Variance Balanced Block Designs With Unequal Block Sizes

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ABSTRACT. This article gives a general method of construction of variance balanced block designs with unequal block sizes under a heteroscedastic model in which it is assumed that the intra block variances are proportional to non-negative real power of block sizes. It has been shown that all variance-balanced designs can be obtained from a pairwise balanced design. In fact, under the usual homoscedastic model, all the methods of construction of variance balanced designs, hitherto known in the literature, follow from this general method. Some methods of construction of pairwise balanced block designs have also been given.

1 Introduction

Consider a block design with v treatments, b blocks of sizes $k_1, k_2, \ldots, k_b$ and n experimental units, $n = k_1 + k_2 + \cdots + k_b$. For a proper design $k_1 = k_2 = \cdots = k_b = k$ (say). Assuming the usual additive, homoscedastic model associated with block designs, the $\boldsymbol{C}$ - matrix is given by $\boldsymbol{C} = \boldsymbol{R} - \boldsymbol{N}\boldsymbol{K}^{-1}\boldsymbol{N}'$, where $\boldsymbol{R} = diag(r_1, r_2, \ldots, r_v), \boldsymbol{K} = diag(k_1, k_2, \ldots, k_b), \boldsymbol{N} = ((n_{ij}))$ a $v \times b$ incidence matrix with the non-negative integers n_{ij}'s being the number of times the i^{th} treatment appears in the j^{th} block, $r_1, r_2, \ldots, r_v$ being the replication numbers of the treatments, $i = 1, \ldots, v$; $j = 1, \ldots, b$. The row sums of $\boldsymbol{N}$ give the replication numbers and the column sums of

N give the block sizes. A treatment connected block design is called a *pairwise balanced design* if the off-diagonal elements of NN' are constant, i.e., $NN' = D + \psi \mathbf{1}_v \mathbf{1}'_v$, where D is a diagonal matrix, $\mathbf{1}_t$ is a t-component vector with all entries equal to one and ψ is a scalar constant. A treatment connected block design is called *variance balanced* (for treatment effects) if all the normalised treatment contrasts are estimated through the design with the same variance. In a variance balanced block design, the information matrix (the usual C - matrix) for treatment effects is a completely symmetric matrix - i.e., it is of the form, $\alpha \mathbf{I}_v + \beta \mathbf{1}_v \mathbf{1}'_v$, $\alpha + \beta v = 0$, where $\mathbf{I}_t$ is an identity matrix of order t. Hedayat and Federer [11] have shown that pairwise balance is neither necessary nor sufficient for a block design to have variance balance. Mathon and Rosa [22] provide a comprehensive table of Balanced Incomplete Block (BIB) designs of small order. Kiefer [19] has shown that among all the designs with given v, b, k, a Balanced Block Design (BBD), if it exists, is universally optimal. But in many practical situations, blocks of different sizes are a necessity (see e.g., Pearce [26]). Also with the advent of high-speed computers, block designs with unequal block sizes may be particularly useful in large experiments, both in industry and agriculture. In the literature there are various papers on the construction of variance balanced designs, including those of Hedayat and Federer ([10], [11]), Kulshreshtha, Dey and Saha [21], Kageyama ([15], [16], [18]), Tyagi [33], Khatri [20], Gupta and Jones [6], Mukerjee and Kageyama [23], Agarwal and Kumar [1], Calvin [2], Jones, Sinha and Kageyama [14], Calvin and Sinha [3], Sinha and Jones [31], Sinha ([28], [29]), Sinha, Kageyama and Das [32], Pal and Pal [24], Gupta, Das and Dey [9] and Gupta and Kageyama [7]. Although several methods of construction of variance balanced designs have been advanced in the literature, it appears that there is only one unified method of construction of these designs through the use of pairwise balanced designs (see e.g., Hedayat and Stufken [13]). In fact, Hedayat and Stufken [13] showed that the problems of constructing variance balanced block designs and pairwise balanced block designs are equivalent. In section 2 we give a general method of construction of variance balanced designs through the use of pairwise balanced designs under a general heteroscedastic model. All the methods of construction, hitherto known, are particular cases of this method. Some methods of construction of pairwise binary, balanced block designs are also given. An exhaustive catalogue of designs may be obtained from the authors (also see Parsad, Gupta and Khanduri [25]).

