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Block designs for comparing test treatments with control treatments- an overview

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Abstract

The problem of finding efficient block designs for comparing several test treatments with one or more standard (control) treatments has received considerable attention in the recent past. This paper presents a review of certain recent results in this area.

Key words: Reinforced block designs; supplemented balanced designs; augmented designs; BBPBUB designs of Type G; GDBPUB designs of Type G; A-optimality.

1 Introduction

Designed experiments are generally conducted for making all possible paired comparisons among the treatments. However, there are situations when the interest of the experimenter is only in a subset of these paired comparisons. For example, in genetic resources environment, an essential activity is to test or evaluate the new germ plasm / provenance / superior selections (test treatments), etc. with the existing provenance or released varieties (control treatments). Similar situations may also occur in other disciplines of agricultural sciences, industry, etc. The problem here is to design an experiment for the estimation of the test treatments versus control treatments contrasts and the comparisons among the test treatments or among the control treatments are of lesser consequence.

It is well known that for the estimation of elementary contrasts among test and control treatments, conventional designs like balanced incomplete block (BIB) designs are not the best (see Cox, 1958; Federer, 1956; Sinha, 1980, 1982). For these experimental situations, therefore, the problem of choosing an efficient design is important and needs attention. For an earlier review on the designing problem for this experimental setting, see Hedayat, Jacroux and Majumdar (1988) and Majumdar (1996). This paper also addresses the same problem. We begin with the mathematical formulation of the problem.

2 Problem formulation

We have $w+u=v$ treatments divided into two disjoint sets, H and G of respective cardinality w and u. The w test treatments in H are denoted by $1,2,\ldots,w$ and the u control treatments in G are denoted by $w+1, w+2,\ldots,v$. The problem here is to

design an experiment for estimating wu test treatments versus control treatments contrasts with as high a precision as possible. Suppose that n experimental units available for experimentation can be arranged in b blocks of sizes $k_1, k_2, \ldots, k_b$, respectively, $k_1 + k_2 + \cdots + k_b = n$. Let y_{tjl} denote the observation of test or control treatment $t(t = 1, \ldots, v)$ on experimental unit $l(l = 1, \ldots, n_{tj})$ of block $j(j = 1, \ldots, b)$. Assume the usual fixed effects, additive linear model

$$y_{tjl} = \mu + \tau_t + \beta_j + \varepsilon_{tjl} \tag{2.1}$$

where μ is the overall mean, τ_t is the effect of test or control treatment t, β_j is the effect of block j and ε_{tjl} are the random errors normally distributed with zero mean and variance $\sigma^2 k_j^{\alpha}$. Note that homoscedasticity is not assumed. Here $\alpha \geq 0$ is a scalar constant, generally unknown and n_{tj} is the number of replications of test or control treatment t in block j. The $v \times b$ incidence matrix N has elements n_{tj}. For $\alpha = 0$ we get the usual homoscedastic model.

The contrasts of interest are $\tau_g - \tau_h, g \in G, h \in H$. Comparisons of treatments within G and within H are of secondary importance. The contrasts of major interest may be written in matrix notation as $P'\tau$, where the $wu \times v$ matrix P may be expressed as $P = [1_u \otimes I_w \vdots - I_u \otimes 1_w]$ with $P'1_v = 0$. Here 1_t, is a t-component vector with all elements one, I_t is an identity matrix of order t, $\otimes$ denotes the Kronecker product of matrices and τ is a v-component vector of test or control treatment effects.

Let $C = (c_{tt'}) = R - NK^{-1}N'$ be the usual $v \times v$ C-matrix of a block design with v treatments. Here $R = diag(r_1, r_2, \ldots, r_v)$ denotes a diagonal matrix of replication numbers of treatments and $K = diag\ (k_1, k_2, \ldots, k_b)$ denotes a diagonal matrix of block sizes. Partition N as $N = [N_1' \vdots N_2']'$, where $N_1 = ((n_{hj}))$ is a $w \times b$ incidence matrix of test treatments and $N_2 = (n_{gj})$ is a $u \times b$ incidence matrix of control treatments. Similarly, $R_1 = diag(r_1, r_2, \ldots, r_w)$ and $R_2 = diag(r_{w+1}, r_{w+2}, \ldots, r_v)$, denote respectively the diagonal matrices of replications of the test treatments and the control treatments. The information matrix C can then be partitioned as

