi v})$ is the diagonal matrix of replication of treatments as a bottom neighbour.

The joint information matrix for estimating the direct effects, left-neighbour effects, right-neighbour effects, top-neighbour effects and bottom-neighbour effects of treatments is as follows:

$$\mathbf{C} = \begin{pmatrix} \mathbf{C}_{11} & \mathbf{C}_{12} & \mathbf{C}_{13} & \mathbf{C}_{14} & \mathbf{C}_{15} \\ \mathbf{C}_{21} & \mathbf{C}_{22} & \mathbf{C}_{23} & \mathbf{C}_{24} & \mathbf{C}_{25} \\ \mathbf{C}_{31} & \mathbf{C}_{32} & \mathbf{C}_{33} & \mathbf{C}_{34} & \mathbf{C}_{35} \\ \mathbf{C}_{41} & \mathbf{C}_{42} & \mathbf{C}_{43} & \mathbf{C}_{44} & \mathbf{C}_{45} \\ \mathbf{C}_{51} & \mathbf{C}_{52} & \mathbf{C}_{53} & \mathbf{C}_{54} & \mathbf{C}_{55} \end{pmatrix},$$

where,

$$\mathbf{C}_{11} = \mathbf{R}_\tau - (\mathbf{N}_{11}\mathbf{K}^{-1}\mathbf{N}'_{11} + \mathbf{N}_{11}\mathbf{K}^{-1}\mathbf{M}\mathbf{E}^{-}\mathbf{M}'\mathbf{K}^{-1}\mathbf{N}'_{11} - \mathbf{N}_{16}\mathbf{E}^{-}\mathbf{M}\mathbf{K}^{-1}\mathbf{N}'_{11} - \mathbf{N}_{11}\mathbf{K}^{-1}\mathbf{M}\mathbf{E}^{-}\mathbf{N}'_{16} + \mathbf{N}_{16}\mathbf{E}^{-}\mathbf{N}'_{16});$$

$$\mathbf{C}_{12} = \mathbf{N}_{1} - (\mathbf{N}_{11}\mathbf{K}^{-1}\mathbf{N}'_{12} + \mathbf{N}_{11}\mathbf{K}^{-1}\mathbf{M}\mathbf{E}^{-}\mathbf{M}'\mathbf{K}^{-1}\mathbf{N}'_{12} - \mathbf{N}_{16}\mathbf{E}^{-}\mathbf{M}\mathbf{K}^{-1}\mathbf{N}'_{12} - \mathbf{N}_{11}\mathbf{K}^{-1}\mathbf{M}\mathbf{E}^{-}\mathbf{N}'_{17} + \mathbf{N}_{16}\mathbf{E}^{-}\mathbf{N}'_{17});$$

$$\mathbf{C}_{13} = \mathbf{N}_{2} - (\mathbf{N}_{11}\mathbf{K}^{-1}\mathbf{N}'_{13} + \mathbf{N}_{11}\mathbf{K}^{-1}\mathbf{M}\mathbf{E}^{-}\mathbf{M}'\mathbf{K}^{-1}\mathbf{N}'_{13} - \mathbf{N}_{16}\mathbf{E}^{-}\mathbf{M}\mathbf{K}^{-1}\mathbf{N}'_{13} - \mathbf{N}_{11}\mathbf{K}^{-1}\mathbf{M}\mathbf{E}^{-}\mathbf{N}'_{18} + \mathbf{N}_{16}\mathbf{E}^{-}\mathbf{N}'_{18});$$

$$\mathbf{C}_{14} = \mathbf{N}_{3} - (\mathbf{N}_{11}\mathbf{K}^{-1}\mathbf{N}'_{14} + \mathbf{N}_{11}\mathbf{K}^{-1}\mathbf{M}\mathbf{E}^{-}\mathbf{M}'\mathbf{K}^{-1}\mathbf{N}'_{14} - \mathbf{N}_{16}\mathbf{E}^{-}\mathbf{M}\mathbf{K}^{-1}\mathbf{N}'_{14} - \mathbf{N}_{11}\mathbf{K}^{-1}\mathbf{M}\mathbf{E}^{-}\mathbf{N}'_{19} + \mathbf{N}_{16}\mathbf{E}^{-}\mathbf{N}'_{19});$$

$$\mathbf{C}_{15} = \mathbf{N}_{4} - (\mathbf{N}_{11}\mathbf{K}^{-1}\mathbf{N}'_{15} + \mathbf{N}_{11}\mathbf{K}^{-1}\mathbf{M}\mathbf{E}^{-}\mathbf{M}'\mathbf{K}^{-1}\mathbf{N}'_{15} - \mathbf{N}_{16}\mathbf{E}^{-}\mathbf{M}\mathbf{K}^{-1}\mathbf{N}'_{15} - \mathbf{N}_{11}\mathbf{K}^{-1}\mathbf{M}\mathbf{E}^{-}\mathbf{N}'_{20} + \mathbf{N}_{16}\mathbf{E}^{-}\mathbf{N}'_{20});$$

$$\mathbf{C}_{22} = \mathbf{R}_\delta - (\mathbf{N}_{12}\mathbf{K}^{-1}\mathbf{N}'_{12} + \mathbf{N}_{12}\mathbf{K}^{-1}\mathbf{M}\mathbf{E}^{-}\mathbf{M}'\mathbf{K}^{-1}\mathbf{N}'_{12} - \mathbf{N}_{17}\mathbf{E}^{-}\mathbf{M}\mathbf{K}^{-1}\mathbf{N}'_{12} - \mathbf{N}_{12}\mathbf{K}^{-1}\mathbf{M}\mathbf{E}^{-}\mathbf{N}'_{17} + \mathbf{N}_{17}\mathbf{E}^{-}\mathbf{N}'_{17});$$

$$\mathbf{C}_{23} = \mathbf{N}_{5} - (\mathbf{N}_{12}\mathbf{K}^{-1}\mathbf{N}'_{13} + \mathbf{N}_{12}\mathbf{K}^{-1}\mathbf{M}\mathbf{E}^{-}\mathbf{M}'\mathbf{K}^{-1}\mathbf{N}'_{13} - \mathbf{N}_{17}\mathbf{E}^{-}\mathbf{M}\mathbf{K}^{-1}\mathbf{N}'_{13} - \mathbf{N}_{12}\mathbf{K}^{-1}\mathbf{M}\mathbf{E}^{-}\mathbf{N}'_{18} + \mathbf{N}_{17}\mathbf{E}^{-}\mathbf{N}'_{18});$$

$$\mathbf{C}_{24} = \mathbf{N}_{6} - (\mathbf{N}_{12}\mathbf{K}^{-1}\mathbf{N}'_{14} + \mathbf{N}_{12}\mathbf{K}^{-1}\mathbf{M}\mathbf{E}^{-}\mathbf{M}'\mathbf{K}^{-1}\mathbf{N}'_{14} - \mathbf{N}_{17}\mathbf{E}^{-}\mathbf{M}\mathbf{K}^{-1}\mathbf{N}'_{14} - \mathbf{N}_{12}\mathbf{K}^{-1}\mathbf{M}\mathbf{E}^{-}\mathbf{N}'_{19} + \mathbf{N}_{17}\mathbf{E}^{-}\mathbf{N}'_{19});$$

$$\mathbf{C}_{25} = \mathbf{N}_{7} - (\mathbf{N}_{12}\mathbf{K}^{-1}\mathbf{N}'_{15} + \mathbf{N}_{12}\mathbf{K}^{-1}\mathbf{M}\mathbf{E}^{-}\mathbf{M}'\mathbf{K}^{-1}\mathbf{N}'_{15} - \mathbf{N}_{17}\mathbf{E}^{-}\mathbf{M}\mathbf{K}^{-1}\mathbf{N}'_{15} - \mathbf{N}_{12}\mathbf{K}^{-1}\mathbf{M}\mathbf{E}^{-}\mathbf{N}'_{20} + \mathbf{N}_{17}\mathbf{E}^{-}\mathbf{N}'_{20});$$

$$\mathbf{C}_{33} = \mathbf{R}_\gamma - (\mathbf{N}_{13}\mathbf{K}^{-1}\mathbf{N}'_{13} + \mathbf{N}_{13}\mathbf{K}^{-1}\mathbf{M}\mathbf{E}^{-}\mathbf{M}'\mathbf{K}^{-1}\mathbf{N}'_{13} - \mathbf{N}_{18}\mathbf{E}^{-}\mathbf{M}\mathbf{K}^{-1}\mathbf{N}'_{13} - \mathbf{N}_{13}\mathbf{K}^{-1}\mathbf{M}\mathbf{E}^{-}\mathbf{N}'_{18} + \mathbf{N}_{18}\mathbf{E}^{-}\mathbf{N}'_{18});$$

$$\mathbf{C}_{34} = \mathbf{N}_{8} - (\mathbf{N}_{13}\mathbf{K}^{-1}\mathbf{N}'_{14} + \mathbf{N}_{13}\mathbf{K}^{-1}\mathbf{M}\mathbf{E}^{-}\mathbf{M}'\mathbf{K}^{-1}\mathbf{N}'_{14} - \mathbf{N}_{18}\mathbf{E}^{-}\mathbf{M}\mathbf{K}^{-1}\mathbf{N}'_{14} - \mathbf{N}_{13}\mathbf{K}^{-1}\mathbf{M}\mathbf{E}^{-}\mathbf{N}'_{19} + \mathbf{N}_{18}\mathbf{E}^{-}\mathbf{N}'_{19});$$

$$\mathbf{C}_{35} = \mathbf{N}_{9} - (\mathbf{N}_{13}\mathbf{K}^{-1}\mathbf{N}'_{15} + \mathbf{N}_{13}\mathbf{K}^{-1}\mathbf{M}\mathbf{E}^{-}\mathbf{M}'\mathbf{K}^{-1}\mathbf{N}'_{15} - \mathbf{N}_{18}\mathbf{E}^{-}\mathbf{M}\mathbf{K}^{-1}\mathbf{N}'_{15} - \mathbf{N}_{13}\mathbf{K}^{-1}\mathbf{M}\mathbf{E}^{-}\mathbf{N}'_{20} + \mathbf{N}_{18}\mathbf{E}^{-}\mathbf{N}'_{20});$$

$$\mathbf{C}_{44} = \mathbf{R}_\eta - (\mathbf{N}_{14}\mathbf{K}^{-1}\mathbf{N}'_{14} + \mathbf{N}_{14}\mathbf{K}^{-1}\mathbf{M}\mathbf{E}^{-}\mathbf{M}'\mathbf{K}^{-1}\mathbf{N}'_{14} - \mathbf{N}_{19}\mathbf{E}^{-}\mathbf{M}\mathbf{K}^{-1}\mathbf{N}'_{14} - \mathbf{N}_{14}\mathbf{K}^{-1}\mathbf{M}\mathbf{E}^{-}\mathbf{N}'_{19} + \mathbf{N}_{19}\mathbf{E}^{-}\mathbf{N}'_{19});$$

$$\mathbf{C}_{45} = \mathbf{N}_{10} - (\mathbf{N}_{14}\mathbf{K}^{-1}\mathbf{N}'_{15} + \mathbf{N}_{14}\mathbf{K}^{-1}\mathbf{M}\mathbf{E}^{-}\mathbf{M}'\mathbf{K}^{-1}\mathbf{N}'_{15} - \mathbf{N}_{19}\mathbf{E}^{-}\mathbf{M}\mathbf{K}^{-1}\mathbf{N}'_{15} - \mathbf{N}_{14}\mathbf{K}^{-1}\mathbf{M}\mathbf{E}^{-}\mathbf{N}'_{20} + \mathbf{N}_{19}\mathbf{E}^{-}\mathbf{N}'_{20});$$

$$\mathbf{C}_{55} = \mathbf{R}_\chi - (\mathbf{N}_{15}\mathbf{K}^{-1}\mathbf{N}'_{15} + \mathbf{N}_{15}\mathbf{K}^{-1}\mathbf{M}\mathbf{E}^{-}\mathbf{M}'\mathbf{K}^{-1}\mathbf{N}'_{15} - \mathbf{N}_{20}\mathbf{E}^{-}\mathbf{M}\mathbf{K}^{-1}\mathbf{N}'_{15} - \mathbf{N}_{15}\mathbf{K}^{-1}\mathbf{M}\mathbf{E}^{-}\mathbf{N}'_{20} + \mathbf{N}_{20}\mathbf{E}^{-}\mathbf{N}'_{20}).$$

The 5v × 5v matrix **C** is symmetric, non-negative definite with zero row and column sums. From the above, the information matrices for estimating the direct effects ($\mathbf{C}_\tau$), left-neighbour effects ($\mathbf{C}_\delta$), right-neighbour effects ($\mathbf{C}_\gamma$), top-neighbour effects ($\mathbf{C}_\eta$) and bottom-neighbour effects ($\mathbf{C}_\chi$) of treatments can be obtained.

4.3 Method of Construction

Method 4.3.1: Let v be a prime number. Obtain a basic array of v - 1 columns each of size v from the following initial sequence for values of s = 0, 1,..., v – 2 by taking modulo v:

v
s+1
2(s+1)
.

.

.

(v-1)(s+1)

Develop the columns of this array cyclically mod v to get v sets of v-1 columns each. Making design circular by adding border plots results in a row-column design for v treatments in v rows and v(v-1) columns which is strongly balanced for left and right neighbours. It is seen that each of the v treatments has every other treatment as left and right neighbour v-1 times, as top and bottom neighbour v times.

Example 4.3.1.1: Let v = 5 be the number of treatments. Develop the initial array of order 5×4 using the method given. The remaining array can be obtained by cyclically generating each column of the initial array and taking mod 5. Finally add border plots to make the design circular.

	4	3	2	1	5	4	3	2	1	5	4	3	2	1	5	4	3	2	1	5	
4	5	5	5	5	1	1	1	1	2	2	2	2	3	3	3	3	4	4	4	4	5
3	1	2	3	4	2	3	4	5	3	4	5	1	4	5	1	2	5	1	2	3	1
2	2	4	1	3	3	5	2	4	4	1	3	5	5	2	4	1	1	3	5	2	2
1	3	1	4	2	4	2	5	3	5	3	1	4	1	4	2	5	2	5	3	1	3
5	4	3	2	1	5	4	3	2	1	5	4	3	2	1	5	4	3	2	1	5	4
	5	5	5	5	1	1	1	1	2	2	2	2	3	3	3	3	4	4	4	4	

Under Model 4.2.2

The various incidence matrices are as follows:

$\mathbf{N}_1 = \mathbf{N}_3 = \mathbf{N}_5 = (v\text{-}1)\mathbf{J}_v$; $\mathbf{N}_2 = \mathbf{N}_4 = \mathbf{M} = \mathbf{J}_{v \times v(v-1)}$; $\mathbf{R}_\tau = \mathbf{R}_\delta = \mathbf{K} = v(v\text{-}1)\mathbf{I}_v$; $\mathbf{H} = v\mathbf{I}_{v(v-1)}$.

