A TEST FOR SPHERICITY

AND

MIXED MODEL ANALYSIS OF FERTILIZER EXPERIMENTS IN CULTIVATOR'S FIELDS

S.K. SRIVASTAVA

(1961-1962)

INSTITUTE OF AGRICULTURAL RESEARCH STATISTICS

(Indian Council of Agricultural Reseach)

New Delhi.

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By

S. K. SRIVASTAVA

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New Delhi.

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S.K. Srivastava
(S.K. Srivastava)

PART ONE

A TEST FOR SPHERICITY

1. INTRODUCTION

If Y is distributed according to a multivariate normal distribution N (o, ψ), we wish to test the hypothesis H that $\psi = \psi_0$ where ψ_0 is a given positive definite matrix, then this is equivalent to testing the hypothesis H_1: $\Sigma = I$ where Σ is the covariance matrix of a vector X distributed according to N (o, Σ). Thus we are interested in testing the hypothesis that a sample from a normal p-variate population is in fact from a population for which the variances are all equal and the covariances are all zero. A population with such a symmetry is called 'spherical'. Under a linear orthogonal transformation of variates, a spherical population remains spherical and consequently the features of a sample which furnish information relevant to this hypothesis must be invariant under such transformations.

John. W. Mauchly (1940) derived the likelihood ratio criterion for testing the sphericity of a population. He finds the moments of the nth root, L_{sp}, of the likelihood ratio criterion, where n is the sample size, when the hypothesis tested is true. However, its distribution under the alternative cannot easily be obtained for a general p. For p=2, he derives the distribution of L_{s2} under the hypothesis tested. For p=3 and higher values of p, no simple expression for the distribution exists and for p=3 he gives the approximate distribution of L_{s3} under the hypothesis when the sample size n is large. For other values of p also, probably the similar approximations may be found.

The power of the likelihood ratio test cannot be calculated since the distribution of L_{sp} cannot be found under the alternative hypothesis. For small sample sizes practically nothing is known about the likelihood ratio test criterion.

In following chapters a test has been suggested which leads to the F-distribution under the null hypothesis and whose power can be approximately calculated and which is always unbiased. However, this simple test can be recommended only for small samples as it is not perhaps consistent. The test has been modified for large values of n, the sample size, for which case the test is consistent, but this test is only an approximate one. The power function of the approximate test can also be calculated.

An example where such a test is necessary may be given as follows:

Let s students be judged by each of the r judges separately and we are interested in testing the hypothesis that the variances of scores to the students given by judges are equal and their covariances zero. Let y_{ij} denote the score of the jth student given by the ith judge. If the covariance matrix of the variables $\{y_i\}$ be $\Sigma_y = (\sigma_{ii'})$, then we are interested in testing the hypothesis H_0: $\Sigma_y = c\,I$, where c is a constant and I denote the identity matrix. To remove the variation due to students we consider the variables $\{x_i\}$ such that

$$x_{ij} = y_{ij} - y_{rj}, \quad i = 1, 2, \ldots, n, \ (n = r-1)$$

whose covariance matrix is, say, $\Sigma_x = (\lambda_{ii'})$ where

$$\lambda_{ii'} = \sigma_{ii'} - \sigma_{ir} - \sigma_{ri'} + \sigma_{rr} \quad \text{for } i \neq i' = 1, 2, \ldots, n$$

and $\lambda_{ii} = \sigma_{ii} - 2\sigma_{ir} + \sigma_{rr}$ for $i = 1, 2, \ldots, n$

Thus the hypothesis H_0 reduces to H: $\Sigma_x = \Sigma_0$ where Σ_0 is a matrix of order n x n whose diagonal elements are equal to 2c and all other elements are equal to c. Thus

$$\Sigma_0 = c \begin{bmatrix} 2 & 1 & \cdots & 1 \\ 1 & 2 & \cdots & 1 \\ \cdot & \cdot & \cdots & \cdot \\ 1 & 1 & \cdots & 2 \end{bmatrix} = cA \text{ say,}$$

consider the quadratic form of variables $x_1, x_2, \ldots, x_n$,

$$Q = x' A x$$
$$= \sum_1^n x_i^2 + \left(\sum_1^n x_i\right)^2$$

Transforming the variables $\{x_i\}$ into $\{z_i\}$ by the transformation

$$z_1 = \frac{1}{\sqrt{n(n+1)}} \sum_{i=1}^{n} x_i$$

$$z_2 = \frac{x_1 - x_2}{\sqrt{2}}$$

$$z_3 = \frac{x_1 + x_2 - 2\,x_3}{\sqrt{6}}$$

$$\cdots\cdots\cdots$$

$$z_n = \frac{x_1 + x_2 + \cdots\cdots + x_{n-1} - (n-1)\,x_n}{\sqrt{\{(n-1) + (n-1)^2\}}}$$

so that $z_1, z_2, \ldots\ldots, z_n$ are orthogonal ∴ transforms to $\sum_1^n z_i^2 = z'I\,z$ and $\sum_0$ transforms to $c\,I$. Hence, if we denote the covariance matrix of z by $\sum_z$, the hypothesis H reduces to $\sum_z = c\,I$ which can be tested using

$$\{z_{ij}\}\,,\ \begin{matrix} i = 1, 2, \ldots, n. \\ j = 1, 2, \ldots, s. \end{matrix}$$

2. A TEST FOR SPHERICITY.

In what follows we write $\chi^2(\nu)$ to denote a quantity distributed as χ^2 with ν d.f. and $F(\nu_1, \nu_2)$ to denote a quantity distributed as Fisher - Snedecor F with ν_1 and ν_2 degrees of freedom.

Let x be a random vector distributed according to the p - variate normal distribution $N(o, \Sigma)$. To find the distribution of $\sum_1^p x_j^2$.

Let $C'\Sigma C = D$

where C is an orthogonal matrix, then the variable $z = C^{-1}x$ has the distribution $N(o, D)$, and

$$\sum_1^p x_j^2 = x'x = z'C'Cz = z'z = \sum_1^p z_j^2$$

Thus if $z = D^{\frac{1}{2}}y$, then y has the distribution $N(o, I)$ and also

$$(2.1) \qquad \sum_1^p x_j^2 = z'z = y'Dy = \sum_1^p \lambda_j y_j^2$$

where λ_j's are the characteristic roots of D, which are the same as that of Σ.

Thus, $Q = x'x$ is distributed like a quantity $\sum_1^p \lambda_j \chi^2(1)$ where each χ^2 is distributed independently of every other and λ's are the latent roots of the positive definite matrix Σ.