$$C = \begin{bmatrix} A & B \\ B' & D \end{bmatrix} \tag{2.2}$$

where $A = \sum_{j=1}^{b} k_j^{-\alpha}[R_{1j} - k_j^{-1}N_{1j}N_{1j}'], B = -\sum_{j=1}^{b} k_j^{-\alpha-1}N_{1j}N_{2j}'$, and $D = \sum_{j=1}^{b} k_j^{-\alpha}[R_{2j} - k_j^{-1}N_{2j}N_{2j}']$. Also $R_{1j} = diag(n_{ij}, \ldots, n_{hj}, \ldots, n_{wj}), R_{2j} = diag(n_{(w+1)_j}, \ldots, n_{gj}, \ldots, n_{vj}), N_1 = [N_{11} \vdots \cdots \vdots N_{1j} \vdots \cdots \vdots N_{1b}], N_2 = [N_{21} \vdots \cdots \vdots N_{2j} \vdots \cdots \vdots N_{2b}], N_{1j} = (n_{1j}, n_{2j}, \ldots, n_{wj})'$ and $N_{2j} = (n_{(w+1)j}, n_{(w+2)j}, \ldots, n_{vj})'$. For $\alpha = 0$, $A = R_1 - N_1K^{-1}N_1'$, $B = -N_1K^{-1}N_2'$ and $D = R_2 - N_2K^{-1}N_2'$.

Let $\hat{\tau}_g - \hat{\tau}_h$ denote the best linear unbiased estimator (BLUE) of $\tau_g - \tau_h$, $g \in G, h \in H$. The BLUE of contrasts of interest $P'\tau$, is $P'\hat{\tau}$ with dispersion matrix $Cov(P'\hat{\tau}) = \sigma^2 P'C^{-}P$. Here C^{-} is a generalized inverse of C, i.e., $CC^{-}C = C$.

It is assumed that the design is connected and Rank $(C) = v - 1$. It might also be desirable that the comparisons of interest are estimated through the design with

the same variance. A design is said to be variance balanced for the estimation of test treatment versus control treatment contrasts if it permits the estimation of these contrasts with the same variance and the covariance between any two estimated test treatment versus control treatment contrasts is also same. In general we shall call such designs as Balanced Bipartite Block Designs with Unequal Block Sizes (BBPUB designs) of type G and henceforth denote these as BG designs.

Definition 2.1. *An arrangement of v treatments in b blocks of sizes $k_1, k_2, \ldots, k_b$ is said to be a BG design if*

(i) $\sum_{j=1}^{b} k_j^{-\alpha-1} n_{hj} n_{h'j} = L$, *a constant* $\forall\ h \neq h' = 1, \ldots, w$,

(ii) $\sum_{j=1}^{b} k_j^{-\alpha-1} n_{gj} n_{g'j} = L_{00}$, *a constant* $\forall g \neq g' = w+1, \ldots, v$ *and*

(iii) $\sum_{j=1}^{b} k_j^{-\alpha-1} n_{gj} n_{hj} = L_0$, *a constant* $\forall h = 1, \ldots, w, g = w+1, \ldots, v$.

The C matrix of a BG design is

$$C = \begin{bmatrix} (wL + uL_0)I_w - L1_w1_w' & -L_0 1_w 1_u' \\ -L_0 1_u 1_w' & (uL_{00} + wL_0)I_u - L_{00}1_u 1_u' \end{bmatrix}. \tag{2.3}$$

A generalized inverse of C is

$$C^- = \begin{bmatrix} \frac{1}{wL+uL_0}(I_w + (L/uL_0)1_w 1_w') & 0 \\ 0 & \frac{1}{uL_{00}+wL_0}(I - (1/u)1_u 1_u') \end{bmatrix}. \tag{2.4}$$