The joint information matrix ($\mathbf{C}$) is of the following form:

$$\mathbf{C} = \begin{pmatrix} v(v\text{-}1)(\mathbf{I}\text{-}\frac{1}{v}\mathbf{J}) & \mathbf{0} \\ \mathbf{0} & v(v\text{-}1)(\mathbf{I}\text{-}\frac{1}{v}\mathbf{J}) \end{pmatrix}.$$

For this class of designs, $\mathbf{C}_\tau = \mathbf{C}_\delta = v(v\text{-}1)(\mathbf{I}\text{-}\frac{1}{v}\mathbf{J})$.

For the given design in Example 4.3.1.1, $\mathbf{C}_\tau = \mathbf{C}_\delta = 20\,\mathbf{I} - 4\,\mathbf{J}$.

Under Model 4.2.3

The various incidence matrices are as follows:

$\mathbf{N}_1 = \mathbf{N}_2 = \mathbf{N}_3 = \mathbf{N}_4 = \mathbf{N}_5 = \mathbf{N}_6 = (v\text{-}1)\mathbf{J}_v$; $\mathbf{N}_7 = \mathbf{N}_8 = \mathbf{N}_9 = \mathbf{M} = \mathbf{J}_{v \times v(v-1)}$; $\mathbf{R}_\tau = \mathbf{R}_\delta = \mathbf{R}_\gamma = \mathbf{K} = v(v\text{-}1)\mathbf{I}_v$; $\mathbf{H} = v\mathbf{I}_{v(v-1)}$.

The joint information matrix ($\mathbf{C}$) is of the following form:

$$\mathbf{C} = \begin{pmatrix} v(v\text{-}1)(\mathbf{I}\text{-}\frac{1}{v}\mathbf{J}) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & v(v\text{-}1)(\mathbf{I}\text{-}\frac{1}{v}\mathbf{J}) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & v(v\text{-}1)(\mathbf{I}\text{-}\frac{1}{v}\mathbf{J}) \end{pmatrix}.$$

The designs obtained here are variance balanced for estimating the direct effects of contrasts in v treatments and the corresponding information matrix for estimating the direct effect of treatments can be obtained as

$$\mathbf{C}_{\tau} = \mathbf{C}_{11} - \mathbf{C}_{12}\mathbf{C}_{22}^{-}\mathbf{C}_{21}\ ;$$

$$\mathbf{C}_{11} = \text{v(v-1)}(\mathbf{I}-\tfrac{1}{\text{v}}\mathbf{J}),\ \mathbf{C}_{12} = \begin{bmatrix}\mathbf{0} & \mathbf{0}\end{bmatrix} \text{ and } \mathbf{C}_{22} = \begin{pmatrix} \text{v(v-1)}(\mathbf{I}-\tfrac{1}{\text{v}}\mathbf{J}) & \mathbf{0} \\ \mathbf{0} & \text{v(v-1)}(\mathbf{I}-\tfrac{1}{\text{v}}\mathbf{J}) \end{pmatrix}.$$

Similarly, the information matrix for estimating left and right neighbour effects can be obtained by the appropriate partitioning of the $\mathbf{C}$ matrix. It can be found that the design is variance balanced for estimating the direct, left and right neighbour effects. Therefore it can be concluded that for this class of designs $\mathbf{C}_{\tau} = \mathbf{C}_{\delta} = \mathbf{C}_{\gamma} = \text{v(v-1)}(\mathbf{I}-\tfrac{1}{\text{v}}\mathbf{J})$.

Particular case: If any number of rows is deleted and keeping a minimum of 4 rows and treating rows as columns and columns as rows, the resultant design is again variance balanced for estimating all the effects.

Under Model 4.2.4

The various incidence matrices are as follows:

$\mathbf{N}_1 = \mathbf{N}_2 = \mathbf{N}_5 = \mathbf{N}_6 = \mathbf{N}_7 = \mathbf{N}_8 = \mathbf{N}_9 = \mathbf{N}_{11} = \mathbf{N}_{12} = \mathbf{N}_{13} = \mathbf{N}_{14} = \mathbf{N}_{15} = \text{(v-1)}\mathbf{J}_\text{v}$; $\mathbf{N}_3 = \mathbf{N}_4 = \mathbf{N}_{10} = \text{v}(\mathbf{J}_\text{v} - \mathbf{I}_\text{v})$; $\mathbf{N}_{16} = \mathbf{N}_{17} = \mathbf{N}_{18} = \mathbf{N}_{19} = \mathbf{N}_{20} = \mathbf{M} = \mathbf{J}_{\text{v} \times \text{v(v-1)}}$; $\mathbf{R}_\tau = \mathbf{R}_\delta = \mathbf{R}_\gamma = \mathbf{R}_\eta = \mathbf{R}_\chi = \mathbf{K} = \text{v(v-1)}\mathbf{I}_\text{v}$ and $\mathbf{H} = \text{v}\mathbf{I}_{\text{v(v-1)}}$

Further,

$$\mathbf{C} = \begin{pmatrix} \text{v(v-1)}(\mathbf{I}-\tfrac{1}{\text{v}}\mathbf{J}) & \mathbf{0} & \mathbf{0} & -\text{v}(\mathbf{I}-\tfrac{1}{\text{v}}\mathbf{J}) & -\text{v}(\mathbf{I}-\tfrac{1}{\text{v}}\mathbf{J}) \\ \mathbf{0} & \text{v(v-1)}(\mathbf{I}-\tfrac{1}{\text{v}}\mathbf{J}) & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \text{v(v-1)}(\mathbf{I}-\tfrac{1}{\text{v}}\mathbf{J}) & \mathbf{0} & \mathbf{0} \\ -\text{v}(\mathbf{I}-\tfrac{1}{\text{v}}\mathbf{J}) & \mathbf{0} & \mathbf{0} & \text{v(v-1)}(\mathbf{I}-\tfrac{1}{\text{v}}\mathbf{J}) & -\text{v}(\mathbf{I}-\tfrac{1}{\text{v}}\mathbf{J}) \\ -\text{v}(\mathbf{I}-\tfrac{1}{\text{v}}\mathbf{J}) & \mathbf{0} & \mathbf{0} & -\text{v}(\mathbf{I}-\tfrac{1}{\text{v}}\mathbf{J}) & \text{v(v-1)}(\mathbf{I}-\tfrac{1}{\text{v}}\mathbf{J}) \end{pmatrix}.$$

The joint information matrix for estimating the direct, top and bottom neighbour effects is of the form

$$\mathbf{C}_{\tau\eta\chi} = \begin{pmatrix} \text{v(v-1)}(\mathbf{I}-\tfrac{1}{\text{v}}\mathbf{J}) & -\text{v}(\mathbf{I}-\tfrac{1}{\text{v}}\mathbf{J}) & -\text{v}(\mathbf{I}-\tfrac{1}{\text{v}}\mathbf{J}) \\ -\text{v}(\mathbf{I}-\tfrac{1}{\text{v}}\mathbf{J}) & \text{v(v-1)}(\mathbf{I}-\tfrac{1}{\text{v}}\mathbf{J}) & -\text{v}(\mathbf{I}-\tfrac{1}{\text{v}}\mathbf{J}) \\ -\text{v}(\mathbf{I}-\tfrac{1}{\text{v}}\mathbf{J}) & -\text{v}(\mathbf{I}-\tfrac{1}{\text{v}}\mathbf{J}) & \text{v(v-1)}(\mathbf{I}-\tfrac{1}{\text{v}}\mathbf{J}) \end{pmatrix}.$$

The designs obtained here are variance balanced for estimating the direct effects of contrasts in v treatments and the corresponding information matrix for estimating the direct effect of treatments can be obtained as

$$\mathbf{C}_{\tau} = \mathbf{C}_{11} - \mathbf{C}_{12}\mathbf{C}_{22}^{-}\mathbf{C}_{21}\ ;$$

$$\mathbf{C}_{11} = v(v-1)(\mathbf{I}-\tfrac{1}{v}\mathbf{J}),\ \mathbf{C}_{12} = \begin{pmatrix} -v(\mathbf{I}-\tfrac{1}{v}\mathbf{J}) & -v(\mathbf{I}-\tfrac{1}{v}\mathbf{J}) \end{pmatrix},$$

and

$$\mathbf{C}_{22} = \begin{pmatrix} v(v-1)(\mathbf{I}-\tfrac{1}{v}\mathbf{J}) & -v(\mathbf{I}-\tfrac{1}{v}\mathbf{J}) \\ -v(\mathbf{I}-\tfrac{1}{v}\mathbf{J}) & v(v-1)(\mathbf{I}-\tfrac{1}{v}\mathbf{J}) \end{pmatrix}.$$

Similarly, the information matrix for estimating top and bottom neighbour effects can be obtained by the appropriate partitioning of the $\mathbf{C}_{\tau\eta\chi}$ matrix. It can be found that the design is variance balanced for estimating the top and bottom effects.

The joint information matrix for estimating the left and right neighbour effects is as follows:

$$\mathbf{C}_{\delta\gamma} = \begin{pmatrix} v(v-1)(\mathbf{I}-\tfrac{1}{v}\mathbf{J}) & \mathbf{0} \\ \mathbf{0} & v(v-1)(\mathbf{I}-\tfrac{1}{v}\mathbf{J}) \end{pmatrix}.$$

Hence $\mathbf{C}_{\delta} = v(v-1)(\mathbf{I}-\tfrac{1}{v}\mathbf{J}) = \mathbf{C}_{\gamma}$.

Therefore, it can be concluded that for this class of designs $\mathbf{C}_{\boldsymbol{\tau}} = \mathbf{C}_{\eta} = \mathbf{C}_{\chi}$ and $\mathbf{C}_{\delta} = \mathbf{C}_{\gamma}$.

For the given design $\mathbf{C}_{\boldsymbol{\tau}} = \mathbf{C}_{\eta} = \mathbf{C}_{\chi}$ = 16.66 **I** – 3.33 **J** and $\mathbf{C}_{\delta} = \mathbf{C}_{\gamma}$ = 20 **I** – 4 **J**.

Method 4.3.2: Let v be the number of treatments. Obtain a $v^2\times 3$ basic array of the form given as follows:

$$\begin{pmatrix} 1 & s & s \\ 2 & s & s+1 \\ . & . & . \\ . & . & . \\ . & . & . \\ v & s & s+v-1 \end{pmatrix} \text{ where } s = 1, 2, \ldots, v$$

Generate cyclically mod v to get an array of size $v^2\times 3v$. Add border plots to make the design circular. The design obtained here is variance balanced for estimating the direct and neighbour effects of treatments.

Example 4.3.2.1: Let v = 4 be the number of treatments. Develop an initial array of size 16×3 using the procedure given and generate it cyclically mod 4 to get an array of size16×12. Add border plots to make the design circular.

	4	4	3	1	1	4	2	2	1	3	3	2	
4	1	1	1	2	2	2	3	3	3	4	4	4	1
1	2	1	2	3	2	3	4	3	4	1	4	1	2
2	3	1	3	4	2	4	1	3	1	2	4	2	3
3	4	1	4	1	2	1	2	3	2	3	4	3	4
1	1	2	2	2	3	3	3	4	4	4	1	1	1
2	2	2	3	3	3	4	4	4	1	1	1	2	2
3	3	2	4	4	3	1	1	4	2	2	1	3	3

4	4	2	1	1	3	2	2	4	3	3	1	4	4
2	1	3	3	2	4	4	3	1	1	4	2	2	1
3	2	3	4	3	4	1	4	1	2	1	2	3	2
4	3	3	1	4	4	2	1	1	3	2	2	4	3
1	4	3	2	1	4	3	2	1	4	3	2	1	4
3	1	4	4	2	1	1	3	2	2	4	3	3	1
4	2	4	1	3	1	2	4	2	3	1	3	4	2
1	3	4	2	4	1	3	1	2	4	2	3	1	3
2	4	4	3	1	1	4	2	2	1	3	3	2	4
	1	1	1	2	2	2	3	3	3	4	4	4	

Under Model 4.2.2

For this class of design, $\mathbf{N}_1 = \mathbf{N}_3 = 3\,\mathbf{J}_{v\times v^2}$; $\mathbf{N}_2 = \mathbf{N}_4 = v\mathbf{J}_{v\times 3v}$; $\mathbf{M} = \mathbf{J}_{v^2\times 3v}$; $\mathbf{R}_\tau = \mathbf{R}_\delta = 3v^2\mathbf{I}_v$; $\mathbf{K} = 3v\,\mathbf{I}_{v^2}$; $\mathbf{H} = v^2\mathbf{I}_{3v}$. The joint information matrix ($\mathbf{C}$) is of the following form

$$\mathbf{C} = \begin{pmatrix} 3v^2(\mathbf{I}-\frac{1}{v}\mathbf{J}) & \mathbf{0} \\ \mathbf{0} & 3v^2(\mathbf{I}-\frac{1}{v}\mathbf{J}) \end{pmatrix}.$$

The class of design obtained here is variance balanced for estimating both the direct and left neighbour effects of the treatments. It can be concluded that for this class of designs $\mathbf{C}_\tau = \mathbf{C}_\delta$.

Under Model 4.2.3

The various incidence matrices are as follows:

$\mathbf{N}_1 = \mathbf{N}_2 = \mathbf{N}_3 = 3v\mathbf{J}_v$; $\mathbf{N}_4 = \mathbf{N}_5 = \mathbf{N}_6 = 3\,\mathbf{J}_{v\times v^2}$; $\mathbf{N}_7 = \mathbf{N}_8 = \mathbf{N}_9 = v\mathbf{J}_{v\times 3v}$; $\mathbf{M} = \mathbf{J}_{v^2\times 3v}$; $\mathbf{R}_\tau = \mathbf{R}_\delta = \mathbf{R}_\gamma = 3v^2\mathbf{I}_v$; $\mathbf{K} = 3v\,\mathbf{I}_{v^2}$; $\mathbf{H} = v^2\mathbf{I}_{3v}$ with

$$\mathbf{C} = \begin{pmatrix} 3v^2(\mathbf{I}-\frac{1}{v}\mathbf{J}) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 3v^2(\mathbf{I}-\frac{1}{v}\mathbf{J}) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & 3v^2(\mathbf{I}-\frac{1}{v}\mathbf{J}) \end{pmatrix}$$

The designs obtained here are variance balanced for estimating the direct effects of contrasts in v treatments and the corresponding information matrix for estimating the direct effect of treatments can be obtained as

$$\mathbf{C}_\tau = \mathbf{C}_{11} - \mathbf{C}_{12}\mathbf{C}_{22}^{-}\mathbf{C}_{21};$$

$$\mathbf{C}_{11} = 3v^2(\mathbf{I}-\tfrac{1}{v}\mathbf{J}),\ \mathbf{C}_{12} = [\mathbf{0} \quad \mathbf{0}] \text{ and } \mathbf{C}_{22} = \begin{pmatrix} 3v^2(\mathbf{I}-\frac{1}{v}\mathbf{J}) & \mathbf{0} \\ \mathbf{0} & 3v^2(\mathbf{I}-\frac{1}{v}\mathbf{J}) \end{pmatrix}.$$

Similarly, the information matrix for estimating left and right neighbour effects can be obtained by the appropriate partitioning of the $\mathbf{C}$ matrix. It can be found that the design is variance balanced for estimating the direct, left and right effects. Therefore it can be concluded that for this class of designs $\mathbf{C}_\tau = \mathbf{C}_\delta = \mathbf{C}_\gamma$. For the given example, $\mathbf{C}_\tau = \mathbf{C}_\delta = \mathbf{C}_\gamma = 48\,\mathbf{I} - 12\,\mathbf{J}$.