G. E. P. Box (1954) states a theorem for the approximate distribution of a quadratic form which is as follows:

<u>Theorem 2.1.</u> If the column vector z follows a p-variate normal distribution $N(o, V)$, and if $Q = z' M z$ is any real quadratic form of rank $r \leq p$, then

$$(2.2) \qquad Q = z' M z = \sum_{j=1}^{r} \lambda_j \chi^2(\nu_j)$$

is distributed approximately as $g . \chi^2(h)$ where

$$(2.3) \qquad g = \frac{\sum \nu_j \lambda_j^2}{\sum \nu_j \lambda_j} \quad \text{and} \quad h = \frac{(\sum \nu_j \lambda_j)^2}{\sum \nu_j \lambda_j^2}$$

It readily follows from the above theorem that $Q = x' x = \sum \lambda_j \chi^2(1)$ is approximately distributed as $g \chi^2(h)$ where

$$(2.4) \qquad g = \frac{\sum \lambda_j^2}{\sum \lambda_j} \quad \text{and} \quad h = \frac{(\sum \lambda_j)^2}{\sum \lambda_j^2}$$

When $\sum = I$, a unit matrix

$\lambda_j = 1$ for all j

and $\sum_1^p \lambda_j = \sum_1^p \lambda_j^2 = p$

Thus $h = p$.

If we take

$$Q_{(\nu)} = x'_{(1)} x_{(1)} + x'_{(2)} x_{(2)} + \dots + x'_{(\nu)} x_{(\nu)}$$

where $x_{(i)}$ has a p - variate normal distribution $N(o, \sum)$, then $Q_{(\nu)}$ is distributed approximately as $g_{(\nu)}\chi^2(h(\nu))$ where

$$g_{(\nu)} = \frac{\sum \nu \lambda_j^2}{\sum \nu \lambda_j} = g,$$

(2.5)

$$h_{(\nu)} = \frac{(\sum \nu \lambda_j)^2}{\sum \nu \lambda_j^2} = \nu h.$$

Also if we take

(2.6)

$$Q_1 = x'_{(1)} x_{(1)} + x'_{(2)} x_{(2)} + \cdots + x'_{(\nu_1)} x_{(\nu_1)} \quad \text{and}$$

$$Q_2 = x'_{(\nu_1+1)} x_{(\nu_1+1)} + \cdots + x'_{(\nu_1+\nu_2)} x_{(\nu_1+\nu_2)}$$

then Q_1 and Q_2 are distributed approximately as $g_{(\nu_1)}\chi^2(h(\nu_1))$ and $g_{(\nu_2)}\chi^2(h(\nu_2))$ respectively, where

(2.7) and

$$g_{(\nu_1)} = g_{(\nu_2)} = g$$

$$h_{(\nu_1)} = \nu_1 h ; \quad h_{(\nu_2)} = \nu_2 h.$$

Thus $\dfrac{\nu_2 \; Q_1}{\nu_1 \; Q_2}$ has approximately a F distribution with degrees of freedom ($\nu_1 h$, $\nu_2 h$) which become ($\nu_1 p$, $\nu_2 p$) when $\sum = I$. The area to the right of a point x of the F distribution with (m, n) d.f. is always smaller than the corresponding area of the F distribution with d.f. (m_1, n_1) where $m_1 < m$ and $n_1 < n$. In above case since $h \leq p$, the power of the test is not less than the first kind of error for any alternative.

Thus if $x_{(1)}, x_{(2)}, \ldots, x_{(\nu_1+\nu_2)}$ is a sample of size $(\nu_1+\nu_2)$ from a p-variate normal population $N(0, \Sigma)$, to test the hypothesis H_0: $\Sigma = I$, we can use the statistic

(2.8) $$F = (\nu_2 Q_1)/(\nu_1 Q_2)$$

which is distributed approximately as a F distribution with d.f. $(\nu_1 h, \nu_2 h)$ which becomes $(\nu_1 p, \nu_2 p)$ under the hypothesis H_0: $\Sigma = I$. And since h is not greater than p, the test is always unbiased.

3. POWER:

The exact power of the test for any given particular alternative hypothesis may be calculated using G.E.P. Box (1954) Theo. 4.3, which can be stated as follows:

Theorem: If $\lambda'_1, \lambda'_2, \ldots, \lambda'_{r'}$, and $\lambda_1, \lambda_2, \ldots, \lambda_r$ are all positive and the ν_j and ν'_j are all even then

(3.1) $$\Pr\left[\left\{\sum_{j=1}^{r'} \lambda'_j \cdot \chi^2(\nu'_j)\right\} \Big/ \left\{\sum_{j=1}^{r} \lambda_j \chi^2(\nu_j)\right\} > Y_0\right] = \sum_{i=1}^{r'} \sum_{s=1}^{g_i} \alpha_{is}$$

where $g_j = \frac{1}{2}\nu'_j$ and the α_{is} $(i = 1, 2, \ldots, r'; s = 1, 2, \ldots, g_j)$ are constants determined as given below.

Consider
$$X = \sum_{j=1}^{r'+r} \zeta_j \chi^2(\nu_j)$$

in which $\zeta_1 = \lambda'_1, \ \zeta_2 = \lambda'_2 \ldots\ldots, \ \zeta_{r'} = \lambda'_{r'};$

$\zeta_{r'+1} = -Y_0 \lambda_1 \ldots\ldots\ldots, \ \zeta_{r'+r} = -Y_0 \lambda_r,$

Denoting hth cumulant by K_h and hth moment about the origin by μ'_h and using the well known equalities expressing the moments in terms of the cumulants we may calculate the constants α_{ig}. For, if we write

(3.2)
$$K_{ih} = (h-1)! \sum_{j \neq i}^{r'+r} \left\{ g_j \left(-\frac{\zeta_j}{\zeta_i - \zeta_j} \right)^h \right\}$$

then
$$\alpha_{ig_i - h} = \left(\frac{\mu'_{ih}}{h!} \right) \alpha_{ig_i}$$

and
$$\alpha_{ig_i} = \prod_{j \neq i}^{r} \left(\frac{\zeta_j}{\zeta_i - \zeta_j} \right)^{g_j}$$

Let us consider an example when $p = 3$. Let $x_{(1)}, x_{(2)}, \ldots, x_{(6)}$ be six observations from a trivariate normal population $N(o, \Sigma)$. To calculate the power of the F test (2.8) for the alternative $H_1: \Sigma = \Sigma_1$ where the latent roots of Σ_1 be $\frac{1}{2}$, 1, and 2.