For a BG design the wu test treatment versus control treatment contrasts are estimated with same variance given by

$$\mathrm{Var}(\hat{\tau}_g - \hat{\tau}_h) = \frac{1}{vL_0}\left[\frac{(v-1)L_0 + L}{wL + uL_0} + \frac{(v-1)L_0 + L_{00}}{uL_{00} + wL_0}\right]\sigma^2, \forall\ g \in G, h \in H.$$

BG designs have been studied extensively in the literature under different names. For $u = 1$, BG designs are Balanced Treatment Incomplete Block (BTIB) designs for proper setting and Balanced Treatment Incomplete Block Designs with Unequal Block Sizes (BTIUB designs) for non proper setting for $\alpha = 0$ and BTIUB designs of type G for $\alpha \neq 0$. For $u > 1$, the BG designs have been termed as Balanced Bipartite Block (BBPB) designs for proper settings and Balanced Bipartite Block Designs with Unequal Block Sizes (BBPUB designs) for non-proper settings with $\alpha = 0$.

Indeed there may exist designs other than BG designs that permit the estimation of test treatment versus control treatment contrasts with the same variance. Group Divisible Bipartite Block Designs with Unequal Block Sizes (GDBPBUB designs) of type G, henceforth denoted as GBG designs, are such designs. The GBG design is defined below:

Definition 2.2. *An arrangement of $v = u + w, w = mn$, treatments in b blocks of sizes $k_1, k_2, \ldots, k_b$ with parameters $u, w, m, n, b, k_1, k_2, \ldots, k_b, L_0, L_1, L_2, L_{00}$ is said to be a GBG design if $1, 2, \ldots, w, w+1, \ldots, w+u$ treatments can be partitioned into $m+1$ disjoint groups $V_1, V_2, \ldots, V_m, V_0$ of respective cardinalities $v_1, v_2, \ldots, v_m, u$, such that*

(i) $V_0 = \{w+1, w+2, \ldots, w+u\}$,

(ii) $v_1 = v_2 = \cdots = v_m = n$,

(iii) $\sum_{j=1}^{b} k_j^{-\alpha-1} n_{hj} n_{h'j} = \begin{cases} L_1, h \neq h' \in V_q, & q = 1, \ldots, m \\ L_2, h \neq h', h \in V_q, h' \in V_{q'}, & q \neq q' = 1, \ldots, m \end{cases}$

(iv) $\sum_{j=1}^{b} k_j^{-\alpha-1} n_{gj} n_{g'j} = L_{00}, \quad g \neq g' \in V_0,$

(v) $\sum_{j=1}^{b} k_j^{-\alpha-1} n_{gj} n_{hj} = L_0, g \in V_0, h = 1, \ldots, w.$

Here L_1, L_2, L_{00}, L_0 are some constants. The C matrix of a GBG design is

$$C = \begin{bmatrix} (A-B) \otimes I_m + B \otimes 1_m 1_m' & D \\ D' & E \end{bmatrix} \tag{2.5}$$

Here $A = [nL_1 + n(m-1)L_2 + uL_0]I_n - L_1 1_n 1_n', E = (wL_0 + uL_{00})I_u - L_{00} 1_u 1_u'$, $B = -L_2 1_n 1_n'$, and $D = -L_0 1_w 1_u'$.

A generalized inverse of C is

$$C^{-} = \begin{bmatrix} X^{-1} & 0 \\ 0' & \frac{1}{wL_0 + uL_{00}} \left[I_u - \frac{1}{u} 1_u 1_u' \right] \end{bmatrix}, \tag{2.6}$$

with $X = (A-B) \otimes I_m + B \otimes 1_m 1_m'$, $X^{-1} = (P-Q) \otimes I_m + Q \otimes 1_m 1_m'$, $Q = -B(A-B)^{-1}[A+(m-1)B]^{-1}$ and $P = [A+(m-2)B](A-B)^{-1}[A+(m-1)B]^{-1}$.