2 Methods of Construction

Gupta [8] has obtained the $\boldsymbol{C}$ - matrix of a block design under a heteroscedastic model where the intra block variances are proportional to a non-negative real power of block sizes, $k_j^{\alpha}, j = 1, \ldots, b$, given by $\boldsymbol{C} = \sum_{j=1}^{b} k_j^{-\alpha}\left[\boldsymbol{R}_j - k_j^{-1}\boldsymbol{N}_j^{*}\boldsymbol{N}_j^{*'}\right]$. Here $\boldsymbol{N}_j^{*}$ is the j^{th} column of $\boldsymbol{N}$ and $\boldsymbol{R}_j = diag(n_{1j}, \ldots, n_{vj})$. A block design is connected iff Rank $(\boldsymbol{C}) = v - 1$. A connected block design under heteroscedastic model is variance balanced block design if $\boldsymbol{C}$ is completely symmetric. Henceforth, we shall consider only connected designs. We now define two classes of variance balanced block designs:

Definition 1. A block design d belonging to a class of designs for given $v, b, k_1, k_2, \ldots, k_b, \alpha$, is called a Generalized Binary Balanced Block (GBBB) design of type α if (i) $|n_{dij} - k_j/v| < 1$ for all $i = 1, \ldots, v; j = 1, \ldots, b$ and (ii) $\sum n_{dij}n_{di'j}/k_j^{\alpha+1} = \lambda$, constant for $i \neq i', i, i' = 1, 2, \ldots, v$.

Definition 2. A block design d belonging to a class of designs for given v, b, n, α, is called a Binary Balanced Block (BBB) design of type α if (i) $n_{dij} = 0$ or 1 for all $i = 1, \ldots, v; j = 1, \ldots, b$ and (ii) $\sum n_{dij}n_{di'j}/k_{dj}^{\alpha+1} = \lambda$, constant for $i \neq i', i, i' = 1, 2, \ldots, v$. For $\alpha = 0$, GBBB designs of type α and BBB designs of type α are same as that of GBBB designs and BBB designs respectively of Gupta, Das and Dey [9]. For $\alpha = 1$, a BBB design of type α with block sizes differing by one is a BBB design of type 1 of Gupta [8]. Gupta, Das and Dey [9] showed that a BBB design is universally optimal in the class of all connected designs with given v, b, n. Gupta [8] has shown that a GBBB design of type α is universally optimal in the class of all connected designs with given $v, b, k_1, k_2, \ldots, k_b, \alpha$. For $\alpha = 1$, a BBB design of type 1 is universally optimal in the class of all connected designs with given v, b, n, and with block sizes differing by one. Binary and generalised binary balanced block designs are universally optimal in some specific class of designs and, therefore, construction of these designs is important. We now give below a general method of construction of variance balanced block designs: Let $\boldsymbol{N} = [\boldsymbol{N}_1 \vdots \boldsymbol{N}_2 \vdots \cdots \vdots \boldsymbol{N}_p]$ be the $v \times b$ incidence matrix of a pairwise balanced block design d with v treatments, b blocks such that $b_1, b_2, \ldots, b_p$ blocks are of sizes $k_1, k_2, \ldots, k_p$, respectively, $b_1 + b_2 + \cdots + b_p = b$, and $\boldsymbol{N}_t$ is a $v \times b_t$ incidence matrix pertaining to b_t blocks of size $k_t, t = 1, 2, \ldots, p$. For $i \neq i' = 1, 2, \ldots, v$, let $\lambda_{tii'} (\geq 0)$ denote the number of times the pair of treatments i and i' appear together in the b_t blocks of size k_t each. Since the design $\boldsymbol{N}$ is pairwise balanced, $\sum_{t=1}^{p}\lambda_{tii'} = \psi$ (a constant) $\forall\ i \neq i' = 1, 2, \ldots, v$. For $t = 1, 2, \ldots, p$, let $\gamma_t(\geq 0)$ be the non-negative numbers, being the weights, such that $\sum_{t=1}^{p}\frac{\gamma_t\lambda_{tii'}}{k_t^{\alpha+1}} = \delta$, a constant. Then the design d^* with incidence matrix

$$\boldsymbol{N}^* = [\mathbf{1}'_{\gamma_1} \otimes N_1 \vdots \mathbf{1}'_{\gamma_2} \otimes \boldsymbol{N}_2 \vdots \cdots \vdots \mathbf{1}'_{\gamma_p} \otimes \boldsymbol{N}_p]$$