By the cannonical reduction of numerator and denominator the ratio
$$\left\{ \sum_{i=1}^{2} x'_{(i)} x_{(i)} \right\} \Big/ \left\{ \sum_{i=3}^{6} X'_{(i)} x_{(i)} \right\}$$

is seen to be distributed like the quantity

(3.3)
$$\frac{X_1}{X_2} = \frac{\left\{ \sum_{j'=1}^{3} \lambda'_{j'} \chi^2(\nu'_{j'}) \right\}}{\left\{ \sum_{j=1}^{3} \lambda_j \chi^2(\nu_j) \right\}}$$

where

$$\lambda_1' = \lambda_1 = \tfrac{1}{2}$$

$$\lambda_2' = \lambda_2 = 1$$

$$\lambda_3' = \lambda_3 = 2$$

$$\nu_1' = \nu_2' = \nu_3' = 2$$

$$\nu_1 = \nu_2 = \nu_3 = 4$$

If we take the first kind of error to be .05, we have

$$Pr \left\{ \frac{\chi^2(6)}{\chi^2(12)} > 1.6 \right\} = .05$$

since $F_{5\%}\ (6,12) = 3.0$

giving $Y_0 = 1.6$

Hence the power of the test is equal to

(3.4) $$Pr \left[\left\{ \sum_{j=1}^{3} \lambda_j' \chi^2(2) \right\} \Big/ \left\{ \sum_{j=1}^{3} \lambda_j \chi^2(4) \right\} > 1.6 / H_1 \right] = \sum_{i=1}^{3} \alpha_{i1}$$

Here $\zeta_1 = \frac{1}{2}$, $\zeta_2 = 1$, $\zeta_3 = 2$, $\zeta_4 = -.75$, $\zeta_5 = -1.6$, $\zeta_6 =$

From (2.2)

$$\alpha_{11} = \prod_{j=2}^{6} \left(\frac{\zeta_1}{\zeta_1 - \zeta_j} \right)^{g_j}$$

$$= .00007.$$

$$\alpha_{21} = \prod_{j \neq 2}^{6} \left(\frac{\zeta_2}{\zeta_2 - \zeta_j} \right)^{g_j}$$

$$= -.00653.$$

$$\alpha_{31} = \prod_{j \neq 3}^{6} \left(\frac{\zeta_3}{\zeta_3 - \zeta_j} \right)^{g_j}$$

$$= .07369.$$

giving $$\sum_{i=1}^{3} \alpha_{i1} = .06723.$$

Thus the exact power of the test for the given alternative is .067.

Using the approximate distribution of the ratio

$$\frac{4 \sum_{i=1}^{2} x'_{(i)} x_{(i)}}{2 \sum_{i=3}^{6} x'_{(i)} x_{(i)}}$$

as F with degrees of freedom 2h and 4h where h is, under the alternative hypothesis, given by

$$h = \frac{(\frac{1}{2} + 1 + 2)^2}{\frac{1}{4} + 1 + 4} = 2\tfrac{1}{3}$$

and the power is, thus, equal to

$$Pr\left[\left\{\chi^2(4\tfrac{2}{3}) / \chi^2(9\tfrac{1}{3})\right\} > 1.5\right]$$

$$= \frac{1}{B(2\frac{1}{3}, 4\frac{2}{3})} \int_{1.5}^{\infty} \frac{F^{1\frac{1}{3}}}{(1+F)^7} \, dF$$

$$= \frac{1}{B(2\frac{1}{3}, 4\frac{2}{3})} \int_{0}^{0.4} x^{4\frac{2}{3} - 1} (1 - x)^{2\frac{1}{3} - 1} \, dx$$

$$= .074$$

Thus we see that the power calculated by two methods, in this case, does not differ considerably and so the approximation used is quite good.

In another case when 20 observations are available, let us take the statistic as

$$(3.6) \qquad \frac{\frac{1}{10} \sum_{i=1}^{10} x'_{(i)} x_{(i)}}{\frac{1}{10} \sum_{i=11}^{20} x'_{(i)} x_{(i)}}$$

which is distributed like the quantity (3.3). Let the alternative hypothesis H_1 for which the power is to be calculated is such that

$$\lambda'_1 = \lambda_1 = 1,$$
$$\lambda'_2 = \lambda_2 = 4$$

$$\lambda_3' = \lambda_3 = 10$$

Here
$$g_1 = g_2 = g_3 = 5$$
and
$$g_4 = g_5 = g_6 = 5$$

and since $F_{5\%}(30, 30) = 1.84$.

$$Y_0 = 1.84$$

Also $\zeta_1 = 1$, $\zeta_2 = 4$, $\zeta_3 = 10$, $\zeta_4 = -1.84$, $\zeta_5 = -7.36$, $\zeta_6 = -18.4$

Using (3.1), the power is equal to

$$(3.6)\quad Pr\left[\left\{\sum_{j'=1}^{3}\lambda_{j'}'\chi^2(10)\right\}\Big/\left\{\sum_{j=1}^{3}\lambda_j\chi^2(10)\right\} > 1.84 \;/\; H_1\right] = \sum_{i=1}^{3}\sum_{s=1}^{g_i}\alpha_{is}$$

By the relations (3.2) we have

$$\alpha_{35} = .00321$$

and
$$K_{31} = 0!\sum_{j\neq 3}^{6} g_j\left\{\frac{-\zeta_j}{\zeta_3-\zeta_j}\right\}$$

$$= 2.24739$$

Similarly,

$$K_{32} = 5.40222$$

$$K_{33} = .54246$$

$$K_{34} = 12.20314$$

Using the well known relation between moments and cumulants, we get.

$$\mu_{31}' = K_{31} \qquad = 2.24739$$

$$\mu_{32}' = K_{32} + K_{31}^2 \qquad = 10.45298$$

$$\mu_{33}' = K_{33} + 3K_{32}K_{31} + K_{31}^3 \qquad = 48.31618$$

$$\mu_{34}' = K_{34} + 4K_{33}K_{31} + 3K_{32}^2 + 6K_{32}K_{31}^2 + K_{31}^4 = 293.85368$$

Using (3.2), we have

$$\sum_{s=1}^{5} \alpha_{3s} = \left\{ 1 + \frac{\mu'_{31}}{1!} + \frac{\mu'_{32}}{2!} + \frac{\mu'_{33}}{3!} + \frac{\mu'_{34}}{4!} \right\} \alpha_{35}$$

$$= .089$$

$$\alpha_{25} = \prod_{j \neq 2}^{6} \left(\frac{4}{4 - \zeta_j} \right)^{s_j}$$

is negligible and also α_{15} is negligible. Hence the power (3.6) of the test is .089.

Using the approximate distribution of the ratio (3.5) as a F with degrees of freedom 10h and 10h under the alternative hypothesis, where

$$h = \frac{(1+4+10)^2}{1+16+100} = 1.923$$

the power of the test is

$$Pr \left\{ (\chi^2(19.23) / \chi^2(19.23)) > 1.84 \right\}$$

$$= \frac{1}{B(9.615,\ 9.615)} \int_0^{0.352} x^{8.615} (1-x)^{8.615} dx$$

$$= .098$$

Thus in this case also, the approximation taken is sufficiently good.

Many authors have compared the exact distribution and approximate distribution of Quadratic forms and have shown that such an approximation is fairly good. Table 3.1 compares the powers of the test based on the statistic (2.8), using the exact distribution of the ratio of quadratic forms and using the approximate distribution of quadratic forms for different sample sizes and shows that the approximation taken is fairly good for calculating the powers.