Method 4.3.3: Let v be the number of treatments. Construct a v × v square with v replicates of v distinct treatments by placing the treatment number c at $(ij)^{th}$ cell, i.e., in i^{th} row and j^{th} column, which satisfies equation ij ≡ c (mod v +1), i, j, c = 1,…,v, where v + 1 should be prime and c (mod v+1) is the unique remainder among the set (1,…,v) on dividing v +1 into ij. i.e., excluding the remainder zero. Given j and c the equation has exactly one solution for i, and similarly, given i and c there is exactly one solution for j, which is the Latin square property implying that each treatment has a different neighbour. Now, take the mirror image of the v × v square and augment to the right or left of the existing design to get a v × 2v array. Add border plots to make the design *circular*. This array results in a NBRC design with parameters v, p = v, q = 2v, r = 2v, $\mu_1 = \mu_1 = 2$.

Example 4.3.3.1: For v = 6, a 6 × 6 array is obtained as follows:

Columns / Rows	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

The NBRC design with parameters v = 6, p = 6, q =12, r = 12 $\mu_1 = 2$ and $\mu_2 = 2$ is obtained as follows:

	1	2	3	4	5	6	6	5	4	3	2	1	
1	1	2	3	4	5	6	6	5	4	3	2	1	1
2	2	4	6	1	3	5	5	3	1	6	4	2	2
3	3	6	2	5	1	4	4	1	5	2	6	3	3
4	4	1	5	2	6	3	3	6	2	5	1	4	4
5	5	3	1	6	4	2	2	4	6	1	3	5	5
6	6	5	4	3	2	1	1	2	3	4	5	6	6
	6	5	4	3	2	1	1	2	3	4	5	6	

Under Model 4.2.2

The various incidence matrices are as follows:

$$\mathbf{N}_1 = \mathbf{N}_3 = \mathbf{N}_5 = 2\mathbf{J}_v;\ \mathbf{N}_2 = \mathbf{M} = \mathbf{N}_5 = \mathbf{J}_{v\times 2v};\ \mathbf{R}_\tau = \mathbf{R}_\delta = \mathbf{K} = 2v\mathbf{I}_v \text{ and } \mathbf{H} = v\mathbf{I}_{2v}.$$

The joint information matrix is of the following form:

$$\mathbf{C} = \begin{pmatrix} 2v(\mathbf{I}-\frac{1}{v}\mathbf{J}) & \mathbf{0} \\ \mathbf{0} & 2v(\mathbf{I}-\frac{1}{v}\mathbf{J}) \end{pmatrix}.$$

The designs obtained here are variance balanced for estimating the direct effects of contrasts in v treatments and the corresponding information matrix for estimating the direct effect of treatments can be obtained as

$$\mathbf{C}_{\tau} = \mathbf{C}_{11} - \mathbf{C}_{12}\mathbf{C}_{22}^{-}\mathbf{C}_{21};$$

Similarly, the information matrix for estimating left neighbour effects can be obtained by the appropriate partitioning of the $\mathbf{C}$ matrix. It can be found that the design is variance balanced for estimating the direct effects and left neighbour effects with $\mathbf{C}_{\tau} = \mathbf{C}_{\delta} = 2v(\mathbf{I}-\frac{1}{v}\mathbf{J})$. For the above given example $\mathbf{C}_{\tau} = \mathbf{C}_{\delta} = 12\,\mathbf{I} - 2\,\mathbf{J}$.

Under Model 4.2.3

Under this the design obtained is variance balanced for estimating the direct effects of treatments. For the given example, it is found that $\mathbf{C}_{\tau} = 12\,\mathbf{I} - 2\,\mathbf{J}$.

Method 4.3.4: Consider a Balanced Incomplete Block (BIB) design with parameters $v, b, r, k>2, \lambda=1$. Repeat each treatment in the block (considered as row) once at the positions adjacent to the existing positions. From each row of this design, develop 2k-1 more rows by rotating the treatments clockwise resulting into 2bk rows. Add border plots to make the design *circular*. The resultant design is a strongly neighbour balanced row-column design with incomplete rows and with parameters $v = v$, $p = 2bk$, $q = 2k$, $r = 4rk$, $\mu_1 = 2k$ and $\mu_2 = 2kr$.

Example 4.3.4.1: Consider the BIB design with parameters $v = b = 7$, $r = k = 3$, $\lambda = 1$. Following is the strongly balanced incomplete row-column design with parameters $v = 7$, $p = 42$, $q = 6$, $r = 36$, $\mu_1 = 6$ and $\mu_2 = 18$:

	3	7	7	1	1	3	
4	1	1	2	2	4	4	1
1	1	2	2	4	4	1	1
1	2	2	4	4	1	1	2
2	2	4	4	1	1	2	2
2	4	4	1	1	2	2	4
4	4	1	1	2	2	4	4
5	2	2	3	3	5	5	2
2	2	3	3	5	5	2	2
2	3	3	5	5	2	2	3
3	3	5	5	2	2	3	3
3	5	5	2	2	3	3	5
5	5	2	2	3	3	5	5

6	3	3	4	4	6	6	3
3	3	4	4	6	6	3	3
3	4	4	6	6	3	3	4
4	4	6	6	3	3	4	4
4	6	6	3	3	4	4	6
6	6	3	3	4	4	6	6
7	4	4	5	5	7	7	4
4	4	5	5	7	7	4	4
4	5	5	7	7	4	4	5
5	5	7	7	4	4	5	5
5	7	7	4	4	5	5	7
7	7	4	4	5	5	7	7
1	5	5	6	6	1	1	5
5	5	6	6	1	1	5	5
5	6	6	1	1	5	5	6
6	6	1	1	5	5	6	6
6	1	1	5	5	6	6	1
1	1	5	5	6	6	1	1
2	6	6	7	7	2	2	6
6	6	7	7	2	2	6	6
6	7	7	2	2	6	6	7
7	7	2	2	6	6	7	7
7	2	2	6	6	7	7	2
2	2	6	6	7	7	2	2
3	7	7	1	1	3	3	7
7	7	1	1	3	3	7	7
7	1	1	3	3	7	7	1
1	1	3	3	7	7	1	1
1	3	3	7	7	1	1	3
3	3	7	7	1	1	3	3
	1	1	2	2	4	4	

For this design under Model 4.2.2, $\mathbf{C}_{\boldsymbol{\tau}} = \mathbf{C}_{\delta} = 24.01\mathbf{I} - 3.43\mathbf{J}$ and under Model 4.2.3, $\mathbf{C}_{\boldsymbol{\tau}} = 21\,\mathbf{I} - 3\,\mathbf{J}$ and $\mathbf{C}_{\delta} = \mathbf{C}_{\gamma} = 11.5\,\mathbf{I} - 1.5\,\mathbf{J}$.

4.4 Minimal Neighbour Balanced Row-Column Designs

A row-column design with four sided neighbour effects is said to be minimally balanced if every treatment has every other treatment appearing as a neighbour exactly once in all the four directions. This design will be balanced and with minimum number of experimental units.

Method 4.4.1: Construct an initial row of size v.

1	v	2	v-1	3	v-2	...	(v+3)/2	(v+1)/2

Develop the columns by proceeding from top to bottom filling only odd numbered cells by adding 1 to the previous odd numbered cell and taking modulo v. Thus the array obtained will be as follows:

1	v	2	v-1	3	v-2	• • •	(v+3)/2	(v+1)/2
2	1	3	v	4	v-1	• • •	(v+5)/2	(v+3)/2
⋮			⋮					⋮
(v-1)/2	(v-3)/2	(v+1)/2	(v-5)/2	(v+3)/2	(v-7)/2	• • •	v	v-1
(v+1)/2	(v-1)/2	(v+3)/2	(v-3)/2	(v+5)/2	(v-5)/2	• • •	1	v

The empty positions in the even numbered cells are filled in the reverse direction (from bottom to top) by adding 1 to the previous cell and taking modulo v. The resultant design is a row-column designs balanced for spatial indirect effects with parameters v, p = v, q = r = v, μ = 1.

Example 4.4.1.1: The minimally neighbour balanced row-column design obtained for v = 7 is as follows:

1	7	2	6	3	5	4
7	6	1	5	2	4	3
2	1	3	7	4	6	5
6	5	7	4	1	3	2
3	2	4	1	5	7	6
5	4	6	3	7	2	1
4	3	5	2	6	1	7

Chapter V

EXPERIMENTAL DESIGNS BALANCED FOR TEMPORAL INDIRECT EFFECTS

5.1 Designs Balanced for Temporal Indirect Effects

Treatments applied in a particular period may influence the response of the units not only in the period of their direct application but also in the subsequent periods. The treatments leave carryover effects (temporal indirect effects) in the periods following the periods of their direct application. The designs involving sequences of treatments are more popularly known as Crossover designs (CODs)

Method 5.1.1: Let number of treatments v be a prime number. Obtain a basic array of v - 1 columns each of size v from the following initial sequence:

$$\omega + 1$$
$$2(\omega + 1)$$
$$.$$
$$.$$
$$.$$
$$(v-1)(\omega + 1)$$

For $\omega = 0, 1, ..., v - 2$ modulo v

Now, develop the columns of this array cyclically mod v to get v sets of v-1 columns each. The resultant designs obtained are balanced for temporal indirect effects up to order two in incomplete units. The parameters of the design so obtained are v (prime) treatments, p = v-1 periods and n = v(v-1) experimental units. The design developed is found to be variance balanced for estimating direct and temporal indirect effects.

Example 5.1.1.1: Let v =5

Initial Array			
1	2	3	4
2	4	1	3
3	1	4	2
4	3	2	1

	Units																			
Periods	1	2	3	4	2	3	4	5	3	4	5	1	4	5	1	2	5	1	2	3
	2	4	1	3	3	5	2	4	4	1	3	5	5	2	4	1	1	3	5	2
	3	1	4	2	4	2	5	3	5	3	1	4	1	4	2	5	2	5	3	1
	4	3	2	1	5	4	3	2	1	5	4	3	2	1	5	4	3	2	1	5

5.2 Analysis of Experiments when Treatments Exhibit Temporal Indirect Effects

The analysis of COD balanced for temporal indirect effect is illustrated through simulated data on milk production of dairy cows, for the COD with number of treatments (v) = 5 (0,1,2,3, and 4), number of periods (p) = 9, the number of experimental units (s) = 5 and with one pre-period.

		Units				
		1	2	3	4	5
Pre-period	0	(0) 9.76	(1) 16.42	(2) 10.63	(3) 16.48	(4) 7.66
Periods	1	(0) 12.45	(1) 15.49	(2) 7.92	(3) 14.20	(4) 10.47
	2	(4) 11.85	(0) 9.80	(1) 14.33	(2) 6.71	(3) 16.42
	3	(1) 16.25	(2) 9.68	(3) 17.33	(4) 8.54	(0) 8.03
	4	(3) 20.477	(4) 9.98	(0) 7.62	(1) 12.60	(2) 12.26
	5	(2) 12.42	(3) 19.83	(4) 10.25	(0) 11.30	(1) 18.49
	6	(3) 19.03	(4) 9.87	(0) 8.32	(1) 16.66	(2) 11.56
	7	(1) 15.14	(2) 10.55	(3) 15.92	(4) 9.11	(0) 8.59
	8	(4) 13.11	(0) 9.95	(1) 13.90	(2) 9.48	(3) 20.18
	9	(0) 9.18	(1) 17.94	(2) 10.40	(3) 14.39	(4) 8.94

*Numbers in the parenthesis indicates treatments

ANOVA

Source	D.F.	S.S.	M.S.	F
Periods	8	30.98	3.87	4.6419
Units	4	49.77	12.44	14.9133
Direct effects ignoring residual effects	4	508.83	127.21	
Residual effects eliminating direct effects	4	35.20	8.80	10.54**
OR				
Residual effects ignoring residual effects	4	52.70	13.18	
Direct effects eliminating direct effects	4	491.32	122.83	147.21**
Error	24	20.02	0.83	
Total	44	644.81		

**indicate significant at 1% level

Here both the direct and residual effects are seen to be significant.

5.3 Crossover Designs with Factorial Treatment Structure

In many animal experiments, it is often required to measure the effect of response from two or more factors over various temporal environments, like studying the effect of simultaneous application of various fertilizers on a crop or, the effect of feeds and environments on milk yield of cows. CODs having two or more factors are suitable for such situations. Several researchers have contributed to the construction and related aspects of CODs for the simultaneous application of more than one factor to experimental units over periods (Fletcher, 1987; Fletcher and John, 1985; Fletcher *et al*., 1990; Dwivedi *et al*., 2008 and Mason and Hinkelmann, 1971). Most of these studies assumed the interaction among the factors to be present. Some studies were made without considering the presence of residual effects (*i.e*., the carry over effects of treatments that persist even after the period of application of treatments) and some considered the situations where one of the factors exhibit residual effects.

Sometimes, different levels of two unrelated factors are to be applied to experimental units simultaneously and their joint effect after each period of application is measured, but as observations are taken over different time periods from the units, first order residual effects of the levels of both the factors may be present. For example, in an experiment to study the effect of different methods of shearing and various feeds on body weight of sheep, method of shearing is not related to type of feed. Here, observation (body weight) has to be taken from each unit during each period and both the factors may exhibit residual effects. Some classes of such factorial CODs assuming the absence of interactions were given by Lewis and Russell (1998) and Varghese *et al.* (2009).

In experimental situations, wherein experimental units are required to perform a series of tasks one after another under various environmental conditions such as different types of lighting or temperature or equipments, it is difficult to change the environmental conditions. Thus, each subject is required to perform all the assigned tasks under one set of conditions during one session. The conditions are altered from one session to another. Designs with nested structure having experimental conditions treated as levels of first factor and different tasks treated as levels of nested factor are suitable for such experiments. This experimental situation has been considered in literature by few researchers (Dean *et al*., 1999; Raghavarao and Xie; 2003). The experimental setting is such that there is a gap between each main session and hence it is assumed that no carry over effect transfers from a main session to another. But carry over effects are assumed to be present within main sessions (from sub-session to sub-session). Here, a class of designs involving sequences of treatment combinations with nested factors has been obtained.

5.3.1 Designs Involving Sequence of Treatment Combinations Balanced for One Factor

We first give here some definitions of designs involving sequence of treatments having factorial treatment structure followed by the model and the class of designs proposed.

Uniform: A factorial COD with two factors F_1 and F_2 (where F_2 is nested within F_1) having levels f_1 and f_2 respectively, is called uniform on periods if every treatment combination occurs in each period the same number of times, say χ_1. A necessary condition for this to hold is that the number of units, $n = \chi_1 v$, v being the number of treatment combinations (= $f_1 f_2$). A factorial COD with two factors F_1 and F_2 having levels f_1 and f_2 respectively, is called uniform on units if every treatment combination is applied to each experimental unit the same number of times, say χ_2. This can occur only if the number of periods, $p = \chi_2 v$. A design is called uniform if it is uniform on both periods and units.