obtained by taking the union of γ_t copies of $\boldsymbol{N}_t, t = 1, 2, \ldots, p$, is a variance balanced design under a heteroscedastic model. Here $\gamma_t = \frac{k_t^{\alpha+1}}{c}, t = 1, 2, \ldots, p$ are integers and c is the highest common divisor of $k_1^{\alpha+1}, k_2^{\alpha+1}, \ldots, k_p^{\alpha+1}$. However, it may happen that the pairwise balanced design itself has been obtained by taking different number of copies of blocks of sizes $k_1, k_2, \ldots, k_p$, say $\delta_1, \delta_2, \ldots, \delta_p$, respectively. In that case it may be possible to get a variance balanced design in fewer blocks. This may be achieved by taking $\gamma_t = \frac{\delta_t k_t^{\alpha+1}}{c}, t = 1, 2, \ldots, p$. For $\alpha = 1$, we get the variance balanced designs under a heteroscedastic set up where the intra block variances are assumed to be proportional to the block sizes and for $\alpha = 0$, we get the variance balanced designs under the usual homoscedastic model. All the variance-balanced designs obtained in the literature have indeed been obtained by this method. The pairwise balanced designs can be obtained in many ways. Some methods of obtaining pairwise balanced designs are given below:

Method 1. For $t = 1, 2, \ldots, p$, let $\boldsymbol{N}_t$ be the incidence matrix of a partially balanced incomplete block (PBIB) design with two associate classes, same association scheme and parameters as $v, b_t, r_t, k_t, \lambda_{t1}, \lambda_{t2}$ such that $\sum_{t=1}^{p}\vartheta_t\lambda_{t1} = \sum_{t=1}^{p}\vartheta_t\lambda_{t2} = \psi$. Then the design obtained by taking the union of ϑ_t copies of $\boldsymbol{N}_t$ is a pairwise balanced design with parameters v, $\sum_{t=1}^{p}\vartheta_t b_t$, $\sum_{t=1}^{p}\vartheta_t r_t$, k_1, $k_2, \ldots, k_p$, ψ. Here ϑ_t is a positive integer.

Remark 1. The variance balanced block designs of Gupta and Jones [6], Agarwal and Kumar [1], Jones, Sinha and Kageyama [14], Kageyama([17], [18]) and Ghosh, Divecha and Kageyama [5] are all particular cases of this

method. The GBBB designs of type α obtained through Method 2.2 of Gupta [8] also follow as a particular case for $p = 2$. It is interesting to note that sometimes we get a variance balanced design in fewer experimental units compared to the corresponding pairwise balanced design. For example, consider two PBIB designs with two associate classes having the same association scheme as
$D1 : SR36 : v = 8, b = 8, r = 4, k = 4, m = 4, n = 2, \lambda_{11} = 0, \lambda_{12} = 2$, and
$D2 : R30 : v = 8, b = 32, r = 8, k = 2, m = 4, n = 2, \lambda_{21} = 2, \lambda_{22} = 1$.
Using the above two designs a pairwise balanced block design can be obtained by taking union of the SR36 and two copies of R30. However, a variance balanced block design can be obtained by taking union of SR36 and R30. Through Method 1 it is possible to obtain many variance-balanced designs using the PBIB designs given in the catalogue prepared by Clatworthy [4] and Sinha [30].

Method 2. For any non-negative integer u, and v treatments take $\binom{v-u-2}{k-2}$ copies of a block of size $(v-u)$ containing any $(v-u)$ treatments. Form additional $\sum_{t=1}^{k}\binom{u}{t}\binom{v-u}{k-t}$ blocks of size $k(< v-u)$ each in the following manner: (i) Form all possible $\binom{u}{t}$ t-plets of each of the remaining u treatments. (ii) Form all possible $\binom{v-u}{k-t}$ $(k-t)$-plets of $(v-u)$ treatments. (iii) Combine the treatments in each of the $\binom{u}{t}$ t-plets with the treatments in each of the $\binom{v-u}{k-t}$ $(k-t)$-plets to form blocks of size k. The resulting design is a pairwise balanced design with parameters $v, b = \binom{v-u-2}{k-2} + \sum_{t=1}^{k}\binom{u}{t}\binom{v-u}{k-t}, k_1 = v-u, k_2 = k, \psi = \binom{v-2}{k-2}, r' = \left\| \left[w_1 \mathbf{1}'_{(v-u)} \;\; w_2 \mathbf{1}'_u \right] \right\|$ with $w_1 = \binom{v-u-2}{k-2} + \sum_{t=1}^{k}\binom{u}{t}\binom{v-u-1}{k-t-1}$ and $w_2 = \sum_{t=1}^{k}\binom{u-1}{t-1}\binom{v-u}{k-t}$. For $k > (v-u)$, we get an unreduced BIB design.