For a GBG design the wu test treatment versus control treatment contrasts are estimated with same variance given by

$$\begin{aligned} \text{Var}\,(\hat{\tau}_g - \hat{\tau}_h) &= \left[\frac{1}{nL_1 + wL_2 - nL_2 + uL_0} + \frac{L_1 - L_2}{(nL_1 + wL_2 - nL_2 + uL_0)(uL_0)} \right. \\ &+ \left. \frac{L_2}{uL_0(wL_2 + uL_0)} + \frac{u-1}{u(uL_{00} + wL_0)} \right] \sigma^2 \end{aligned}$$

For $L_1 = L_2$, these designs are same as BG designs. Some interesting special cases of GBG designs have been studied in the literature. For $u = 1$, GBG designs are termed as Group Divisible Treatment Designs (GDTD) for proper setting and Group Divisible Treatment Designs with Unequal Block Sizes (GDTUB designs) for non proper setting for $\alpha = 0$ and GDTUB designs of type G for $\alpha \neq 0$.

3 Method of construction of BG designs

We now give a general method of construction BG designs. The methods of construction, hitherto known in the literature, fall out as special cases of this method.

Method 3.1 Suppose that there exists a pairwise balanced binary block (PBBB) design with parameters $w, b_1, b_2, \ldots, b_p, k' = (k_1 1'_{b_1}, k_2 1'_{b_2}, \ldots, k_p 1'_{b_p})$, $\lambda, b = b_1 + b_2 + \cdots + b_p$ and incidence matrix $N = [N_1 \vdots N_2 \vdots \cdots \vdots N_p]$, N_s being the $w \times b_s$ matrix corresponding to the incidence of treatments in the blocks of size k_s satisfying $N_s 1_w = r_s$. Then the design with incidence matrix

$$N^* = \begin{bmatrix} N_1 \otimes 1_{\theta_1} & N_2 \otimes 1'_{\theta_2} & \ldots & N_p \otimes 1'_{\theta_p} \\ a_1 1_u 1'_{b_1\theta_1} & a_2 1_u 1'_{b_2\theta_2} & \ldots & a_p 1_u 1'_{b_p\theta_p} \end{bmatrix}$$

is a BG design with parameters $w,\ u,\ b^* = b_1\theta_1 + b_2\theta_2 + \ldots + b_p\theta_p,\ k^{*\prime} = ((k_1 + a_1 u)1'_{b,\theta_1}, (k_2 + a_2 u)1'_{b_2\theta_2}, \ldots, (k_p + a_p u)1'_{b_p\theta_p})$, whee $a_1, a_2, \ldots, a_p$ are some non-negative integers such that $\sum_{s=1}^{p} a_s r_s = a1$, a being a scalar and $\theta_1, \theta_2, \ldots, \theta_p$ are taken in the ratio

$$((k_1 + a_1 u)^{\alpha+1} : (k_2 + a_2 u)^{\alpha+1} : \cdots : (k_p + a_p u)^{\alpha+1}).$$

For $u = 1$, an efficient block design for making test treatment versus control treatment comparisons can be obtained by taking $a_s \leq \text{int}(k_s/2);\ \forall\ s = 1, \ldots, p$. For a review of the methods of construction of PBBB designs, see Parsad, Gupta and Khanduri (2000).

Remark 3.1 If N is the incidence matrix of a $(\mu_1, \mu_2, \ldots, \mu_s, \ldots, \mu_p)$ - resolvable PBBB design, i.e., $N_s 1_w = \mu_s 1,\ s = 1, \ldots, p$. Then the design with incidence matrix

$$N^{**} = \begin{bmatrix} N_1 \otimes 1'_{\theta_1} & N_2 \otimes 1'_{\theta_2} & \cdots & N_p \otimes 1'_{\theta_p} \\ & \sum^{+} a_s 1'_{b_s\theta_s} & & \end{bmatrix}$$

is a BG design with parameters $w, u, b^* = b_1\theta_1 + b_2\theta_2 + \ldots + b_p\theta_p,\ k^{*\prime} = ((k_1 + a_1)1'_{b_1\theta_1}, (k_2 + a_2)1'_{b_2\theta_2}, \ldots, (k_p + a_p)1'_{b_p\theta_p})$, where $a_1, a_2, \ldots, a_p$ are some nonnegative integers such that $a_s\mu_s = a,\ s = 1, \ldots, p, a$ being a scalar and $\theta_1, \theta_2, \ldots, \theta_p$ are taken in the ratio

$$((k_1 + a_1)^{\alpha+1} : (k_2 + a_2)^{\alpha+1} : \cdots : (k_p + a_p)^{\alpha+1}).$$