Balanced: A COD with two factors F_1 and F_2 having levels f_1 and f_2 respectively, with levels of F_2 nested within levels of F_1, is said to be balanced if every combination of the two factors is preceded by each level of the nested factor F_1 (except the level appearing in the combination) an equal number of times.

Strongly balanced: A COD with two factors F_1 and F_2 having levels f_1 and f_2 respectively, with levels of F_2 nested within levels of F_1, is said to be strongly balanced if every combination of the two factors is preceded by every other level of the nested factor (excluding the level appearing in the combination) equally often say, Δ_1 times and by the level of the nested factor appearing in the combination Δ_2 times. Δ_1 and Δ_2 may or may not be equal.

Variance Balanced: A COD with two factors F_1 and F_2 is said to be variance balanced if all elementary contrasts pertaining to direct effects of various treatment combinations consisting of levels of both the factors are estimated with a constant variance.

5.3.2 Designs Involving Sequences of Treatment Combinations with Nested Factors

Let two factors F_1 and F_2 have number of levels f_1 (represented by 1, 2, 3,…) and f_2 (represented by a, b, c,…) giving rise to $f_1 \times f_2$ treatment combinations where the levels of F_2 are nested within the levels of F_1. First consider a balanced COD for f_1 levels of first factor in p_1 periods (main sessions) and n_1 experimental units. Within each cell (period-unit intersection) of this design, consider another balanced/strongly balanced COD for f_2 levels of second factor in p_2 periods (sub-sessions) and n_2 experimental units. The resultant design will have $p_1 p_2$ periods and $n_1 n_2$ units and each sub-cell receives a treatment combination out of $f_1 f_2$ possible combinations. It is assumed that there is a gap between each main session and hence no carry over effect is assumed from one main session to another main session. But carry over effects are assumed to be present within main sessions (from sub-session to sub-session). The resultant design is balanced/ strongly balanced depending on the design considered for the nested factor.

To explain the general method described above, we take up two illustrations using two classes of designs wherein the second case is just the reverse arrangement of the first. In the first case, we make use of Williams (1949) Latin squares as the main session design and a two treatment design as the sub-session design. An easy method of obtaining Williams Latin squares was given by Sharma (1975). The steps involved are described below:

- Construct one (or two) $f_1 \times f_1$ table(s) in which columns refer to experimental units and rows to periods according to even (or odd) f_1.
- In both the squares, number the periods from 1 to f_1 successively.
- Assign the levels of first factor 1, 2, …, f_1 successively to the f_1 cells in the first column of both the squares by proceeding from top to bottom, entering only in odd-numbered cells in the first and even numbered cells in the second square, and then reversing the direction, filling in even-numbered cells in the first and odd numbered cells in the second square.
- Obtain the successive columns of the squares by adding integer 1 to each element of the previous column and reducing the elements, if necessary, by mod f_1.

It is to be noted that in each of the constructed squares every level occurs in each row and in each column precisely once. Moreover, when f_1 is even, each level is preceded exactly once by other level in either of the two squares. Thus, in this case either of the two squares may be used. This situation occurs in neither of the two squares if f_1 is odd. However, when both the squares are considered together, one after another horizontally, each level is preceded by every other level exactly twice. Consequently, both the squares must be used in this case.

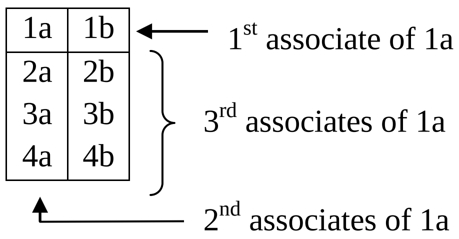
The sub-session design considered is a two treatment COD involving 2 periods and 4 units obtained by taking all possible pairs (including the identical pairs) between the two symbols.

Example 5.3.2.1: Let there are two factors with 4 levels of main session factor denoted by (1, 2, 3, 4) and 2 levels of sub-session factor denoted by (a, b). Hence there are a total of 8 (= $f_1 f_2$ = 4 × 2) treatment combinations. Using a Williams square design as the main session factor design and the above mentioned two period design as the sub-session factor design, we get a nested COD in 8 (= $p_1 p_2$ = 4 × 2) periods and 16 (= $n_1 n_2$ = 4 × 4) experimental units with 2 sub-sessions nested within 4 main sessions as given below:

		Experimental Unit															
		i	**ii**	**iii**	**iv**	**v**	**Vi**	**vii**	**viii**	**ix**	**x**	**xi**	**xii**	**xiii**	**xiv**	**xv**	**xvi**
Period	**1**	1a	1a	1b	1b	2a	2a	2b	2b	3a	3a	3b	3b	4a	4a	4b	4b
	2	1a	1b	1a	1b	2a	2b	2a	2b	3a	3b	3a	3b	4a	4b	4a	4b
	3	4a	4a	4b	4b	1a	1a	1b	1b	2a	2a	2b	2b	3a	3a	3b	3b
	4	4a	4b	4a	4b	1a	1b	1a	1b	2a	2b	2a	2b	3a	3b	3a	3b
	5	2a	2a	2b	2b	3a	3a	3b	3b	4a	4a	4b	4b	1a	1a	1b	1b
	6	2a	2b	2a	2b	3a	3b	3a	3b	4a	4b	4a	4b	1a	1b	1a	1b
	7	3a	3a	3b	3b	4a	4a	4b	4b	1a	1a	1b	1b	2a	2a	2b	2b
	8	3a	3b	3a	3b	4a	4b	4a	4b	1a	1b	1a	1b	2a	2b	2a	2b

It can be seen that this design is uniform and combinatorially strongly balanced. The design is partially variance balanced following the rectangular association scheme given by Vartak (1955) as described below:

Association Scheme: Two treatment combinations $\phi\varphi$ and $\phi'\varphi'$ ($\phi \neq \phi'=1,2,\ldots, f_1$; $\varphi \neq \varphi' = 1,2,\ldots, f_2$) are said to be first associates if $\phi = \phi'$ *i.e.*, the combinations with same level of first factor and different levels of second factor are first associates. Two treatment combinations $\phi\varphi$ and $\phi'\varphi'$ are said to be second associates if $\varphi = \varphi'$ *i.e.*, the combinations with same level of second factor and different levels of first factor are second associates, and remaining are third associates. For the Example 3.1, the arrangement of 8 treatment combinations is as follows:



For the given association scheme for $f_1 \times f_2$ treatment combinations, number of first associates $= f_2 - 1$, number of second associates $= f_1 - 1$, and number of third associates $= f_1 f_2 - f_1 - f_2 + 1$.

Reverse Arrangement: By reversing the roles of F_1 and F_2 in the above example, we get another design belonging to a different class of designs involving sequences of treatment combinations with nested factors with same number of experimental periods and units.

Example 5.3.2.2: Let there are two factors with 2 levels of main session factor denoted by (1, 2) and 4 levels of sub-session factor denoted by (a, b, c, d). Hence there are 8 (= $f_1 \times f_2 = 2 \times 4$) treatment combinations. By the method of construction described above, using the above two period design and Williams square design as the main session factor design and nested factor design respectively, we get a nested COD in 8 (= $p_1 \times p_2 = 2 \times 4$) periods and 16 (= $n_1 \times n_2 = 4 \times 4$) experimental units as given below:

		Experimental Unit															
		i	**ii**	**iii**	**iv**	**v**	**vi**	**vii**	**viii**	**ix**	**x**	**xi**	**xii**	**xiii**	**xiv**	**xv**	**xvi**
Period	**1**	1a	1b	1c	1d	1a	1b	1c	1d	2a	2b	2c	2d	2a	2b	2c	2d
	2	1d	1a	1b	1c	1d	1a	1b	1c	2d	1a	2b	2c	2d	1a	2b	2c
	3	1b	1c	1d	1a	1b	1c	1d	1a	2b	2c	2d	2a	2b	2c	2d	2a
	4	1c	1d	1a	1b	1c	1d	1a	1b	2c	2d	2a	2b	2c	2d	2a	2b
	5	1a	1b	1c	1d	2a	2b	2c	2d	1a	1b	1c	1d	2a	2b	2c	2d
	6	1d	1a	1b	1c	2d	1a	2b	2c	1d	1a	1b	1c	2d	1a	2b	2c
	7	1b	1c	1d	1a	2b	2c	2d	2a	1b	1c	1d	1a	2b	2c	2d	2a
	8	1c	1d	1a	1b	2c	2d	2a	2b	1c	1d	1a	1b	2c	2d	2a	2b

This design is uniform and combinatorially balanced. In terms of variance of estimate of contrasts pertaining to direct as well as residual effects of treatment combinations, the design was seen to be partially variance balanced following rectangular association scheme. The precision of estimation of direct and residual effects is more in the design having more number of levels of the sub-session factor.

5.4 Experimental Designs with Temporal Indirect Effects in the Presence of Systematic Trend

In many agricultural experiments where observations are recorded over periods, experimental units may exhibit time trend. For example, in agricultural experiment where aim is to study the effect of growth stimulating treatments on plant growth rate, the plant growth rate exhibits trend over periods as it will be high when the plant is young and declines gradually as the plant becomes older.

Another example could be nutritional experiment with dairy cattle, where experimenter wants to study the effect of feeds on the milk yield of dairy cows. Here, the milk yield within lactation exhibits time trend, as it first increases for certain number of periods till it reaches its peak and then declines gradually.

Therefore, it is necessary to account for these possible trends while carrying out analysis of data and/or designing experiments for such situations. Patterson (1950) pointed out that the analysis suggested by Cochran *et al.* (1941) and Williams (1949) for the cross-over designs taking into account the first residual effects of the treatments may give rise to biased estimates of error variance for the comparisons of certain treatment effects in the case of dairy cattle feeding experiments where the experimental units exhibit time trend in yields over the periods. He then suggested a procedure of analysis which gives unbiased estimates of error variance in such situations. The procedure given by Patterson (1950) consists of first computing the linear, quadratic, … components of time trends for each experimental unit and then analyzing these quantities in order to obtain the estimates of treatment effects and error variance. Subsequently, Patterson (1951) suggested a method which is equivalent to making a separate analysis for each degree of polynomial, each of which provides some information on the parameters and then pooling the sets of estimates using their precision as weights. It was concluded that if linear, quadratic, … components of the time trends for each experimental unit are independent and have unequal variances, the weighted analysis provides more precise estimates of treatment effects as compared to the un-weighted analysis. Lucas (1951) working independently, estimated the bias from some sets of data from dairy cow feeding experiments using CODs of Latin-squares type and concluded that the bias is of no practical significance in 3×3 designs but might be serious for 4×4 designs. Vijaya (1992) dealt with the analysis of CODs by incorporating the linear trend variable in the model.

Another way of taking care of time trends is to use trend-free designs. Designs which allow the estimation of treatment effects orthogonal to trend effects are called *trend-free* designs.

Trend-free designs for factorial and fractional factorial have been studied by Daniel and Wilcoxon (1966). Bradley and Yeh (1980) and Yeh and Bradley (1983) have studied trend-free block designs in detail. However, in context of CODs with carry-over effects, no serious attempt seems to have been made perhaps due to complexity of the problem.

Here, an attempt has been made to obtain conditions for a balanced CODs that allow estimation of treatment effects contrasts orthogonal to trend effects. Methods of constructing two-period trend-free balanced CODs for odd as well as for even number of the form 2^s of treatments have been given. Method of analysis of these designs has also been presented along with an illustration.

First, we give some definitions that will be used in subsequent sections.

Totally balanced cross-over designs: Cross-over designs permitting the estimation of direct effects in the presence of first order residual effects is called variance balanced, if the variance of elementary contrasts among the direct effects is the same, say α and the variance of elementary contrasts among the residual effects is also same, say β. The constant α need not be equal to β. A cross-over design is called *totally balanced,* if $\alpha = \beta$ (Dey and Balachandran, 1976).

Dey and Balachandran (1976) constructed a class of totally balanced cross-over designs from a series of BIB designs by using a pre-period. Varghese and Sharma (2000) constructed a series of totally balanced cross-over designs for v treatments using v experimental units and (2v-1) periods.

Trend-free cross-over designs: A cross-over design is said to be *trend-free*, if the sum of squares due to treatments (direct as well as residual) under the model considering trend effects besides treatment and unit effects, is same as that obtain under the model considering treatment and unit effects ignoring trend effects. In these designs treatment effects contrasts are estimated orthogonal to trend effects.

5.4.1 Conditions for a Variance Balanced Crossover Design to be Trend-Free

Let a crossover design for v treatments with p periods and n experimental units be represented by COD (v, p, n). We assume that the experimental units exhibit time trend over the periods and the trend effects are represented by orthogonal polynomials of various degrees. Assuming presence of first residual effects of treatments, the additive fixed effect model M_1 for the observations from this design, in matrix notation, can be written as

$$M_1: \quad \mathbf{Y} = \mathbf{X}_1\boldsymbol{\theta}_1 + \mathbf{X}_2\boldsymbol{\theta}_2 + \boldsymbol{\varepsilon} \qquad \dots (5.4.1)$$

where $\mathbf{Y}$ is a (np×1) vector of observations, $\mathbf{X}_1 = [\boldsymbol{\Delta}_1 \quad \boldsymbol{\Delta}_2]$, $\boldsymbol{\Delta}_1$ (np×v) is the design matrix for direct effects, $\boldsymbol{\Delta}_2$ (np×v) is the design matrix for residual effects, $\boldsymbol{\theta}_1$ {= $[\boldsymbol{\tau} \quad \boldsymbol{\rho}]'$} is the coefficient vector of parameters of interest with $\boldsymbol{\tau}$ as vector of direct effects, $\boldsymbol{\rho}$ vector of first residual effects, $\mathbf{X}_2 = [\,\mathbf{S}\ \ \mathbf{Z}\ \ \mathbf{1}\,]$, $\mathbf{S}$ (np×n) is the design matrix for experimental unit effects, $\mathbf{Z}$

($np \times q$; $q \leq p-1$) is the design matrix for trend effects, **1** is the ($np \times 1$) column vector of unities. $\boldsymbol{\theta}_2$ {= [$\boldsymbol{\Psi}$ $\boldsymbol{\alpha}$ μ]′} is the coefficient vector of other factors, namely experimental unit effects $\boldsymbol{\Psi}$, trend effects $\boldsymbol{\alpha}$ and general mean μ, $\boldsymbol{\varepsilon}$ is the ($np \times 1$) error vector ~ N ($\mathbf{0}$, σ^2 $\mathbf{I}_{np \times np}$). The trend effects are assumed to be the same for all the units. The matrix **Z** can be written

$$\mathbf{Z} = \mathbf{1_n} \otimes \boldsymbol{\xi}_{\mathbf{p \times q}}$$

where $\boldsymbol{\xi}_{\mathbf{p \times q}}$ is the $p \times q$ matrix of orthonormal polynomials which implies $\boldsymbol{\xi}'\boldsymbol{\xi} = \mathbf{I_q}$.