Remark 2. The variance balanced designs of Kageyama [15], Tyagi [33] and Khatri [20] are a particular case of this general method. The GBBB designs of type α, obtainable through Method 2.1 of Gupta [8], also follow as a particular case of this method. The drawback of these methods is that the designs obtained are such that the block sizes are a multiple of each other. However, the Method 2, described above, has the flexibility of the choice of block sizes. As a matter of fact, many more variance balanced designs can be generated from this method.

Method 3. Let N be the incidence matrix of a pairwise balanced design d with parameters $v, b, r, k_1, k_2, \ldots, k_b, \psi$. Delete any $u(< v)$ treatments from the design d. The resulting design is again a pairwise balanced design with parameters $v-u, b, r, k_1^*, k_2^*, \ldots, k_b^*, \psi$.

Remark 3. The equireplicated variance balanced designs obtained by Sinha [28] from BIB designs with parameters $v_0, b_0, r_0, k_0, \lambda_0$ follow as a particular case of this method. Indeed many equireplicated as well as unequal replicated variance balanced designs can be obtained from this method.

Note. Following the method of Sinha [28], for p = 1, we get a BBB design with parameters $v = v_0 - 1, r = r_0k_0 - \lambda_0, k_1 = k_0, k_2 = k_0 - 1, b_1 = k_0(b_0 - r_0), b_2 = r_0(k_0 - 1), \psi = \lambda_0$. As given in an example explained by Sinha [28], for $p = 1$, a BIB design with parameters $v_0 = 8, b_0 = 14, r_0 = 7, k_0 = 4, \lambda_0 = 3$ gives a BBB design with parameters $v = 7, r = 25, k_1 = 4, k_2 = 3, b_1 = 28, b_2 = 21, \psi = 3$. This method, however, is restrictive in the sense that the number of experimental units increases too rapidly. However, if one takes a solution of BIB design that is resistant (globally or locally) with respect to loss of a treatment, then we can get a design in smaller number of experimental units. In general, a BIB design is said to be resistant of degree p, if after the loss of p distinct treatments from the design, the resulting design remains variance balanced. It is said to be globally resistant if it is resistant to the loss of any of the $\binom{v}{p}$ sets of p-treatments and locally resistant if it is resistant for a particular set of p-treatments. For $p = 1$, the resistant BIB designs have been introduced by Hedayat and John [12]. Therefore, it can easily be said that a BBB design with $v - p$ treatments is always existent provided a resistant BIB design of degree p exists. In fact, the resultant design will be simultaneously a pairwise balanced and variance balanced in this case as it will be a union of several BIB designs. Fortunately, for the example we are discussing for $v = 8, b = 14, r = 7, k = 4, \lambda = 3$, a resistant solution of degree one is given by

1	2	3	4	5	6	7	3	4	5	6	7	1	2
2	3	4	5	6	7	1	5	6	7	1	2	3	4
4	5	6	7	1	2	3	6	7	1	2	3	4	5
8	8	8	8	8	8	8	7	1	2	3	4	5	6

The above design is resistant with respect to treatment 8. Therefore, by deleting treatment 8 from the above design, we get a pairwise and variance balanced design

1	2	3	4	5	6	7	3	4	5	6	7	1	2
2	3	4	5	6	7	1	5	6	7	1	2	3	4
4	5	6	7	1	2	3	6	7	1	2	3	4	5
							7	1	2	3	4	5	6

Using a resistant variance balanced block design in v treatments, one gets a variance balanced block design in $v-1$ treatments by deleting the treatment with respect to which the design is resistant. For more details one may refer to Singh and Gupta [27]. Indeed, using Method 3 and the note below it, many equireplicated as well as varying replications BBB designs can be obtained.

Method 4. Suppose there exists an s-resolvable BIB design D with parameters $v_1, b_1 = t\beta, r_1 = st, k_1, \lambda_1 = sp$ and $t = ap$, where a, p, t, s, β are positive integers. For an s-resolvable design with above parameters, the blocks can be grouped into t sets such that there are β blocks within each set. Now if one wants to get a BBB design in $v_1 + a$ treatments, then regroup the t sets of blocks into "a" sets of blocks such that there are βp blocks within each set. Let the incidence matrix $\boldsymbol{N}$ of $\boldsymbol{D}$ be rewritten as $\boldsymbol{N} = [\boldsymbol{N}_1\ \boldsymbol{N}_2\ \cdots\ \boldsymbol{N}_a]$, where $\boldsymbol{N}_i$ is a $v_1 \times \beta p$ matrix such that $\boldsymbol{N}_i \mathbf{1}_{\beta p} = sp\mathbf{1}_v, \boldsymbol{N}_i'\mathbf{1}_v = k\mathbf{1}_{\beta p}$, $i = 1, \ldots, a$. Then N^*, given below, is the incidence matrix of a pairwise balanced block design