Table 3.1 Comparison of Powers calculated using the exact and approximate distribution of ratio of quadratic forms.

No. of terms in numerator	denominator	1	2	3	Powers calculated using the exact distribution.	approx. distribution
2	4	½	1	2	.067	.074
10	10	1	4	10	.089	.098
2	12	1	4	10	.081	.084
4	20	1	2	10	.101	.108
6	18	1	2	10	.108	.100
10	30	1	2	10	.096	.109
4	20	1	2	100	.140	.143
6	18	1	2	100	.160	.164

To see if the proposed test is consistent or not, let us take 2n term in numerator and 2nk terms in the denominator, i.e.,

$$F = \frac{2nk \sum_{i=1}^{2n} x'_{(i)} x_{(i)}}{2n \sum_{i=2n+1}^{2n+2nk} x'_{(i)} x_{(i)}}$$

The 5% point of the F distribution with 2n and 2nk degrees of freedom tends to 1.0 as $n \to \infty$, and let us assume that under the alternative hypothesis the d.f. becomes $(6n\alpha,\ 6nk\alpha)$, $\frac{1}{2} \le \alpha \le 1$. Then the power will be given by

$$\Pr\left\{\frac{\chi^2(6n\alpha)}{\chi^2(6nk\alpha)} > \frac{1}{k}\right\}$$

$$= \frac{\int_0^{\frac{k}{k+1}} x^{3nk\alpha-1}(1-x)^{3n\alpha-1}dx}{\int_0^1 x^{3nk\alpha-1}(1-x)^{3n\alpha-1}dx}$$

the limit of which as $n \to \infty$ is not easily found, but the power remains small for quite large values of n.

If we take two terms in the numerator and 2n in the denominator, the power will be given by

(3.7) $$\Pr\left\{\frac{\chi^2(6\alpha)}{(6n\alpha)} > \frac{2.1}{n}\right\}$$

since $\qquad F_{5\%}(6,\ \infty) = 2.1$

Also let $\qquad \alpha = \frac{1}{3}$, then

Power $$= \Pr\left\{\frac{\chi^2(2)}{\chi^2(2n)} > \frac{2.1}{n}\right\}$$

$$= \frac{\int_0^{\frac{n}{n+2.1}} x^{n-1}\,dx}{\int_0^1 x^{n-1}\,dx}$$

$$= (n/(n+2.1))^n \quad = \left(1+\frac{2.1}{n}\right)^{-n}$$

which tends to $e^{-2.1}$ as $n \to \infty$.

Also if $\alpha = \frac{2}{3}$, (3.7) reduces to

$$\Pr\left(\frac{\chi^2(4)}{\chi^2(4n)} > \frac{2.1}{n}\right)$$

$$= \frac{\int_0^{\frac{n}{n+2.1}} x^{2n-1}(1-x)\,dx}{\int_0^1 x^{2n-1}(1-x)\,dx}$$

$$= (2n+1)\left(\frac{n}{n+2.1}\right)^{2n} - 2n\left(\frac{n}{n+2.1}\right)^{2n+1}$$

$$= \left(\frac{n}{n+2.1}\right)^{2n}\left(\frac{5.2n+2.1}{n+2.1}\right)^{2n+1}$$

which tends to $5.2\,e^{-4.2}$ as $n \to \infty$.

Thus the power does not tend to 1 as the sample size tends to infinity, and so the test is not consistent.

Thus the proposed test can be used to good results only for small sample sizes. For large sample sizes this test is not good as the power does not increase for large sample sizes.

4. A TEST FOR LARGE SAMPLES.

For large sample sizes we can calculate several F's from the given sample. Each F is calculated by taking k_1 terms in numerator and k_2 in denominator, giving

$$F(k_1 p, k_2 p) = \frac{\frac{1}{k_1 p}\sum_{i=1}^{k_1} x'_{(i)}\, x_{(i)}}{\frac{1}{k_2 p}\sum_{i=k_1+1}^{k_1+k_2} x'_{(i)}\, x_{(i)}}$$

and the following statistic is used for testing the hypothesis. Define

(4.1) $$\bar{X} = \frac{X_1 + X_2 + \cdots + X_N}{N}$$

where $X = \sqrt{F(m,n)}$, ; $m = k_1 p$, $n = k_2 p$

and use the statistic,

(4.2) $$t = \frac{\bar{X} - \mu}{\sigma/\sqrt{N}}$$

which for large values of N follows the normal distribution for testing the given hypothesis. The values of μ and σ can be calculated under the null as well as under the alternative hypothesis. Since t follows the normal distribution, the test is obviously consistent.

To see that, for how large N, the statistic (4.2) follows a normal distribution, we shall try to find the moments of t. The moments of $X = \sqrt{F(m,n)}$ can be obtained as follows:

$$E(X^r) = E\left(\sqrt{F(m,n)}\right)^r$$

$$= \left(\frac{n}{m}\right)^{r/2} E \; \frac{\left(\chi^2(m)\right)^{r/2}}{\left(\chi^2(n)\right)^{r/2}}$$

and

$$E\left(\chi^2(m)\right)^{r/2} = \frac{1}{\Gamma(\frac{1}{2}m)} \int_0^\infty z^{r/2} e^{-z/2} (z/2)^{\frac{m-2}{2}} \, d(z/2)$$

$$= \frac{2^{r/2}}{\Gamma(\frac{1}{2}m)} \int_0^\infty e^{-z/2} (z/2)^{\frac{m+r-2}{2}} \, d(z/2)$$

$$= 2^{r/2} \frac{\Gamma(\frac{m+r}{2})}{\Gamma(\frac{m}{2})}$$

$$E\left(\frac{1}{\chi^2(n)}\right)^{r/2} = \frac{1}{\Gamma(\frac{1}{2}n)} \int_0^\infty \frac{1}{z^{r/2}} e^{-z/2} (z/2)^{\frac{n-2}{2}} \, d(z/2)$$

$$= \frac{1}{2^{r/2}\Gamma(\frac{1}{2}n)} \int_0^\infty e^{-z/2} (z/2)^{\frac{n-r-2}{2}} \, d(z/2)$$

$$= \frac{\Gamma(\frac{n-r}{2})}{2^{r/2}\Gamma(\frac{n}{2})}$$

Thus

$$(4.3) \quad E\left(\sqrt{F(m,n)}\right)^r = \left(\frac{n}{m}\right)^{r/2} \frac{\Gamma\left(\frac{m+r}{2}\right)\Gamma\left(\frac{n-r}{2}\right)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)}$$

and

$$(4.4) \quad \begin{cases} \mu'_1 = \left(\frac{n}{m}\right)^{\frac{1}{2}} \dfrac{B\left(\frac{m+1}{2}, \frac{n-1}{2}\right)}{B\left(\frac{m}{2}, \frac{n}{2}\right)} \\ \mu'_2 = \dfrac{n}{n-2} \\ \mu'_3 = \left(\frac{n}{m}\right)^{3/2} \dfrac{B\left(\frac{m+3}{2}, \frac{n-3}{2}\right)}{B\left(\frac{m}{2}, \frac{n}{2}\right)} \\ \mu'_4 = \dfrac{n^2(m+2)}{m(n-2)(n-4)} \end{cases}$$

which exist when $n \geqslant 5$.