We consider another model M_2,

$$M_2: \quad \mathbf{Y} = \mathbf{X}_1\boldsymbol{\theta}_1 + \mathbf{X}_3\boldsymbol{\theta}_3 + \boldsymbol{\varepsilon} \qquad \ldots (5.4.2)$$

where $\mathbf{X}_3$ = [**S** **1**] and $\boldsymbol{\theta}_3$ = [$\boldsymbol{\Psi}$ μ]′

The COD(v, p, n) is said to be trend-free, if the sum of squares due to fitting of $\boldsymbol{\theta}_1$ after eliminating the effect of $\boldsymbol{\theta}_2$, R ($\boldsymbol{\theta}_1$ | $\boldsymbol{\theta}_2$) is the same as the sum of squares due to fitting of $\boldsymbol{\theta}_1$ after eliminating the effect of $\boldsymbol{\theta}_3$, R ($\boldsymbol{\theta}_1$ | $\boldsymbol{\theta}_3$), i.e.

$$R(\boldsymbol{\theta}_1 \mid \boldsymbol{\theta}_2) = R(\boldsymbol{\theta}_1 \mid \boldsymbol{\theta}_3). \qquad \ldots (5.4.3)$$

Under model M_1, the sum of squares due to fitting of $\boldsymbol{\theta}_1$ after eliminating the effect of $\boldsymbol{\theta}_2$ is seen to be

$$R(\boldsymbol{\theta}_1 \mid \boldsymbol{\theta}_2) = \mathbf{Y}' \mathbf{H}_1 \mathbf{X}_1 \mathbf{C}_1^- \mathbf{X}_1' \mathbf{H}_1 \mathbf{Y} \qquad \ldots (5.4.4)$$

where, $\mathbf{H}_1 = \mathbf{I} - \mathbf{X}_2(\mathbf{X}_2'\mathbf{X}_2)^- \mathbf{X}_2'$, $\mathbf{C}_1 = \mathbf{X}_1'\mathbf{H}_1\mathbf{X}_1$ and $\mathbf{C}_1^-$ and $(\mathbf{X}_2'\mathbf{X}_2)^-$ are generalized inverses of $\mathbf{C}_1$ and $(\mathbf{X}_2'\mathbf{X}_2)$ respectively. Similarly, the sum of squares due to fitting of $\boldsymbol{\theta}_1$ after eliminating the effect of $\boldsymbol{\theta}_3$ under model M_2 is seen to be

$$R(\boldsymbol{\theta}_1 \mid \boldsymbol{\theta}_3) = \mathbf{Y}' \mathbf{H}_2 \mathbf{X}_1 \mathbf{C}_2^- \mathbf{X}_1' \mathbf{H}_2 \mathbf{Y} \qquad \ldots (5.4.5)$$

where $\mathbf{H}_2 = \mathbf{I} - \mathbf{X}_3(\mathbf{X}_3'\mathbf{X}_3)^- \mathbf{X}_3'$, $\mathbf{C}_2 = \mathbf{X}_1'\mathbf{H}_2\mathbf{X}_1$ and $\mathbf{C}_2^-$ and $(\mathbf{X}_3'\mathbf{X}_3)^-$ are generalized inverses of $\mathbf{C}_2$ and $(\mathbf{X}_3'\mathbf{X}_3)$ respectively.

In view of (5.4.4) and (5.4.5), condition (5.4.3) becomes

$$\mathbf{Y}' \mathbf{H}_1 \mathbf{X}_1 \mathbf{C}_1^- \mathbf{X}_1' \mathbf{H}_1 \mathbf{Y} = \mathbf{Y}' \mathbf{H}_2 \mathbf{X}_1 \mathbf{C}_2^- \mathbf{X}_1' \mathbf{H}_2 \mathbf{Y}; \text{ for all values of } \mathbf{Y}. \qquad \ldots (5.4.6)$$

which implies

$$\mathbf{H}_1 \mathbf{X}_1 \mathbf{C}_1^- \mathbf{X}_1' \mathbf{H}_1 = \mathbf{H}_2 \mathbf{X}_1 \mathbf{C}_2^- \mathbf{X}_1' \mathbf{H}_2 \qquad \ldots (5.4.7)$$

Pre and post multiplication of both sides of above equation by $\mathbf{X}_1'$ and $\mathbf{X}_1$ respectively, give

$$\mathbf{X}_1'\mathbf{H}_1 \mathbf{X}_1 \mathbf{C}_1^- \mathbf{X}_1' \mathbf{H}_1 \mathbf{X}_1 = \mathbf{X}_1'\mathbf{H}_2 \mathbf{X}_1 \mathbf{C}_2^- \mathbf{X}_1' \mathbf{H}_2 \mathbf{X}_1$$

i.e.,

$$\mathbf{C}_1 = \mathbf{C}_2 \qquad \ldots (5.4.8)$$

since $\mathbf{C}_1 = \mathbf{X}_1'\mathbf{H}_1\mathbf{X}_1$ and $\mathbf{C}_2 = \mathbf{X}_1'\mathbf{H}_2\mathbf{X}_1$ and using $\mathbf{AA^-A=A}$; $\mathbf{A}^-$ being a g-inverse of $\mathbf{A}$. Thus the condition (5.4.8) becomes

$$\mathbf{X}_1' (\mathbf{H}_1 - \mathbf{H}_2)\, \mathbf{X}_1 = \mathbf{0}.$$

which implies

$$\mathbf{X}_1'[\mathbf{X}_2(\mathbf{X}_2'\mathbf{X}_2)^{-}\mathbf{X}_2' - \mathbf{X}_3(\mathbf{X}_3'\mathbf{X}_3)^{-}\mathbf{X}_3']\mathbf{X}_1 = \mathbf{0}. \qquad \ldots (5.4.9)$$

Now for COD (v, p, n), the matrix

$$\mathbf{X}_2'\mathbf{X}_2 = \begin{bmatrix} \mathbf{S'S} & \mathbf{S'Z} & \mathbf{S'1} \\ \mathbf{Z'S} & \mathbf{Z'Z} & \mathbf{Z'1} \\ \mathbf{1'S} & \mathbf{1'Z} & \mathbf{1'1} \end{bmatrix}$$

$$= \begin{bmatrix} \mathrm{p}\,\mathbf{I}_{\mathrm{n}} & \mathbf{0}_{\mathrm{n}\times\mathrm{q}} & \mathrm{p}\,\mathbf{1}_{\mathrm{n}} \\ \mathbf{0}'_{\mathrm{q}\times\mathrm{n}} & \mathrm{n}\,\mathbf{I}_{\mathrm{q}} & \mathbf{0}_{\mathrm{q}\times 1} \\ \mathrm{p}\,\mathbf{1}'_{\mathrm{n}} & \mathbf{0}_{1\times\mathrm{q}} & \mathrm{np} \end{bmatrix}.$$

It can be seen that a generalized inverse of $\mathbf{X}_2'\mathbf{X}_2$ is

$$(\mathbf{X}_2'\mathbf{X}_2)^{-} = \begin{bmatrix} \mathrm{p}^{-1}\mathbf{I}_{\mathrm{n}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathrm{n}^{-1}\mathbf{I}_{\mathrm{q}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

Therefore,

$$\mathbf{X}_2(\mathbf{X}_2'\mathbf{X}_2)^{-}\mathbf{X}_2' = \mathrm{p}^{-1}\,\mathbf{SS}' + \mathrm{n}^{-1}\,\mathbf{ZZ}' \qquad \ldots (5.4.10)$$

Again,

$$\mathbf{X}_3'\mathbf{X}_3 = \begin{bmatrix} \mathbf{S'S} & \mathbf{S'1} \\ \mathbf{1'S} & \mathbf{1'1} \end{bmatrix} = \begin{bmatrix} \mathrm{p}\,\mathbf{I}_{\mathrm{n}} & \mathrm{p}\,\mathbf{1}_{\mathbf{n}} \\ \mathrm{p}\,\mathbf{1}'_{\mathrm{n}} & \mathrm{np} \end{bmatrix}$$

And a g-inverse of $\mathbf{X}_3'\mathbf{X}_3$ is

$$(\mathbf{X}_3'\mathbf{X}_3)^{-} = \begin{bmatrix} \mathrm{p}^{-1}\,\mathbf{I}_{\mathrm{n}} & \mathbf{0} \\ \mathbf{0}' & 0 \end{bmatrix}.$$

It can be easily seen that

$$\mathbf{X}_3(\mathbf{X}_3'\mathbf{X}_3)^{-}\mathbf{X}_3' = \mathrm{p}^{-1}\,\mathbf{SS}' \qquad \ldots (5.4.11)$$

In view of (5.4.10) and (5.4.11), the condition (5.4.9) becomes

$$\mathbf{X}_1'\,\mathbf{Z}\,\mathbf{Z}'\,\mathbf{X}_1 = \mathbf{0}$$

giving

$$\begin{bmatrix} \mathbf{\Delta}_1'\mathbf{ZZ}'\mathbf{\Delta}_1 & \mathbf{\Delta}_1'\mathbf{ZZ}'\mathbf{\Delta}_2 \\ \mathbf{\Delta}_2'\mathbf{ZZ}'\mathbf{\Delta}_1 & \mathbf{\Delta}_2'\mathbf{ZZ}'\mathbf{\Delta}_2 \end{bmatrix} = \mathbf{0}.$$

i.e.,

$$\left.\begin{array}{l} \mathbf{\Delta}_1'\mathbf{Z} = \mathbf{0} \\ \text{and} \\ \mathbf{\Delta}_2'\mathbf{Z} = \mathbf{0} \end{array}\right\} \qquad \ldots (5.4.12)$$

Thus, the COD (v, p, n) is trend-free, if it satisfies the conditions (5.4.12).

In the following two sections, we present methods of constructing CODs that are simultaneously totally balanced as well as trend-free for odd number of treatments and even number of the form 2^s of treatments.

5.4.2 Construction of Totally Balanced Trend-Free CODs for Odd Number of Treatments

Let the number of treatments, v, be odd. Denote these treatments by the symbols 0, 1, 2, …, v-1. From the initial sequences {0, i, 2i}, i = 1, 2, …, (v-1) develop v-1 rectangles in mod(v), each with three rows and v columns. Number the rows of each rectangle as 0, 1, 2 and columns of first, second, …, (v-1)th rectangle in succession with numbers 1 to v(v-1). Now if the rows represent the periods and columns, the experimental units, then v-1 rectangles form a two-period totally balanced trend-free cross-over designs in two periods and v(v-1) experimental units for v treatments; the 0th period representing the pre-period.

As each treatment occurs v-1 times in each of the periods, therefore, $\mathbf{\Delta}_1'\mathbf{Z} = \mathbf{0}$. Similarly, as the residual effect of each treatment occurs (v-1) times in each period, $\mathbf{\Delta}_2'\mathbf{Z} = \mathbf{0}$. Hence the designs are trend-free.

Example 5.4.2.1: Let v = 5. The two-period totally balanced trend-free cross-over design with 20 (= 5×4) experimental units developed from the four bold face initial sequences, is the following:

		Experimental units																			
		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
	0	**0**	1	2	3	4	**0**	1	2	3	4	**0**	1	2	3	4	**0**	1	2	3	4
Pds	1	**1**	2	3	4	0	**2**	3	4	0	1	**3**	4	0	1	2	**4**	0	1	2	3
	2	**2**	3	4	0	1	**4**	0	1	2	3	**1**	2	3	4	0	**3**	4	0	1	2

0th period represents the pre-period.

5.4.3 Construction of Totally Balanced Trend-Free CODs for $v = 2^s$

Let $v = 2^s$, where s is a positive integer ≥ 2. Construct a set of v – 1 mutually orthogonal Latin-squares of order v. Form a rectangle consisting of any three consecutive rows from all the Latin-squares of this set each row having v (v-1) treatment symbols. Number the rows as 0, 1, 2 and columns as 1, 2, 3, …, v(v-1). If the rows represent the periods (0^{th} row representing the pre-period), and columns, the experimental units, then the rectangle form a two-period totally balanced trend-free cross-over design with v(v-1) experimental units.

As the direct effect and residual effect of each treatment occur v-1 times in each period, the conditions given in (5.4.12) are satisfied. Thus, the design obtained by the above method is also trend-free.

Example 5.4.3.1: Let v = 4 (= 2^2). The two-period totally balanced trend-free crossover design formed from first three rows of the set of mutually orthogonal Latin-squares given in Fisher and Yates (1963, p. no. 88-89) is the following:

		Experimental Units											
		1	2	3	4	5	6	7	8	9	10	11	12
Periods	0	0	1	2	3	0	1	2	3	0	1	2	3
	1	1	0	3	2	2	3	0	1	3	2	1	0
	2	2	3	0	1	3	2	1	0	1	0	3	2

5.4 Designs Balanced for Spatial and Temporal Indirect Effects

There may exist experimental situation where both spatial as well as temporal effects are suspected. Designs which are balanced with respect to both spatial and temporal indirect effects will be useful for such situations. A series of experimental designs balanced for spatial and temporal indirect effects of treatments has been constructed and named as Neighbour balanced crossover designs.

Method 5.4.1: Construct an initial row of size v as follows:

1	v	2	v-1	3	v-2	...	v/2	(v/2)+1

Develop the columns by proceeding from top to bottom filling only odd numbered cells by adding 1 to the previous odd numbered cell and taking modulo v. The array obtained is as given below.

1	v	2	v-1	3	v-2	…	v/2	(v/2)+1
2	1	3	v	4	v-1	…	(v/2)+1	(v/2)+2
. .			. .					. .

.			.					.
(v/2)+1	(v/2)	(v/2)+2	(v/2)-2	(v/2)+3	(v/2)-3	…	(v/2)+2	v-1
(v/2)	(v/2)-1	(v/2)+1	(v/2)-2	(v/2)+2	(v/2)-3	…	v-1	v

The empty positions in the even numbered cells are filled in the reverse direction (from bottom to top) by adding 1 to the previous cell and taking modulo v.

1	v	2	v-1	3	v-2	…	v/2	(v/2)+1
V	v-1	1	v-2	2	v-3	…	(v/2)-1	(v/2)
2	1	3	v	4	v-1	…	(v/2)+1	(v/2)+2
v-1	v-2	v	v-3	1	v-4	…	(v/2)-2	(v/2)-1
. . .			. . .					. . .
(v/2)+2	(v/2)+1	(v/2)+3	(v/2)	(v/2)+4	(v/2)-1	…	1	2
(v/2)	(v/2)-1	(v/2)+1	(v/2)-2	(v/2)+2	(v/2)-3	…	v-1	v
(v/2)+1	(v/2)	(v/2)+2	(v/2)-1	(v/2)+3	(v/2)-2	…	v	1

The design obtained so is neighbour balanced crossover design with parameters v treatments = p periods = n units/sequences and μ = 1.