Remark 3.2 Suppose there exists a binary block design with incidence matrix $N = [N_1 \vdots N_2 \vdots \cdots \vdots N_p]$ and parameters $v, b_1, b_2, \ldots, b_p, k' = (k_1 1'_{b_1}, \ldots, k_p 1'_{b_p}), \lambda_1, \lambda_2$ and in which the w test treatments can be divided into m disjoint sets $V_1, V_2, \ldots, V_m$ of cardinality n each such that

$$\sum_{s=1}^{p} \lambda_{shh'} = \begin{cases} \lambda_1, h \neq h' \in V_q, q = 1, \ldots, m \\ \lambda_2, h \neq h', h \in V_q, h' \in V_{q'} q \neq q' = 1, \ldots, m. \end{cases}$$

Here $\lambda_{shh'}$ is the concurrence of treatments h and h' in N_s. One way of obtaining such a design is to replace every treatment in an equireplicate PBBB design in m treatments with a group of n new treatments.

Following the procedure given in Method 3.1 for BG designs on N gives a GBG design.

4 Optimality results

We now present some results on optimal designs for the problem. In the present context, the most appropriate optimality criteria are A- and MV-optimality criteria. A design d^* belonging to a certain class of competing designs D is said to be A-optimal if it minimizes $\sum\sum_{g\in G, h\in H} \mathrm{Var}(\hat{\tau}_g - \hat{\tau}_h)$ over all designs $d \in \mathcal{D}$. Let m_L, $1 \leq L \leq wu$ be the Lth diagonal element of $P'C_d^{-}P$, the dispersion matrix of $P'\hat{\tau}$ under the design d. Then one has to find a design $d^* \in \mathcal{D}$ that minimizes $\sum_{L=1}^{wu} m_L/wu$ over $\mathcal{D}$. From the arithmetic mean - harmonic mean inequality it follows that

$$\sum_{L=1}^{wu} m_L/wu \geq wu/\sum_{L=1}^{wu} 1/m_L.$$

The equality is attained when $m_L = m$ for all $L = 1, \ldots, wu$ and this holds for GBG and BG designs. These designs are, therefore, candidates for the most efficient designs for test treatment versus control treatment comparisons according to A-optimality criterion.

A design d^* belonging to a certain class of competing designs $\mathcal{D}$ is said to be MV-optimal if d^* has the least value of the maximum variance of BLUE of elementary contrasts among test treatments and control treatments as compared to any other design $d \in \mathcal{D}$. It may be mentioned here that for the present problem all the A-optimal designs are MV-optimal as well.

Result 1 (Majumdar, 1986) : For one-way heterogeneity setting, for comparing w test treatments with u control treatments using $n = bk$ experimental units arranged in b blocks of size k each, a design is A-optimal if $k = 0 \ (\mathrm{mod} w + \sqrt{wu})$, wu is a perfect square, $k = 0 \ (\mathrm{mod} u + \sqrt{wu})$ and

$$n_{hj} = \frac{k}{w + \sqrt{wu}}, \ n_{gj} = n_{hj}\sqrt{w/u}, \ \forall\, h = 1, \ldots, w, g = w + l, \ldots, v, j = 1, \ldots, b.$$

For non-orthogonal designs the following optimality results are available in the literature.

Result 2: Constantine (1983) showed that a reinforced BIB design obtained by adding a control treatment once in every block of a BIB design is A-optimal in the restricted class of block designs having a single replication of the control treatment in each block.

Result 3 : Jacroux (1984) showed that in the restricted class of block designs with a single replication of control treatment in each block, a design obtained by adding control treatment once to each block of a most balanced group divisible design is A-optimal.

Designs in which a standard treatment is reinforced in each block of the design were termed as Standard reinforced (SR-) designs.