Let us take F's each with 5 terms in the numerator and 5 in the denominator. Thus if P=3, m= n = 15 and from (4.4)

$$\mu'_1 = \frac{B(8, 7)}{B(7.5, 7.5)} = 1.036$$

$$\mu'_2 = \frac{15}{13} = 1.154$$

$$\mu'_3 = \frac{B(9, 6)}{B(7.5, 7.5)} = 1.382$$

$$\mu'_4 = \frac{15 \times 17}{13 \times 11} = 1.783$$

and so the moments about the mean of X are

$\mu_1 = 0$

$\mu_2 = .081$

$\mu_3 = .012$

$$\mu_4 = .031$$

and the moments of t (4.2) become

$$\mu_1(t) = 0$$

$$\mu_2(t) = 1$$

$$\mu_3(t) = \frac{\mu_3}{\sigma^3 \sqrt{N}} = \frac{.783}{\sqrt{N}}$$

and $\mu_4(t) = 3 + \frac{1}{N}\left(\frac{\mu_4}{\sigma^4} - 3\right) = 3 + \frac{1.726}{N}$

giving

$$\gamma_1 = \frac{.783}{\sqrt{N}}$$

and $\gamma_2 = \frac{1.726}{N}$

which will give the value of N for which t will be distributed approximately as N (o, 1). For N $\geq$ 8, the approximation will be sufficiently good.

If $m = n = 9$

we find that the moments of X are

$$\mu_1 = 0$$

$$\mu_2 = .154$$

$$\mu_3 = -.067$$

and $\mu_4 = .170$

giving the moments of t (4.2) as

$$\mu_1(t) = 0$$

$$\mu_2(t) = 1$$

$$\mu_3(t) = -\frac{1.109}{\sqrt{N}}$$

and $\mu_4(t) = 3 + \frac{4.083}{N}$

and $\gamma_1(t) = - \frac{1.109}{\sqrt{N}}$

$\gamma_2(t) = \frac{4.083}{N}$

Hence for $N \geqslant 16$, the normal approximation of t will be sufficiently good.

Thus for large sample sizes we can use this test effectively. For smaller samples we can always use the F test (2.8).

If we take 6 terms in the numerator and 6 terms in the denominator of each F, we find that γ_1 and γ_2 of t (4.2) calculated on the basis of N such F's. are

$\gamma_1 = \frac{.80}{\sqrt{N}}$

and $\gamma_2 = \frac{1.64}{N}$

and thus the sample size required to make the statistic (4.2) N (o, 1), when we take 6 terms in numerator and 6 in the denominator of each F, is more than when we take only 5 terms each in the numerator and denominator of each F. Similarly if we take 7 terms each in the numerator and denominator of every F, we find that the sample size required to make the statistic (4.2) N (o,1) is more than when we take only 5 terms each in the numerator and denominator and $\mu_4(t) = 3 + \frac{4.083}{N}$

and $\gamma_1(t) = -\frac{1.102}{\sqrt{N}}$

$\gamma_2(t) = \frac{4.083}{N}$

Hence for $N \geqslant 16$, the normal approximation of t will be sufficiently good.

Thus for large sample sizes we can use this test effectively. For smaller samples we can always use the F test (2.8).

If we take 6 terms in the numerator and 6 terms in the denominator of each F, we find that γ_1 and γ_2 of t (4.2) calculated on the basis of N such F's, are

$$\gamma_1 = \frac{.80}{\sqrt{N}}$$

and $$\gamma_2 = \frac{1.64}{N}$$

and thus the sample size required to make the statistic (4.2) N (o, 1), when we take 6 terms in numerator and 6 in the denominator of each F, is more than when we take only 8 terms each in the numerator and denominator of each F. Similarly if we take 7 terms each in the numerator and denominator of every F, we find that the sample size required to make the statistic (4.2) N (o,1) is more than when we take only 8 terms each in the numerator and denominator

8 CALCULATION OF THE POWER:

Let us take 5 terms in numerator and 5 in denominator for each F. Then under the null hypothesis Ho: m=n=15

and $E\left(\sqrt{F(m,n)}\right) = 1.036$

$\therefore E(\bar{X}/H_0) = 1.036$

also $E\left(\sqrt{F(m,n)}\right)^2/H_0 = 1.154$

$\therefore \text{Var.}(\bar{X}/H_0) = \dfrac{.081}{N}$

and $\text{s.e.}(\bar{X}/H_0) = .0735$

when N=15, i.e., the sample size is 150.

Let the alternative hypothesis H_1 be such that

$\lambda_1 = 1, \ \lambda_2 = 2, \ \lambda_3 = 10$

then , $h = \dfrac{(\sum \lambda_j)^2}{\sum \lambda_j^2} = \dfrac{169}{105}$

and $5h = 8$

Thus under the alternative H_1: m=n=8

and $E(\bar{X}/H_1) = 1.074$

$E\left\{\left(\sqrt{F(m,n)}\right)^2/H_1\right\} = 1.333$

$\therefore \text{s.e.}(\bar{X}/H_1) = .1096$ when N = 15

Since under the alternative hypothesis the d.f. for F cannot exceed the d.f. for F under the null hypothesis, $E(\bar{X})$ under the null hypothesis will always be less than that under the alternative hypothesis. Thus only the upper tail of the normal distribution will be used for testing the hypothesis H_o. In above case we have to test the hypothesis $H_o: \mu = 1.036$ against all alternatives H_1 under which $\mu > 1.036$.

Fixing the first kind of error at 5%, the test gives

$$Pr\left\{ \frac{\bar{X} - 1.036}{.0735} > 1.6449 \;/\; H_o \right\} = .05$$

$$\text{i.e., } Pr\left(\bar{X} > 1.1569 \;/\; H_o \right) = .05$$

The power for the alternative hypothesis $H_1 : \mu_1 = 1.074$, is

$$Pr\left(\bar{X} > 1.1569 \;/\; H_1 \right)$$

$$= Pr\left(\frac{\bar{X} - 1.074}{.1098} > .7571 \;/\; H_1 \right) = .23$$

Thus the power of the test for the given alternative is .23

6. POWER OF THE LIKELIHOOD RATIO TEST

Mauchly (1940) gives the likelihood ratio test for testing the sphericity of a population. The distribution of the likelihood ratio criterion is not known, in general, under the alternative hypothesis and so we cannot calculate the power of the test for any given alternative. However, in a bivariate case when the correlation is zero, we can find out a lower bound for the power of the likelihood ratio test for any given alternative hypothesis.