Example 5.4.1.1: For v = 6, following is a neighbour balanced crossover design in r rows, 6 columns and μ = 1:

1	6	2	5	3	4
6	5	1	4	2	3
2	1	3	6	4	5
5	4	6	3	1	2
3	2	4	1	5	6
4	3	5	2	6	1

Chapter VI

WEB GENERATION OF EXPERIMENTAL DESIGNS BALANCED FOR INDIRECT EFFECTS OF TREATMENTS (WEB-DBIE)

6.1 Introduction

It is seen from the previous chapters that Neighbour Balanced Designs (NBDs) are used for the situations when spatial indirect effects are suspected from the treatments applied in the neighbouring experimental units whereas Crossover Designs are used when temporal indirect effects consisting of residual or carryover effects from the treatments applied in the previous period are present. These designs ensure that each treatment occurs adjacent to every other treatment spatially or temporarily same number of times.

A large number of NBDs and Crossover Designs are developed in the literature. For details of NBDs, one can refer to Azais *et al.* (1993), Monod and Bailey (1993), Azais and Druilhet (1997), Azais *et al.* (1998), Bailey (2003), Bailey and Druilhet (2004), Tomar *et al*. (2005), Jaggi *et al.* (2006), Jaggi *et al.* (2007) and Pateria *et al.* (2007). For details of crossover designs, one can refer to Williams (1949), Patterson and Lucas (1962), Balaam (1968), Sharma (1975), Dey and Balachandran (1976), Sharma (1981), Sharma (1982), Afsarinejad (1990), Varghese and Sharma (2000), Sharma *et al.* (2002) and Sharma *et al*. (2003) and Bose and Dey (2009).

For easy accessibility of these designs by the experimenters, it is required that these designs are compiled and presented at one place. A software solution for the generation of these designs, like the one for partially balanced incomplete block designs developed by Sharma *et al.* (2013), is required. A web-enabled software *Web Generation of Experimental Designs Balanced for Indirect Effects of Treatments (WEB-DBIE)* has been developed and deployed at www.iasri.res.in/webdbie which generates useful classes of designs in the presence of indirect effects of treatments. An online catalogue of these designs incorporating indirect effects of treatments is also developed. The web solution developed for generation of NBDs and Crossover designs is described below along with an online catalogue of the designs within a permissible range.

6.2 Architecture of WEB-DBIE

The software contains three main components namely user interface management, input data management and statistical engine for generation of NBD and Crossover designs. Any communication to software from users is handled through user interface at client side and input data handling is done by data management module. Statistical engine is implemented at server side and it contains the various procedures required for generation. User interface has been separated from the statistical engine to free software developers from interface problem. The user interface management has been developed using HTML, CCS and javascript (Yehuda and Tomer, 1998). Input data management component has been developed using ASP.NET and C# (Ulman *et al*., 2002). Web generation engine has been developed using C# language. This engine contains the Dynamic Link Libraries (DLL) for generation and randomization of designs.

WEB-DBIE has been developed for web platform and programming has been done with the ASP.NET and C#.NET programming language. C# provides a complete set of tools for creation of rapid and powerful graphical user interface (GUI) based web applications. Microsoft Visual Studio 2010 integrative development environment has been used as a platform for development of the software. Fig. 6.2.1 shows the architecture of the software.

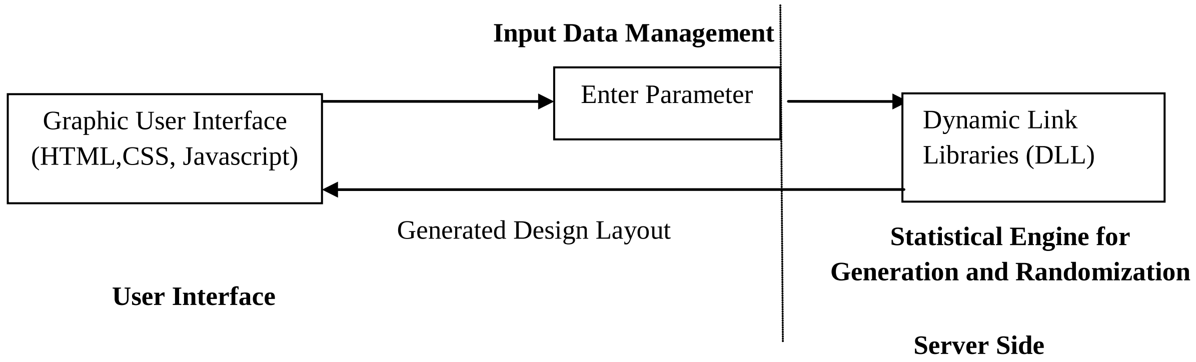


Fig. 6.2.1: Architecture of WEB-DBIE

6.3 Software Process Model and Design

Software process model is the framework that describes the activities to be performed at each stage of the software development. Since most of the requirements were understood in advance, waterfall model was used for development of WEB-DBIE. Software designing identified four major modules namely generation of various classes of NBDs and Crossover designs, catalogue of NBDs and Crossover designs, about NBDs and Crossover designs and user management.

6.4 WEB-DBIE Description

This software is available at www.iasri.res.in/webdbie. The hierarchical structure chart for the design of the software is shown in Fig. 6.4.1.

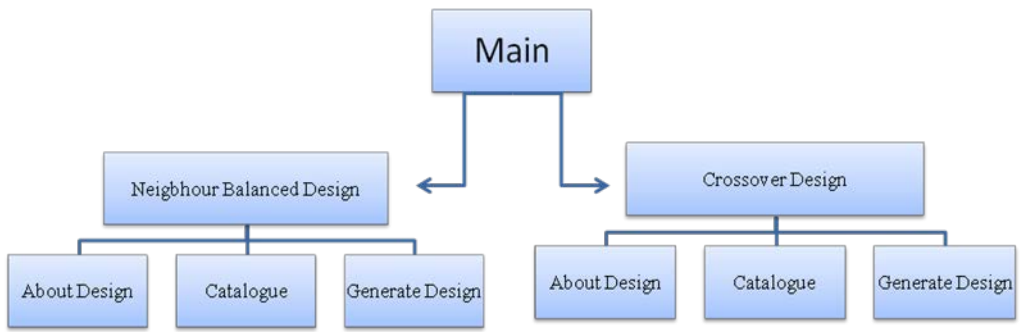


Fig. 6.4.1: Design of WEB-DBIE

The site requires the creation of user profile before accessing it (Fig. 6.4.2). After entering a correct user id and password, a web page appears with the links for generation of NBB and Crossover designs (Fig. 6.4.3).

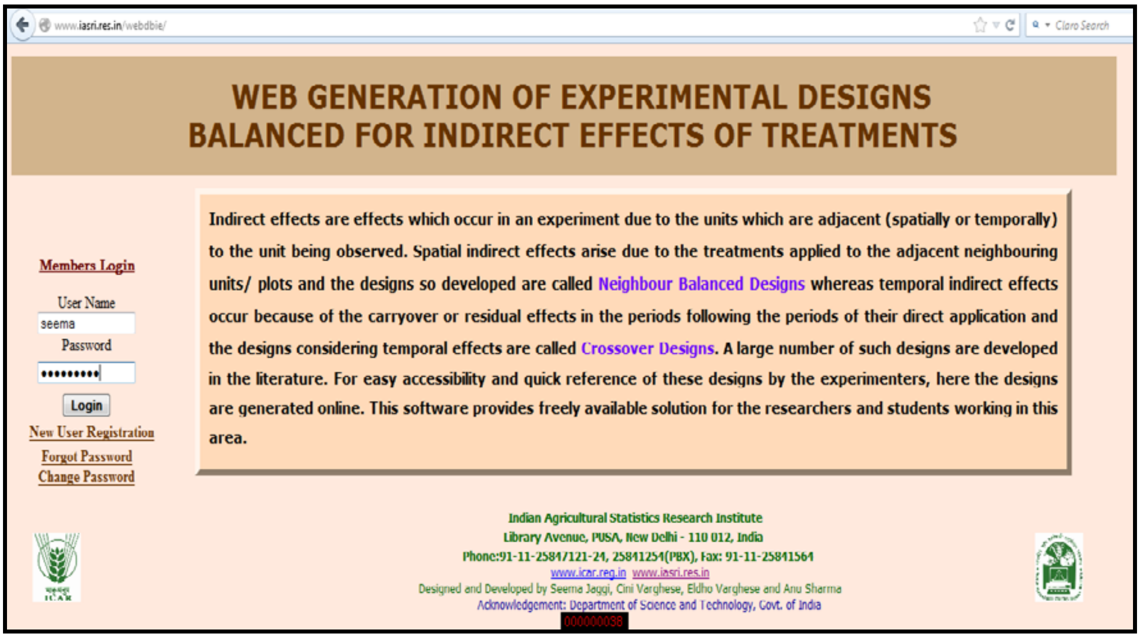


Fig. 6.4.2: Login Page

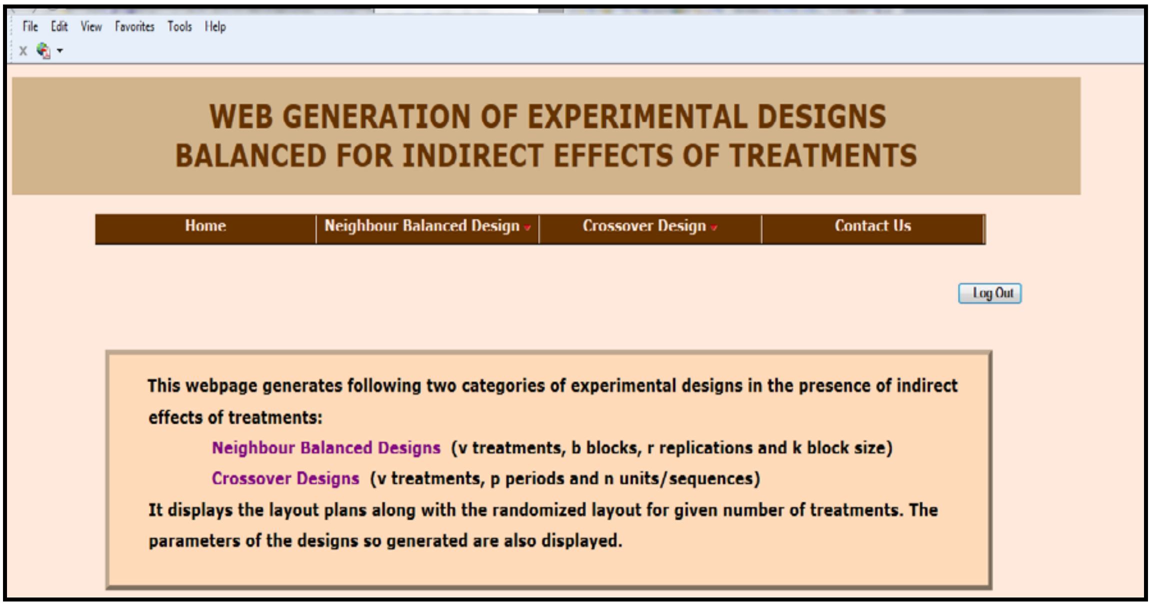


Fig. 6.4.3: Home Page of Software

6.5 Generation and Randomization of NBD and Crossover Designs

WEB-DBIE generates design and randomized layout for various classes of NBD (Fig. 6.5.1) and Crossover designs (Fig. 6.5.2). It generates totally balanced/ partially balanced, complete/ incomplete NBD (v treatments, b blocks, r replications and k block size).

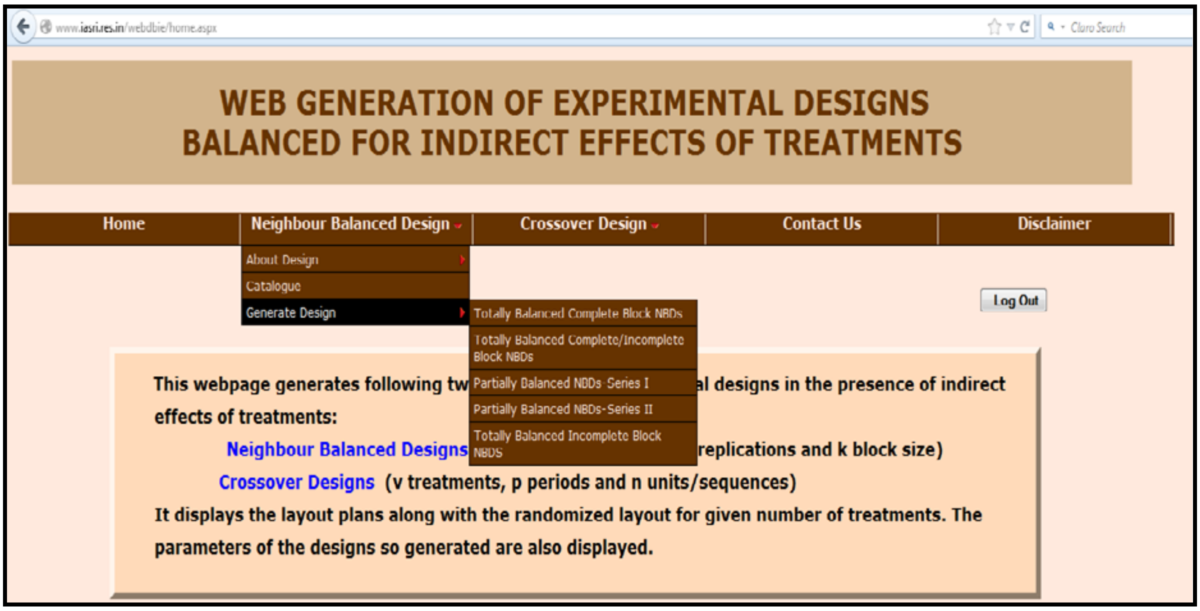


Fig. 6.5.1: Neighbour Balanced Designs

The classes of crossover designs (v treatments, p periods and n units/sequences) generated are Williams square, extra period Williams square, two-period designs, minimal balanced, strongly balanced, totally balanced designs in complete/ incomplete sequence, designs with complete/ incomplete sequence using MOLS.

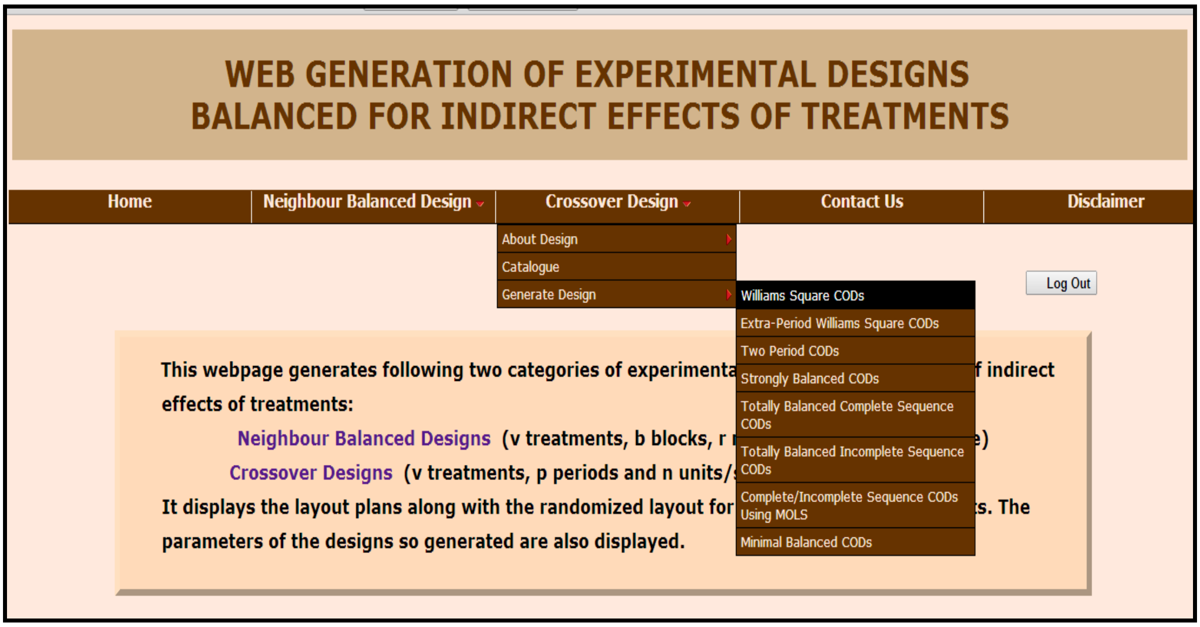


Fig. 6.5.2: Crossover Designs

Various web forms have been designed and developed for generation and randomization of the above listed designs. User can enter the number of treatments (Fig. 6.5.3) and then click "Generate Design" to see the design layout. The parameters of the designs so generated are also displayed (Fig. 6.5.4 and Fig. 6.5.5).