Result 4 : Majumdar and Notz (1983) showed that a BTIB design, binary in test treatments, and satisfying certain conditions is A-optimal. Hedayat and Majumdar (1984) utilized these conditions to give a stronger definition of BTIB designs. They classified the BTIB designs as (a) Rectangular or R- type BTIB designs (equal replication of the control treatment in all the blocks) and (b) Step or (S-type) BTIB designs (the replications of the control treatment in the blocks differs by one). Stufken (1988) gave the bounds to the A-efficiency of BTIB designs. Cheng, Majumdar, Stufken and Ture (1989) studied the A- and MV-optimality of S-type BTIB designs and gave an algorithm to obtain these designs. Das (1986) and Kisan (1987) also studied the optimality aspects of these designs and gave some general methods of their construction.

Hedayat and Majumdar (1985) obtained a sufficient condition for A-optimality in the form of an inequality involving number of test treatments and block size. This sufficient condition is helpful in obtaining A-optimal R-type BTIB designs having single replication of the control treatment in each block. Stufken (1987) extended the sufficient condition to the case of R-type BTIB designs having $t \geq 1$ replications of the control treatment in each block.

Result 5 (Stufken, 1987) : A BTIB design obtained by adding a control treatment t times to each block of a BIB design with parameters $w, b, r, k-t, \lambda$ in test treatments is A-optimal whenever $(k-t-1)^2 + 1 \leq wt^2 \leq (k-t)^2$.

Gupta (1989) obtained a simpler sufficient condition to search an A-optimal design among the class of all connected binary block designs in terms of elements of information matrix.

Sinha (1992) gave general methods of construction of BTIB designs by merging treatments in a group divisible design. Parsad, Gupta and Prasad (1995) gave general methods of construction of BTIB designs and investigated their optimality using sufficient condition of Hedayat and Majumdar (1984) and Gupta (1989). Jacroux (1987a) introduced Group Divisible Treatment Designs (GDTD). A computer intensive sufficient condition for a GDTD to be A-optimal is given by Hedayat, Jacroux and Majumdar (1988). Jacroux (1987b, 1987c, 1988, 1989), Ting and Notz (1988), Giovagnoli and Wynn (1985) and Stufken (1991) have also provided some interesting results. Jacroux and Majumdar (1989) gave optimal block designs for comparing test treatments with a control treatment when block size is greater than the number of test treatments. Bhaumik (1990) and Cutler (1993) have studied the problem when the errors are correlated.

All these studies are restricted to situations when there is a single control treatment. For more than one control treatment optimality aspects have been studied by Majumdar (1986), Jaggi (1992), Jaggi, Gupta and Parsad (1996), Jacroux (2000) and Solorzano and Spurrier (2001). Majumdar (1986) gave an algorithm to obtain A-optimal BBPB designs and a catalogue of A-optimal BBPB designs for small block sizes. Jaggi, Parsad and Gupta (1996) extended the result of Constantine (1983) to more than one control treatment situation and also studied the A-optimality of BBPB designs in the restricted class of designs in which all control treatments appear equally frequently in a block or do not appear at all. Jacroux (2000) gave methods for determining and constructing MV-optimal and highly efficient orthogonal and nearly orthogonal block designs for comparing test treatments

with several control treatments under the restriction that replication number of control treatments is fixed. Solorzano and Spurrier (2001) obtained some results on construction and A-optimality of BBPB designs for small values of u and w.

The studies just described relate to proper setting under fixed effects model. In an incomplete block design, the block effects may be random. Pandey (1993) and Gupta, Pandey and Parsad (1998) have obtained sufficient conditions for generating A-optimal incomplete block designs for making test treatments versus control treatment comparisons under a two-way classified, additive, linear, mixed effects model. It has been shown empirically that an A-optimal/efficient design under a fixed effects model remains A-optimal/efficient under a mixed effects model also. Catalogues of A-efficient/optimal designs have also been given.

The problem of characterization and construction of A- and MV -optimal designs for making test treatment versus control treatment comparisons was till now restricted to proper setting. However, non-proper experimental settings do exist and it is required to generate efficient designs under these situations as well. Prasad (1989) investigated the optimality of designs with unequal block sizes in a very restricted class of designs when the control replications are taken as constant and intra-block variances are assumed to be constant.