$$\boldsymbol{N}^* = \begin{bmatrix} \boldsymbol{N}_1 & \boldsymbol{N}_2 & \cdots & \boldsymbol{N}_a & \mathbf{1}'_{\lambda_1} \otimes \mathbf{0}_a \\ \boldsymbol{M}_1 & \boldsymbol{M}_2 & \cdots & \boldsymbol{M}_a & \mathbf{1}'_{\lambda_1} \otimes \mathbf{1}_a \end{bmatrix}$$

where $\mathbf{0}_a$ is a vector of zeros and $\boldsymbol{M}_i$ is an $a \times \beta p$ matrix having zeros everywhere except the i^{th} row which contains all unities, $i = 1, \ldots, a$. The parameters of the pairwise balanced binary block design are $v' = v_1 + a, b_1' = b_1, k_1' = k_1 + 1, b_2' = \lambda_1, k_2' = a, \boldsymbol{r}' = [r_1\mathbf{1}'_{v_1}\ (\beta p + \lambda_1)\mathbf{1}'_a], \psi = \lambda_1$. For $\alpha = 0, p = 1, t = a, \lambda_1 = s$, this method is same as Theorem 2 of Gupta and Kageyama [7].

Method 5. For $t = 1, 2$, let $\boldsymbol{N}_t$ be the incidence matrix of a partially balanced incomplete block (PBIB) design with two associate classes with same association scheme and with parameters $v, b_t, r_t, k_t, \lambda_{t1}, \lambda_{t2}$ such that

$$\sum_{t=1}^{2} \vartheta_t \lambda_{t1} = \sum_{t=1}^{2} \vartheta_t \lambda_{t2} = \sum_{t=1}^{2} \vartheta_t i_t r_t = \psi, \quad \text{a constant}$$

and $k_1 + i_1 \neq k_2 + i_2$, where $i_1, i_2\{i_1, i_2 \in (0, 1)$ and $i_1 + i_2 \neq 0\}$ denotes the number of times a new treatment has been added to the blocks of sizes

k_1 and k_2, respectively. Then the design

$$\left[\mathbf{1}'_{\vartheta_1} \otimes \begin{bmatrix} N_1 \\ i_1 \mathbf{1}'_{b_1} \end{bmatrix} \vdots \mathbf{1}'_{\vartheta_2} \otimes \begin{bmatrix} N_t \\ i_2 \mathbf{1}'_{b_2} \end{bmatrix} \right]$$

obtained by taking the union of ϑ_t copies of $\begin{bmatrix} N_t \\ i_t \mathbf{1}'_{b_t} \end{bmatrix}$ is a pairwise balanced block design with parameters $v^* = v+1, b^* = \vartheta_1 b_1 + \vartheta_2 b_2, r^{*'} = [(\vartheta_1 r_1 + \vartheta_2 r_2)\mathbf{1}'_v, (i_1 \vartheta_1 b_1 + i_2\, \vartheta_2 b_2)], \boldsymbol{k}^{*'} = [(k_1+i_1)\mathbf{1}'_{\vartheta_1 b_1}, (k_2+i_2)\mathbf{1}'_{\vartheta_2 b_2}], \psi$. However, if $\vartheta_1 r_1 + \vartheta_2 r_2 = i_1 \vartheta_1 b_1 + i_2 \vartheta_2 b_2$, then we get an equireplicated binary pairwise balanced block design. This kind of reinforcement and unionisation has been used in the construction of efficient block designs for comparing test treatments with control treatments. In some of these cases, it turns out to be a pairwise balanced design. For PBIB designs as group divisible designs and $i_1 = 0$ and $i_2 = 1$, this method reduces to the Method 1 of Gupta and Kageyama [7].

Remark 4. If we take BIB designs instead of PBIB designs then for $i_1 = 1, i_2 = 0$, the Method 5 reduces to corollary of Theorem 2.1 of Kulshreshtha, Dey and Saha [21] for obtaining BBB designs under a homoscedastic model. In fact, for $\alpha = 0$, it is also same as Theorem 3.2 of Hedayat and Federer [11], and when $k_2 > k_1 + 1$, it reduces to the method given by Hedayat and Federer [10]. If $\boldsymbol{N}_1$ is a BIB design, $\boldsymbol{N}_2$ is a BBB design and $i_1 = 1, i_2 = 0$, then it is same as that of Theorem 14.8 of Kageyama [15].

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