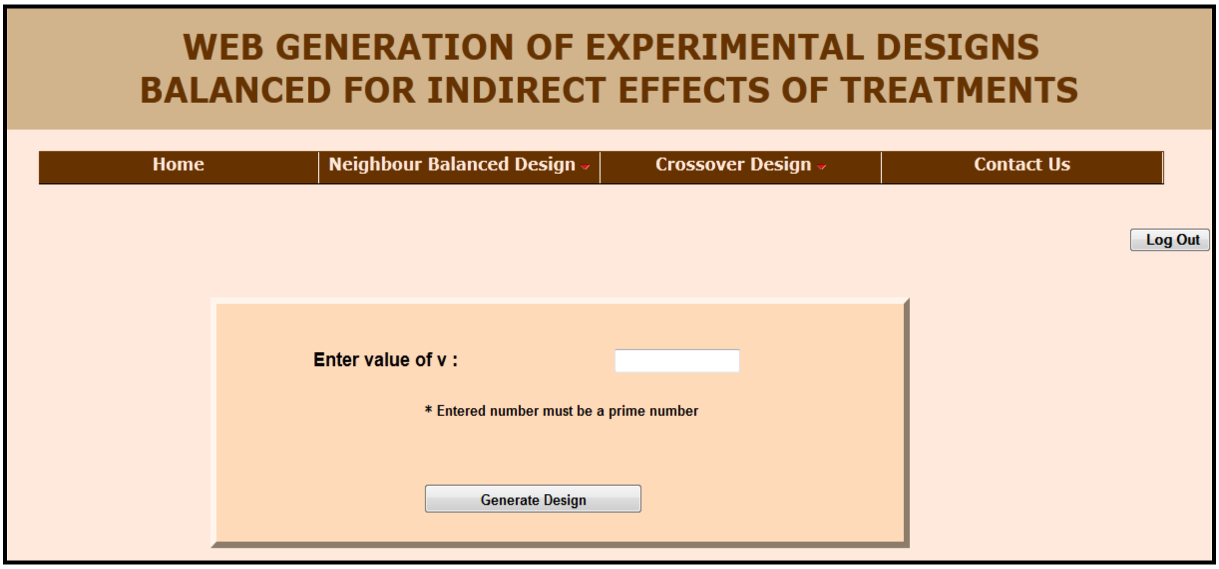
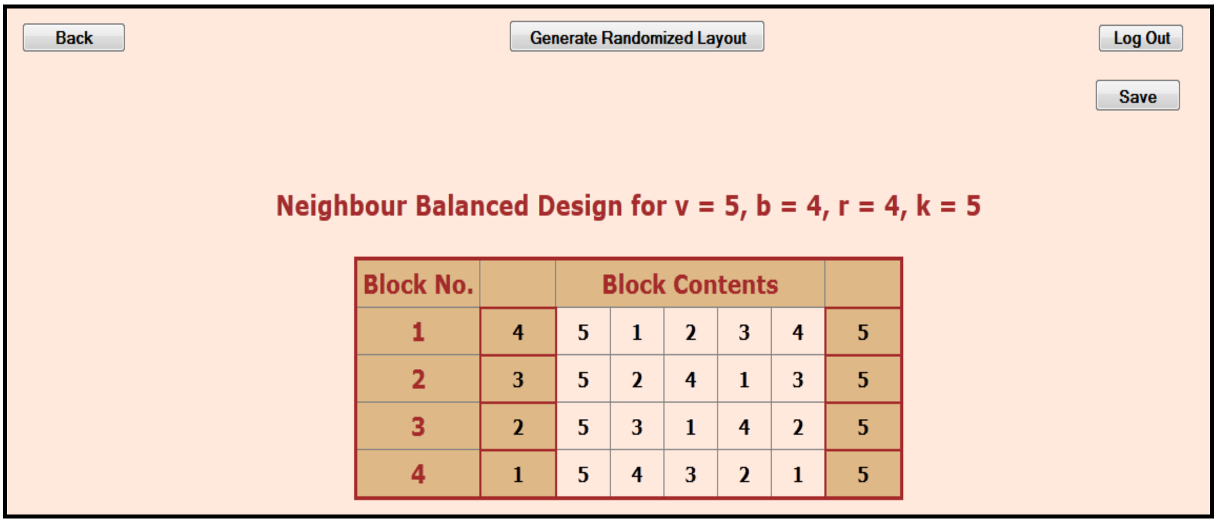
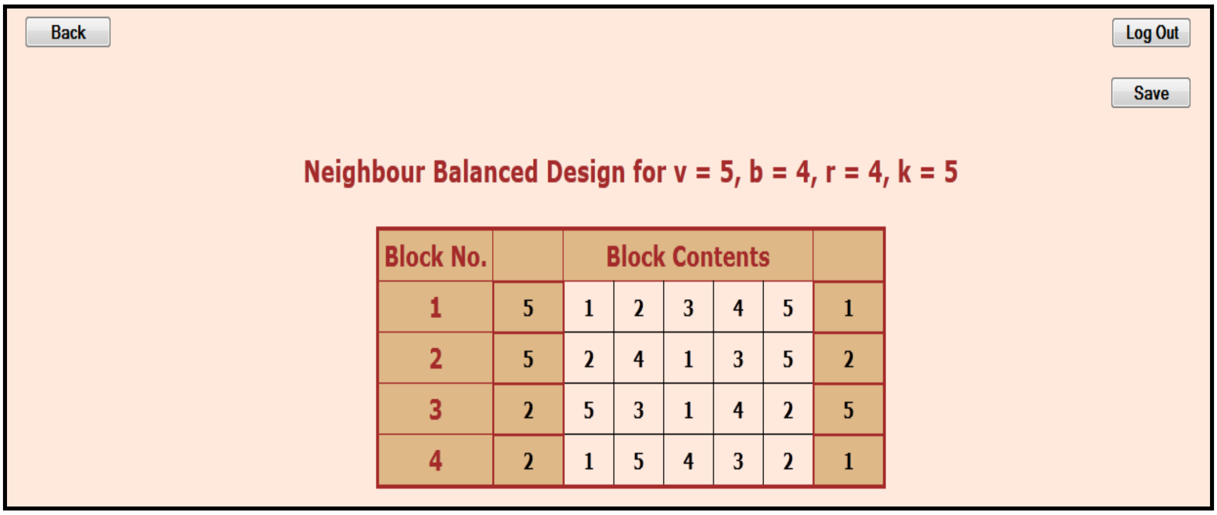


Fig. 6.5.3: Generation of NBD



Block No.		Block Contents					
1	4	5	1	2	3	4	5
2	3	5	2	4	1	3	5
3	2	5	3	1	4	2	5
4	1	5	4	3	2	1	5

Fig. 6.5.4: NBD for v = 5



Block No.		Block Contents					
1	5	1	2	3	4	5	1
2	5	2	4	1	3	5	2
3	2	5	3	1	4	2	5
4	2	1	5	4	3	2	1

Fig. 6.5.5: Randomized Layout of Design

Output can be exported to MS-Excel spread sheet for further use (Fig. 6.5.6).

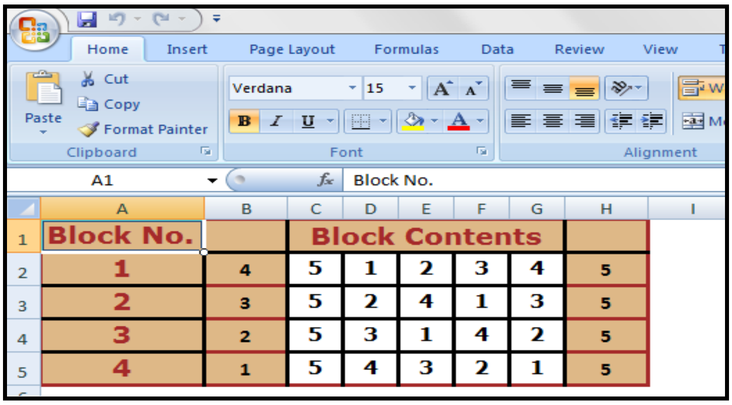


Fig. 6.5.6: Saving in MS-Excel

Online catalogue for v ≤ 20 of NBDs and Crossover Designs has been developed and is included in the software. Search facility of all designs and designs for some particular value of parameters has been provided along with showing the layout of the design. Fig. 6.5.7 and Fig. 6.5.8 shows the catalogue and generation of NBDs.

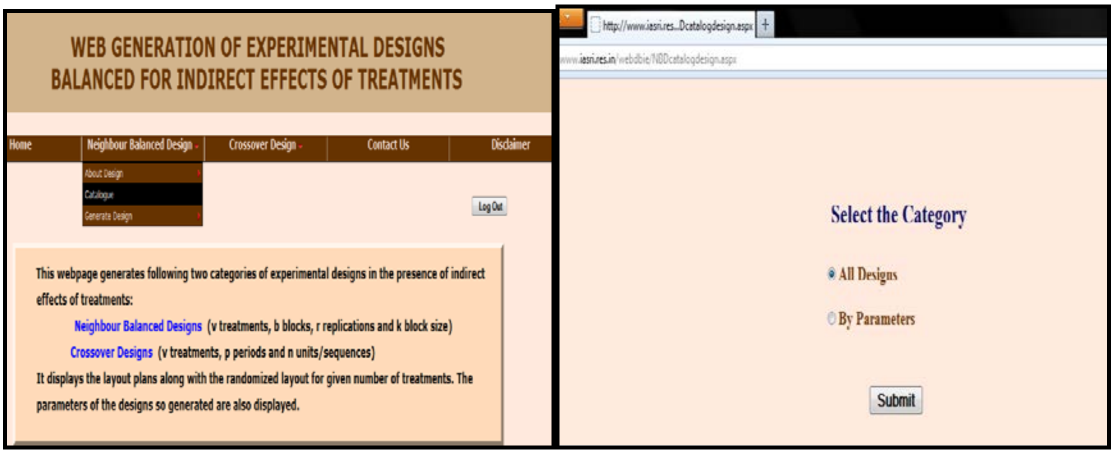


Fig. 6.5.7: Catalogue of NBDs

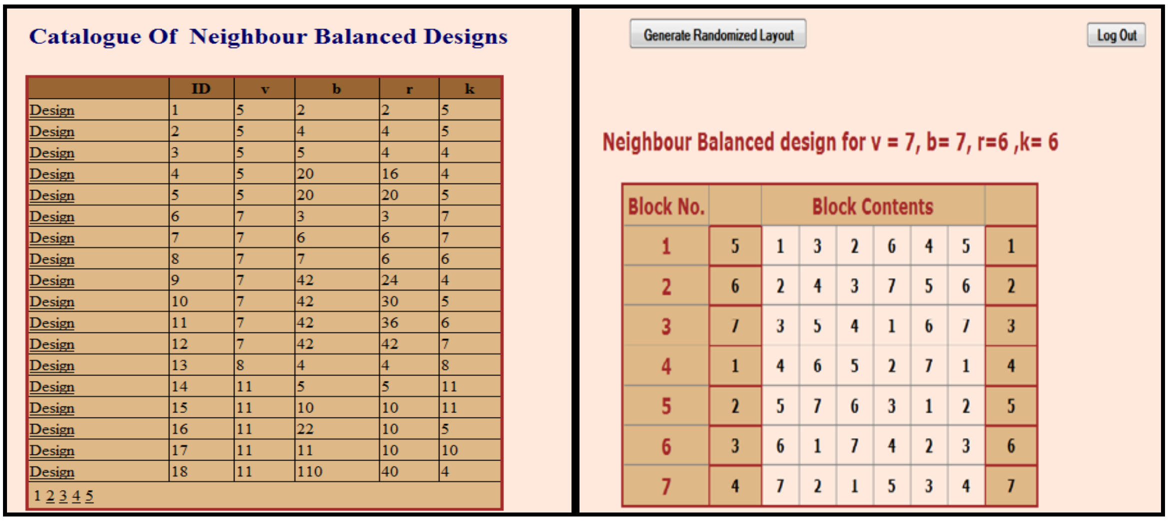
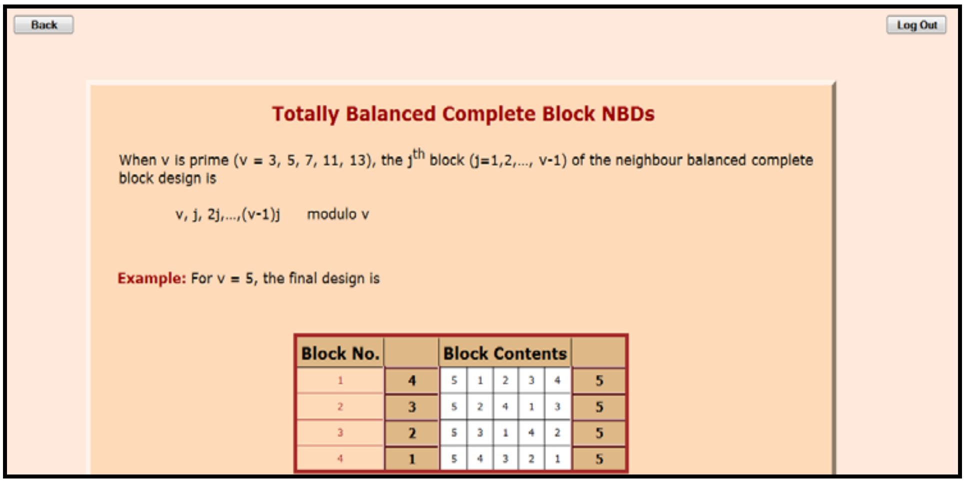


Fig. 6.5.8: Catalogue and Generation of NBDs

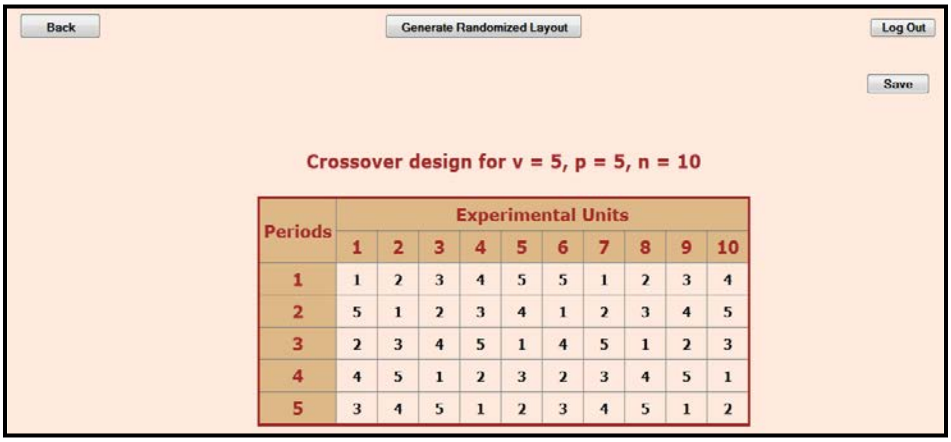
Fig. 6.5.9 displays the details of one series of NBDs.