For the bivariate case the likelihood ratio test criterion is

$$\lambda = \frac{\begin{vmatrix} s_1^2 & rs_1s_2 \\ rs_1s_2 & s_2^2 \end{vmatrix}}{\frac{1}{2}(s_1^2 + s_2^2)}$$

$$(6.1) \qquad = \frac{2\sqrt{1-r^2}}{\left(\frac{s_1}{s_2} + \frac{s_2}{s_1}\right)}$$

We have to get the distribution of λ, when the sample of size n is taken from the bivariate normal population $N(o, \Sigma)$, where

$$\Sigma = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \sigma_1\sigma_2\rho & \sigma_2^2 \end{bmatrix}$$

The joint distribution of s_1, s_2, and r is given by

$$(6.2) \quad dF \propto \exp\left[-\frac{n}{2(1-\rho^2)}\left\{\frac{s_1^2}{\sigma_1^2} - \frac{2\rho\, r s_1 s_2}{\sigma_1\sigma_2} + \frac{s_2^2}{\sigma_2^2}\right\}\right] s_1^{n-2}\, s_2^{n-2}\, (1-r^2)^{\frac{n-4}{2}}\, ds_1\, ds_2\, dr.$$

It is not easy to find out the distribution of λ (6.1) but when $\rho = o$, we may be able to find out the lower bound for the probability $P(\lambda < \lambda')$ under any given alternative. The joint distribution of s_1, s_2 when $\rho = 0$ is given by

$$(6.3) \quad C_1 \exp\left\{-\frac{n}{2}\left(\frac{s_1^2}{\sigma_1^2} + \frac{s_2^2}{\sigma_2^2}\right)\right\} s_1^{n-2}\, s_2^{n-2}\, ds_1\, ds_2.$$

Putting $\frac{s_1}{s_2} = u$; $s_2\, du = ds_1$

(6.3) reduces to

$$C_1 \exp\left\{-\frac{n}{2}\left(\frac{u^2}{\sigma_1^2}+\frac{1}{\sigma_2^2}\right) s_2^2\right\} s_2^{2n-3}\, u^{n-2}\, du\, ds_2 \tag{6.4}$$

Integrating out s_2 from (6.4), we get the distribution of u as

$$C_1 u^{n-2} du \int_0^\infty s_2^{2n-3} \exp\left\{-\frac{n}{2}\left(\frac{u^2}{\sigma_1^2}+\frac{1}{\sigma_2^2}\right) s_2^2\right\} ds_2$$

$$= C\, u^{n-2}\left(\frac{u^2}{\sigma_1^2}+\frac{1}{\sigma_2^2}\right)^{1-n} du, \quad 0 \le u \le \infty \tag{6.5}$$

Now put $u + \frac{1}{u} = v$

or $u = \frac{v \pm \sqrt{v^2-4}}{2}$

Thus corresponding to a given v, there are two values of u, and

$$du = \frac{v + \sqrt{v^2-4}}{2\sqrt{v^2-4}}\, dv \quad \text{when } u = \frac{v+\sqrt{v^2-4}}{2}$$

and $du = \frac{v - \sqrt{v^2-4}}{2\sqrt{v^2-4}}\, dv$ when $u = \frac{v-\sqrt{v^2-4}}{2}$

From (6.5), the probability density at $u = \frac{v+\sqrt{v^2-4}}{2}$

transforms to

$$C\left(\frac{v+\sqrt{v^2-4}}{2}\right)^{n-2}\left\{\left(\frac{v+\sqrt{v^2-4}}{2}\right)^2 \frac{1}{\sigma_1^2}+\frac{1}{\sigma_2^2}\right\}^{1-n} \frac{v+\sqrt{v^2-4}}{2\sqrt{v^2-4}}\, dv$$

and that at $u = \frac{v-\sqrt{v^2-4}}{2}$ transforms to

$$C\left(\frac{v-\sqrt{v^2-4}}{2}\right)^{n-2}\left\{\left(\frac{v-\sqrt{v^2-4}}{2}\right)^2\frac{1}{\sigma_1^2}+\frac{1}{\sigma_2^2}\right\}^{1-n}\frac{v-\sqrt{v^2-4}}{2\sqrt{v^2-4}}\,dv.$$

and so the total probability density of v is the sum of the above two expressions, i.e.,

$$(6.6)\quad C\left[\left(\frac{v+\sqrt{v^2-4}}{2}\right)^{n-1}\left\{\left(\frac{v+\sqrt{v^2-4}}{2}\right)^2\frac{1}{\sigma_1^2}+\frac{1}{\sigma_2^2}\right\}^{1-n}\right.$$

$$\left.+\left(\frac{v-\sqrt{v^2-4}}{2}\right)^{n-1}\left\{\left(\frac{v-\sqrt{v^2-4}}{2}\right)^2\frac{1}{\sigma_1^2}+\frac{1}{\sigma_2^2}\right\}^{1-n}\right]\frac{dv}{\sqrt{v^2-4}},\quad 2\le v\le\infty$$

And since $\left(\frac{v+\sqrt{v^2-4}}{2}\right)\left(\frac{v-\sqrt{v^2-4}}{2}\right) = 1$

the distribution of v reduces to

$$(6.7)\quad C\left(\frac{v+\sqrt{v^2-4}}{2}\right)^{n-1}\left[\left\{\left(\frac{v+\sqrt{v^2-4}}{2}\right)^2\frac{1}{\sigma_1^2}+\frac{1}{\sigma_2^2}\right\}^{1-n}+\right.$$

$$\left.\left\{\left(\frac{v+\sqrt{v^2-4}}{2}\right)^2\frac{1}{\sigma_2^2}+\frac{1}{\sigma_1^2}\right\}^{1-n}\right]\frac{dv}{\sqrt{v^2-4}},\quad 2\le v\le\infty$$

and putting $y = \frac{1}{v}$, we get the distribution of

$y = \left(\frac{s_1}{s_2}+\frac{s_2}{s_1}\right)^{-1}$ as

$$(6.8)\quad C.\left(\frac{1+\sqrt{1-4y^2}}{2y}\right)^{n-1}\left[\left\{\left(\frac{1+\sqrt{1-4y^2}}{2y}\right)^2\frac{1}{\sigma_1^2}+\frac{1}{\sigma_2^2}\right\}^{1-n}+\right.$$

$$\left.\left\{\left(\frac{1+\sqrt{1-4y^2}}{2y}\right)^2\frac{1}{\sigma_2^2}+\frac{1}{\sigma_1^2}\right\}^{1-n}\right]\frac{dy}{y\sqrt{1-4y^2}},\quad 0\le y\le\tfrac{1}{2}$$

The distribution of r, the sample correlation coefficient from a bivariate normal population, when $\rho = 0$, is

(6.9) $f(r)\,dr = k.\,(1-r^2)^{\frac{n-4}{2}}\,dr, \qquad -1 \le r \le 1.$

where k is a function of n, the sample size.