For comparing test treatments with a control treatment in block designs with unequal blocks, the concept of Balanced Treatment Incomplete Block Designs with unequal block sizes (BTIUB) was given by Angelis and Moyssiadis (1991) as a natural extension of BTIB designs. They also gave a sufficient condition for establishing the A-optimality of BTIUB designs. Angelis and Moyssiadis (1991), Angelis, Moyssiadis and Kageyama (1993) and Gupta and Kageyama (1993) gave some methods of constructing A-efficient BTIUB designs. Jacroux (1992) studied the A-and MV-optimality of block designs with two distinct block sizes where block sizes may be greater than the number of test treatments for comparing several test treatments with a control treatment. These studies were also carried out under the assumption that intra block variances are constant. Parsad (1991), Parsad and Gupta (1994a) introduced BTIUB designs of Type G and obtained a sufficient condition for A-optimality of non - proper incomplete block designs for comparing test treatments with a control treatment assuming that intrablock variances are proportional to non - negative real power of block sizes. Parsad and Gupta (1994b) introduced GDTUB designs of type G and a sufficient condition for A-optimality of GDTUB designs of type G in the class of block designs that are binary in test treatments and in which the control treatment is added same number of times to each block of same size. A catalogue of A-optimal GDTUB designs of type G has also been given. Srivastava, Gupta and Parsad (2000) have studied the A-optimality of non-proper block designs for comparing test treatments with a control treatment when the block sizes may be larger than the number of test treatments. Jaggi (1996) and Jaggi and Gupta (1997a, 1997b) have studied the A-optimality aspects of the designs for comparing several test treatments with several control treatments under a non-proper block design setting where intra block variances have been assumed to be constant. The results are obtained in a restricted class of designs in which all controls appear equally frequently in a block or do not appear at all and block sizes are large. For small block sizes, the condition of Majumdar (1986) has been obtained for non-proper settings. Jaggi, Parsad and Gupta (1999) gave methods of

construction of BBPUB designs. Parsad, Gupta and Singh (1996) have obtained a sufficient condition for a block design to be A-optimal for comparing two disjoint sets of treatments under the above heteroscedastic set up.

Result 6: A BG design is A-optimal for comparing w test treatments with u control treatments in the class of designs binary in test treatments and control treatments if $L_0/L = 1 + \sqrt{\frac{w+u}{u}}$. This is a fairly general condition and the conditions for all other designs useful for test treatment versus control treatment comparisons fall out as a particular case of this. A procedure to obtain an efficient design for the experimental situations for which this condition does not hold well has also been suggested using the concept of lower bound to the average variance.

The problem of obtaining efficient designs for comparing test treatments with several control treatments under an unrestricted class is still unsolved and as has been seen in the discussion above only partial solution to the problem is available. Further, all the results are available for a class of deisgns in which the pairwise comparisons within a set are made with same variance, and between treatments from different sets with same variance. Kuriakose (1999) has introduced Group Divisible Bipartite Block Designs and studied the A-optimality of these designs for $k = 2$.

5 Weighted A-optimal designs for test treatments versus control treatments comparisons

In certain problems, it is necessary to generate designs that estimate contrasts of interest with differential precision and minimize the weighted sum of variances of the BLUE of the contrasts of interest. It may indeed be possible to obtain exact optimal designs for these experimental settings. In the most general set up we may restate the problem as follows:

We consider again the experimental setting described in Section 2. Let $\mathcal{D}$ denote the class of competing designs. Find a design $d^* \in \mathcal{D}$ that minimizes

$$\sum_{g\in G}\beta_g \sum_{h\in H} \text{Var}\,(\hat{\tau}_{dg} - \hat{\tau}_{dh}) + \alpha \sum_{h=1}^{w-1}\sum_{h'=h+1}^{w} \text{Var}\,(\hat{\tau}_{dh} - \hat{\tau}_{dh'}) + \gamma \sum_{g=w+1}^{v-1}\sum_{g'=g+1}^{v} \text{Var}\,(\hat{\tau}_{dg} - \hat{\tau}_{dg'}). \tag{5.1}$$

Here $\beta_1, \beta_2, \ldots, \beta_u, \alpha, \gamma \geq 0$ are scalar constants or the weights attached with the precision of various comparisons and satisfy $\beta_1 + \beta_2 + \cdots + \beta_u + \alpha + \gamma = 1$.