Block No.		Block Contents					
1	4	5	1	2	3	4	5
2	3	5	2	4	1	3	5
3	2	5	3	1	4	2	5
4	1	5	4	3	2	1	5

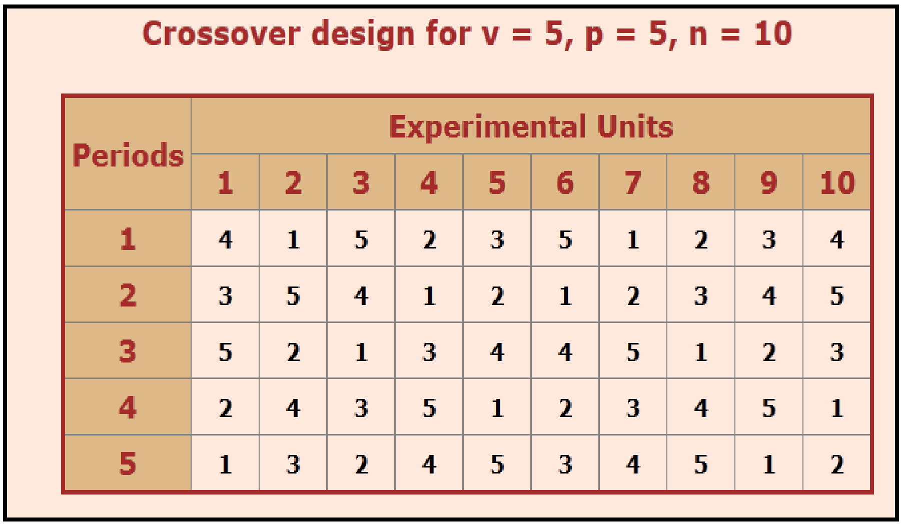
Fig. 6.5.9: About NBDs

Fig. 6.5.10 and Fig. 6.5.11 shows the snapshots of a crossover design for v = 5, p = 5 and n = 10 and its randomized layout.



Periods	Experimental Units									
	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	5	1	2	3	4
2	5	1	2	3	4	1	2	3	4	5
3	2	3	4	5	1	4	5	1	2	3
4	4	5	1	2	3	2	3	4	5	1
5	3	4	5	1	2	3	4	5	1	2

Fig. 6.5.10: Crossover (Williams Square) Design for v = 5



Periods	Experimental Units									
	1	2	3	4	5	6	7	8	9	10
1	4	1	5	2	3	5	1	2	3	4
2	3	5	4	1	2	1	2	3	4	5
3	5	2	1	3	4	4	5	1	2	3
4	2	4	3	5	1	2	3	4	5	1
5	1	3	2	4	5	3	4	5	1	2

Fig. 6.5.11: Randomized Layout

Fig. 6.5.12 and Fig. 6.5.13 shows the catalogue and generation of Crossover designs.

Catalogue Of Crossover Designs

	ID	v	p	n
Design	1	3	2	9
Design	2	3	2	6
Design	3	3	3	6
Design	4	3	3	6
Design	5	3	4	6
Design	6	3	5	3
Design	7	3	6	3
Design	8	4	2	16
Design	9	4	4	4
Design	10	4	5	4
Design	11	4	7	4
Design	12	4	8	4
Design	13	5	2	25
Design	14	5	2	10
Design	15	5	3	20
Design	16	5	3	10
Design	17	5	4	20
Design	18	5	5	10

1 2 3 4 5 6 7 8 9 10

Fig. 6.5.12: Catalogue of Crossover Designs

Catalogue By Parameters

Select the Parameter Period (p)

Select Operator equal

Value 5

Submit

Total Number of Records Found for p = 5is 11

	ID	v	p	n
Design	6	3	5	3
Design	10	4	5	4
Design	18	5	5	10
Design	19	5	5	20
Design	33	7	5	42
Design	46	9	5	18
Design	59	11	5	22
Design	60	11	5	110
Design	80	13	5	156
Design	114	17	5	272
Design	141	19	5	342

Fig. 6.5.13: Catalogue of Crossover Designs: Select by Parameters

SUMMARY

Indirect effects are very common in agricultural experiments. For a given experiment, the experimental layout and the treatment being tested primarily determines the significance of any indirect or interference effects. In varietal trials, the varieties of a given crop are planted in adjacent plots. As varieties generally differ in their ability to compete, plants in a plot will be subjected to different environmental conditions depending upon location relative to adjacent plots. In fertilizer trials, the plants in an unfertilized plot may affect the share of the plants in a nearby heavily fertilized plot. To avoid the bias while comparing the effects of treatments in such situation, it is important to ensure that no treatment is unduly disadvantaged by its neighbour. Understanding the structure of these effects helps in minimizing the bias in treatments to great extent. Hence, for proper model specification, indirect effects resulting in interference effects from neighbouring units must be incorporated into the model where situations demand.

Methodologies pertaining to designing of experiments for different experimental situations in the presence of spatial indirect effects from neighbouring units under block design setup have been developed in Chapter II. Spatial effects from the neighbouring units are very common phenomenon in field experiments. Further, in such experiments, interference effects may arise not only from immediate neighbouring units but also from the units at distance two, three or at higher distance. Under block design setup, some aspects of interference effects from the neighbouring units up to distance 2 or second order interference effects have been discussed. Various methods of constructing balanced or strongly balanced complete/incomplete block design with interference effects up to distance 2 have been discussed. All the designs so obtained are totally balanced for estimating the contrasts pertaining to direct effects of treatments and interference effects up to distance 2 arising from the neighbouring units. In some cases, the information matrix for estimating direct effects and interference effects of treatments from the neighbouring units have been generalized for higher order interference effects. Optimality aspects with interference effects from the left neighbouring units up to distance 2 have been discussed. Series of complete block designs with interference effects up to distance 2 have been identified to be universally optimal under two different situations. If experimenters have the reason to believe that interference effects from the neighbouring units can affect the response of a particular plot of a block, the identified series of designs can be effectively laid out in field experiments under block design setup as these designs satisfy the important statistical property of being universally optimal.

Analytical procedure of agricultural experiments under block design setup when neighbour effects also affects the response of a particular plot of a block has been explained. Here, neighbour effects up to distance 2 (second order) have been considered. The method of analysis has been illustrated using hypothetical data for 11 genotypes. The result shows the significance of incorporating all the neighbour effects up to distance 2 in the analysis.

Most of the work on optimality aspects of block designs incorporating interference effects from the neighbouring units have been carried out under the usual block model with interference effects from the neighbouring units i.e. by using additive model. In the present work, the emphasis is on the optimal estimation of direct effects of treatments in block design setup involving interference effects from the neighbouring units under the non-additive model. A class of designs has been identified as universally optimal for the estimation of direct effects of treatments under the non-additive model. Interestingly, the class of designs also resulted to be universally optimal for estimation of interference effects from both the left and right neighbouring units.

Non-additivity under a block model with block effects, direct effects of treatment, interference effects of treatment from the immediate left neighbouring units and left interference × direct effects has been discussed. A class of complete circular block designs balanced for one-sided interference effects from the neighbouring units has been shown to be universally optimal for the estimation of both direct effects and interference effects among the class of all competing designs under the postulated model. Under this class of design, two series of designs have been obtained out of which the series of designs with larger experimental units are also shown to be universally optimal for the estimation of left interference × direct treatment interaction effects under the non-additive model. Further, direct treatment × block effects non-additivity under block design setup with interference effects from the left neighbouring units has also been explored.

Interference effects may arise from the immediate neighbouring units or it may extend further. But, when there is slope or while dealing with undulating land in hilly areas, this may cause a trend in experimental units. To overcome such situations, trend free block designs balanced for interference effects from the immediate neighbouring units and also from the neighbouring units at distance 2 have been obtained and is given in Chapter III. The designs so obtained are totally balanced for estimating direct and interference effects of treatments and are capable of completely eliminating the effects of a common trend. Thus, the analysis can be done in the usual manner as if no trend effect is present in the experimental material.

Row-column designs incorporating directional neighbour effects have also been studied in Chapter IV. A row-column design is said to be neighbour balanced if every treatment has all other treatments appearing as a neighbour a constant number of times. Three different situations under row-column setup incorporating neighbour effects viz., row-column design with one-sided neighbour effect, two-sided neighbour effect and four-sided neighbour effect have been considered. The information matrices for all the situations for estimating the direct and neighbour effects of treatments have been derived. Methods of constructing neighbour balanced row-column designs have been developed and its characterization properties have been studied. A class of row-column designs which are minimally balanced for spatial indirect has also been obtained.

In Chapter V, the designs with temporal indirect effects have been studied and a class of designs which are balanced for temporal indirect is also provided. Series of designs involving varying temporal environments under factorial treatment structure balanced for one factor have been obtained. These designs are uniform, combinatorially balanced and in terms of variance of estimated contrasts pertaining to direct as well as residual effects of treatment combinations, partially variance balanced following rectangular association scheme. These designs find application in experiments involving more than one factor applied sequentially under different environmental conditions. Conditions for a balanced CODs that allow estimation of treatment effects contrasts orthogonal to trend effects have been obtained. Methods of constructing trend-free balanced CODs for odd as well as for even number of treatments have been given. Besides, the concept of experimental designs in the presence of spatial and temporal indirect effects of treatments has been studied. A series of experimental designs balanced for spatial and temporal indirect effects of treatments has been constructed and named as Neighbour balanced crossover designs.

WEB-DBIE is a web based solution for generation of different classes of Neighbour Balanced Designs and Crossover Designs along with details of these designs and given in Chapter VI. It displays the randomized layout plans for given number of treatments. Online catalogue for selection of an appropriate design with given parameters is also provided. The software is menu driven and provides user-friendly interface for its easy operability. Availability of the WEB-DBIE software, a purpose oriented and user-friendly software for agricultural and allied sciences, will encourage the researchers to conduct experiments using appropriate designs.

LIST OF PUBLICATIONS

Research Papers

Seema Jaggi, Cini Varghese, Eldho Varghese and Anu Sharma. (2015). Web generation of experimental designs balanced for indirect effects of treatments (WEB-DBIE). *Computers and Electronics in Agriculture.* 111, 62–68.

Eldho Varghese, Seema Jaggi and Cini Varghese (2014). Neighbor-Balanced Row-column Designs, *Communications in Statistics - Theory and Methods*, 43 (6), 1261-1276.

Cini Varghese, Eldho Varghese, Seema Jaggi and Arpan Bhowmik. Experimental designs for open-pollination in polycross trials. *Journal of Applied Statistics.* DOI:10.1080/02664763.2015.1043860

Arpan Bhowmik, Seema Jaggi, Cini Varghese and Eldho Varghese (2015). Optimal block designs with interference effects from neighbouring units under a non-additive model. *Communication in Statistics: Theory and Methods*, 44(10), 2092-2103.

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Popular Articles

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vi.kZ HkkSfed] ,Ynks oxhZl] lhek tXxh] fluh oxhZl] ch-ts- xgykSr ,oa fot; fcUny (2015)A mipkjksa ds iHkko dk Lkgh vkdyu gsrq mPp Øe ucj lUrqfyr pØh; Cykd vfHkdYiuk,a (Higher order neighbour balanced circular block designs for proper estimation of treatment effects)A *Hkkjrh; Ñf"k vuqla/kku if=dk* A 30(3), 155-158

Pamphlet

Seema Jaggi, Cini Varghese, Eldho Varghese and Anu Sharma (2014). Web generation of experimental designs balanced for indirect effects of treatments. I.A.S.R.I./B.-01/2014.

Poster Presentation

Following paper was adjudged **First** in the **Hindi Research Paper Poster Presentation** organized at IASRI (10-09-2013) during Hindi Pakhwada celebration:

vi[.k Hkkfed] ,Ynks oxhZl] lhek tXxh] fluh oxhZl ,oa ch- t- xgykr A fudVorhZ bdkb;ksa dk ÁHkkoksa dks lfEefyRk djrs gq, —f"k ijh{k.kksa dk fy, LkUrqfyRk vfHkdYiukvksa dk fuekZ.k

Papers Presented in Conferences

Seema Jaggi, Cini Varghese, Eldho Varghese and Anu Sharma. Web generation of experimental designs balanced for indirect effects of treatments. Paper presented in the International conference on Statistics and Informatics in Agricultural Research organized by the Indian Society of Agricultural Statistics which was held during 18-20 December, 2012 at IASRI.

Eldho Varghese, Seema Jaggi and Cini Varghese. Row-column designs balanced for neighbour effects. Paper presented in the International conference on Statistics and Informatics in Agricultural Research organized by the Indian Society of Agricultural Statistics which was held during 18-20 December, 2012 at IASRI.

Arpan Bowmik, Seema Jaggi, Cini Varghese and Eldho Varghese. Trend resistant second order neighbour balanced block designs. Paper presented in the International conference on Statistics and Informatics in Agricultural Research organized by the Indian Society of Agricultural Statistics which was held during 18-20 December, 2012 at IASRI.

Arpan Bowmik, Seema Jaggi, Cini Varghese and Eldho Varghese (2012). Trend free block design balanced for interference effects. Paper presented in the 11th biennial conference of the International Biometric Society (IBS-IR) which was held during 8-9 March, 2012 hosted by Dept. of statistics, University of Pondicherry.

Arpan Bhowmik, Seema Jaggi, Eldho Varghese and Cini Varghese. Optimal block designs under a non-additive mixed effects interference model. In the Dr. G.R. Seth memorial young scientist award session. Presented in the 67th Annual Conference of Indian Society of Agricultural Statistics organized at Institute of Agricultural Sciences, BHU, Varanasi-221005 during December 18-20, 2013.

Cini Varghese, Eldho Varghese, Seema Jaggi and Arpan Bhowmik. Experimental designs for open-pollination in polycross trials. Paper presented in the poster presentation session in Agriculture and Forestry Sciences section of 102nd Indian Science Congress organized at University of Mumbai during 3-7 January, 2014.

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