Putting $\sqrt{1-r^2} = x$ in (6.9), we have

$$r = \pm\sqrt{1-x^2} \quad \text{and} \quad dr = \mp \frac{x\,dx}{\sqrt{1-x^2}}$$

Thus corresponding to a given x there are two values of r. The probability density at $r = +\sqrt{1-x^2}$ transforms to

$$k.\ \frac{x^{n-3}}{\sqrt{1-x^2}}\ dx$$

and that at $r = -\sqrt{1-x^2}$ transforms to

$$k\ \frac{x^{n-3}}{\sqrt{1-x^2}}\ dx$$

and so the total probability density of x is the sum of the above two expressions, i.e.

(6.10) $$2k.\frac{x^{n-3}}{\sqrt{1-x^2}}\ dx, \qquad 0 \le x \le 1$$

And since s_1, s_2, and r are independent when $\rho = 0$ the joint distribution of x and y becomes

(6.11) $$C'\left(\frac{1+\sqrt{1-4y^2}}{2y}\right)^{n-1}\left[\left\{\left(\frac{1+\sqrt{1-4y^2}}{2y}\right)^2\frac{1}{\sigma_1^2}+\frac{1}{\sigma_2^2}\right\}^{1-n} + \left\{\left(\frac{1+\sqrt{1-4y^2}}{2y}\right)^2\frac{1}{\sigma_2^2}+\frac{1}{\sigma_1^2}\right\}^{1-n}\right]\frac{1}{y\sqrt{1-4y^2}}\,\frac{x^{n-3}}{\sqrt{(1-x^2)}}\,dx\,dy, \quad \begin{matrix} 0 \le x \le 1 \\ 0 \le y \le \end{matrix}$$

To get the distribution of xy, put xy = z, then the joint distribution of y and z is given by

$$(6.12)\quad C' \left(\frac{1+\sqrt{1-4y^2}}{2y}\right)^{n-1}\left[\left\{\left(\frac{1+\sqrt{1-4y^2}}{2y}\right)^2\frac{1}{\sigma_1^2}+\frac{1}{\sigma_2^2}\right\}^{1-n}+\left\{\left(\frac{1+\sqrt{1-4y^2}}{2y}\right)^2\frac{1}{\sigma_2^2}+\frac{1}{\sigma_1^2}\right\}^{1-n}\right]\frac{1}{y\sqrt{1-4y^2}}\;\frac{z^{n-2}}{y^{n-2}\sqrt{(1-z^2/y^2)}}\,dz\,dy,$$

$$0 \leq z \leq \tfrac{1}{2},\; 0 \leq y \leq \tfrac{1}{2}$$

Integrating out y from (6.12) we get the distribution of $z = \dfrac{\sqrt{1-r^2}}{\left(\frac{s_1}{s_2}+\frac{s_2}{s_1}\right)}$, dz

$$(6.13)\quad C'\; z^{n-3}\,dz \int_0^{\frac{1}{2}}\left(\frac{1+\sqrt{1-4y^2}}{2y}\right)^{n-1}\left[\left\{\left(\frac{1+\sqrt{1-4y^2}}{2y}\right)^2\frac{1}{\sigma_1^2}+\frac{1}{\sigma_2^2}\right\}^{1-n}+\left\{\left(\frac{1+\sqrt{1-4y^2}}{2y}\right)^2\frac{1}{\sigma_2^2}+\frac{1}{\sigma_1^2}\right\}^{1-n}\right]\frac{1}{y^{n-3}\sqrt{y^2-z^2}}$$

$$\frac{dy}{y\sqrt{1-4y^2}},\quad 0 \leq z \leq \tfrac{1}{2}$$

The integral in (6.13) cannot be evaluated in simple terms and so we cannot calculate the power from this. But if the sample size is large, r, the sample correlation coefficient may be taken as nearly zero, it being an unbiased estimate of ρ, the population correlation coefficient for large n. However, since $\sqrt{1-r^2} \leq 1$, we can get the lower bound of the power for any given alternative, using the distribution (6.8) of $\dfrac{1}{\left(\frac{s_1}{s_2}+\frac{s_2}{s_1}\right)}$

C in (6.8) can be obtained by integrating (6.7) over v for $2 \leq v \leq \infty$ and equating it with 1. Thus putting

$$(6.14) \qquad \frac{v+\sqrt{v^2-4}}{2} = x \quad , \quad \frac{v+\sqrt{v^2-4}}{2\sqrt{v^2-4}}\, dv = dx$$

we get

$$(6.15) \quad C \left[\int_1^\infty x^{n-2} \left(\frac{x^2}{\sigma_1^2} + \frac{1}{\sigma_2^2}\right)^{1-n} dx + \int_1^\infty x^{n-2} \left(\frac{x^2}{\sigma_2^2} + \frac{1}{\sigma_1^2}\right)^{1-n} dx \right] = 1$$

Writing $\frac{1}{x}$ for x in the second integral in (6.15), we get

$$\frac{1}{C} = \int_1^\infty x^{n-2} \left(\frac{x^2}{\sigma_1^2} + \frac{1}{\sigma_2^2}\right)^{1-n} dx + \int_0^1 x^{n-2} \left(\frac{x^2}{\sigma_1^2} + \frac{1}{\sigma_2^2}\right)^{1-n} dx$$

$$(6.16) \quad = \int_0^\infty x^{n-2} \left(\frac{x^2}{\sigma_1^2} + \frac{1}{\sigma_2^2}\right)^{1-n} dx$$

and putting $x = \frac{\sigma_1}{\sigma_2} z$ in (6.16), we get

$$\frac{1}{C} = (\sigma_1 \sigma_2)^{n-1} \int_0^\infty \frac{z^{n-2}}{(1+z^2)^{n-1}}\, dz$$

$$= \frac{(\sigma_1 \sigma_2)^{n-1}}{2} \int_0^\infty \frac{z^{\frac{n-3}{2}}\, dz}{(1+z)^{n-1}} \, , \quad \text{writing } z \text{ for } z^2$$

$$= \frac{(\sigma_1 \sigma_2)^{n-1}}{2}\, B\left(\frac{n-1}{2}, \frac{n-1}{2}\right)$$

or

$$(6.17) \quad C = \frac{2}{(\sigma_1 \sigma_2)^{n-1}\, B\left(\frac{n-1}{2}, \frac{n-1}{2}\right)}$$

7. THE COMPARISON OF POWERS OF THE TWO TESTS.

The significance of the value of obtained from a given sample of n points is simply