The comparisons among control treatments are of no interest to the experimenter and so $\gamma = 0$. The problem then reduces to finding a design $d^* \in \mathcal{D}$ that minimizes

$$\sum_{g\in G}\beta_g \sum_{h\in H} \text{Var}\,(\hat{\tau}_{dg} - \hat{\tau}_{dh}) + \alpha \sum_{h=1}^{w-1}\sum_{h'=h+1}^{w} \text{Var}\,(\hat{\tau}_{dh} - \hat{\tau}_{dh'}),\ \beta_1 + \beta_2 + \cdots + \beta_u + \alpha = 1. \tag{5.2}$$

When all the controls have the same weights of importance, then the problem is to find a design $d^* \in \mathcal{D}$ that minimizes

$$\beta \sum_{g\in G}\sum_{h\in H} \text{Var}\,(\hat{\tau}_{dg} - \hat{\tau}_{dh}) + \alpha \sum_{h=1}^{w-1}\sum_{h'=h+1}^{w} \text{Var}\,(\hat{\tau}_{dh} - \hat{\tau}_{dh'}), \beta + \alpha = 1. \tag{5.3}$$

A special case of the problem in (5.2) is when $\alpha = 0$. In other words the different controls are given different weights according to their importance but the comparisons among test treatments are not considered. The problem in (5.2) now reduces to find a design $d^* \in \mathcal{D}$ that minimizes

$$\sum_{g \in G} \beta_g \sum_{h \in H} \text{Var}\,(\hat{\tau}_{dg} - \hat{\tau}_{dh}), \beta_1 + \beta_2 + \cdots + \beta_u = 1 \tag{5.4}$$

For $u = 1$, the problem reduces to finding a design $d^* \in \mathcal{D}$ that minimizes

$$\beta \sum_{h=1}^{w} \text{Var}(\hat{\tau}_{d(w+1)} - \hat{\tau}_{dh}) + \alpha \sum_{h=1}^{w-1} \sum_{h'=h+1}^{w} \text{Var}(\hat{\tau}_{dh} - \hat{\tau}_{dh'}),\ \alpha + \beta = 1. \tag{5.5}$$

The problems in (5.3) and (5.5) are the weighted sum of the variances of the BLUE of test treatments versus control treatments contrasts and contrasts among test treatments, respectively with weights as β and α. Since more precision is required for the test treatments versus control treatments comparisons than the comparisons among test treatments, we insist $\beta > \alpha$. For $\alpha = 0, \beta = 1$, the experimental settings in (5.3) and (5.5) reduce to the usual setting of A-optimality of test treatment versus control treatment comparisons. It may be seen that for $\beta = \alpha$ these experimental settings reduce to the usual setting for A-optimality of designs when all the possible paired comparisons among the v treatments are of equal interest. However, there may be situations when more precision is required for comparisons among test treatments than the test treatments versus control treatments comparisons. For this setting, $\beta < \alpha$.

The problem posed in (5.5) has been solved under the block design set up by Gupta, Ramana and Parsad (1999). A catalogue of A-efficient/optimal designs has also been presented.

The problems of obtaining weighted A-efficient designs for many control treatments have been handled in two phases. In the first phase, the problem of obtaining weighted A-efficient designs for several control treatments has been attempted by giving unequal weights to various control treatments. In the choice of an optimal design no consideration is given to the comparisons among test treatments. This refers to the situation in (5.4) [see Gupta, Ramana and Parsad (2001)]. In the second phase not only we consider the problem of obtaining weighted A-efficient designs by giving equal importance to all the control treatments but also considering the estimation of comparisons among test treatments through the same design, though with a smaller precision than that of the test treatments versus control treatments comparisons. This corresponds to problem in (5.3) [see Ramana (1995)]. The most general problem in (5.1) and the problem in (5.2) is seemingly a difficult problem. These may, however, need attention so as to be able to solve completely the problem of test treatments versus control treatments comparisons.

This article provides a limited review on the designing problem for making test treatments versus control treatments comparisons. There are many other studies available on this problem under different experimental set ups. A complete bibliography on this subject is available with the authors.

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