(7.1) $\quad \Pr(\lambda < \lambda') = (\lambda')^{n-2}$

If we take the sample size to be two hundred and the alternative hypothesis H_1 as $\Sigma = \Sigma_1$ where

$$\Sigma_1 = \begin{pmatrix} 9 & 0 \\ 0 & 1 \end{pmatrix}$$

i.e., $\sigma_1 = 3$, $\sigma_2 = 1$, $\rho = 0$

then if the level of significance is 5% , we get

$$\Pr(\lambda<\lambda') = (\lambda')^{198} = .05$$

$$\text{or} \quad \lambda' = .985$$

Thus the lower bound for the power of the likelihood ratio test is given by

(7.2) $\Pr(\lambda < .985/H_1)$

$= \Pr(y < .4925/H_1)$, since $= 2y = \dfrac{2}{\left(\dfrac{s_1}{s_2}+\dfrac{s_2}{s_1}\right)}$

When $y = 0$, $v = \frac{1}{y} = \infty$, and $x = \dfrac{v+\sqrt{v^2-4}}{2} = \infty$

when $y = .4925$, $v = \frac{1}{.4925}$ and $x = \dfrac{v+\sqrt{v^2-4}}{2} = 1.19$

Hence, (7.2) is equal to, using the transformation (6.14),

$$C\int_{1.19}^{\infty} x^{n-2}\left\{\left(\frac{x^2}{\sigma_1^2}+\frac{1}{\sigma_2^2}\right)^{1-n}+\left(\frac{x^2}{\sigma_2^2}+\frac{1}{\sigma_1^2}\right)^{1-n}\right\}dx$$

$$= C.\left[\int_{1.19}^{\infty} x^{n-2}\left(\frac{x^2}{9}+1\right)^{1-n}dx+\int_{1.19}^{\infty} x^{n-2}\left(x^2+\frac{1}{9}\right)^{1-n}dx\right]$$

$$= C.\ \frac{3^{n-1}}{2}\ B\left(\frac{n-1}{2},\frac{n-1}{2}\right)\left\{I_{\frac{9}{9+1.19^2}}\left(\frac{n-1}{2},\frac{n-1}{2}\right)+\right.$$

$$+ I_{\frac{1}{1+9\times 1.19^2}}\left(\frac{n-1}{2}, \frac{n-1}{2}\right)\Big\}$$

$$= I_{.864}(99.5, 99.5) + I_{.073}(99.5, 99.5)$$

$$= 1.0$$

where $I_x(p,q) = \frac{1}{B(p,q)} \int_0^x x^{p-1} (1-x)^{q-1}\, dx$

Thus the power of the likelihood ratio test in this case is one. If we use the normal test proposed in section 4, then under the null hypothesis H_o: m = n = 10, and

$$E(\bar{X} / H_o) = \frac{B(5.5, 4.5)}{B(5,5)} = 1.057$$

$$\text{and } V(\bar{\bar{X}} / H_o) = \frac{.133}{N}$$

$\therefore$ s.e. $(\bar{X} / H_o)$ = .0816 when N = 20, the sample size being 200.

Under the alternative hypothesis H_1: $\Sigma = \Sigma_1$

$\lambda_1 = 9$, $\lambda_2 = 1$.

$$h = \frac{(\Sigma \lambda_i)^2}{\Sigma \lambda_i^2} = \frac{100}{82}$$

$$5h \simeq 6.$$

Thus under H_1: m = n = 6, and

$$E(\bar{\bar{X}} / H_1) = 1.104$$

and s.e. $(\bar{\bar{X}} / H_1)$ = .1186

Taking the level of significance to be 5%, we have

$$Pr\left\{\frac{\bar{\bar{X}} - 1.057}{.0816} > 1.6449 / H_o\right\} = .05$$

or $\Pr(\bar{\bar{X}} > 1.1911) = .05$

Hence the power of the test is equal to

$$\Pr\left(\frac{\bar{\bar{X}} - 1.104}{.1188} > \frac{1.1911 - 1.104}{.1188} / H_1\right)$$

$$= \Pr\left\{N(0, 1) > .733\right\}$$

$= .23.$

Thus the power of this test is only .23.

Here we compared the powers of the two tests for a very particular case of a bivariate normal population when the correlation is zero, and we see that the power of the likelihood ratio test for large samples is very large. The proposed test is thus less powerful, but it is easy to calculate the statistic of this test than that of the likelihood ratio test and no separate tables for the percentage points is necessary, as in the case of likelihood ratio test.

Also nothing is known about the distribution of the likelihood ratio criterion when the hypothesis tested is not true and so we cannot calculate the power of the test in a general case.

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P A R T   T W O

MIXED MODEL ANALYSIS OF FERTILIZER EXPERIMENTS IN CULTIVATOR'S FIELDS

1. Introduction. The need for carrying out fertilizer trials in cultivator's fields under actual farming conditions so as to provide a sound basis for making practical recommendations on fertilizer use, has been felt by many statisticians in India, and fertilizer experiments are now conducted in cultivator's fields distributed over the whole country whose primary aim is' to estimate average response to fertilizers and other improvement measures over the tract and its variation in different parts of the tract.' In a symposium on experiments in cultivators fields held under the auspices of Indian Society of Agricultural Statistics in 1966, a number of speakers emphasised various aspects of this problem and stressed the need of such experiments to verify the results obtained at experimental research stations.

Mainly the experiments in cultivator's fields are needed

(i) To verify results obtained in experimental stations under actual farming conditions.

(ii) To find the response of manures and practices in different soil climatic regions and to select the best manure or practice in the region.

(iii) To demonstrate the use of fertilizers.

The number of experimental stations in a country is usually small and also the fertility of the soil and the level of management at experimental stations are superior to those in the surrounding cultivator's fields. Experimentation at research station cannot, therefore, provide a reliable guide for generalising the results under actual farming conditions and it becomes important to determine the responses to different improved measures under actual farming conditions by experimenting on a representative sample of cultivator's fields.

Dr. S.P. Raychandhury has mentioned in the report 'the use of artificial fertilizers in India' published by I.C.A.R. that no systematic soil survey has been carried out in India to fix the soil type for fertilizer trials and only the broad soil classes are considered. Within each of these soild classes there are a number of soil types showing different responses to manures and fertilizers and thus the estimates are really averages over a number of soil types. This point has also been observed in 'Fertilizer trials on paddy', I.C.A.R. Research Report Series No.1 where it is mentioned that " Comparison of the responses obtained at centres having the same soil classes with those having different soil classes, showed that variation in responses between the centres would not be accounted for by the differences in the broad soil classes to which the centres belong."

This report also makes the following observation on the optimum dose of fertilizers, "The optimum doses given above are for the country as a whole. It is obviously necessary to work out the optimum doses for all the different regions, representing broad soil classes, which can be determined on the basis of responsiveness to fertilizers." More definite conclusions on the optimum doses for different regions would be determined when the results of the series of fertilizer trials in cultivators' fields are available.

Thus apart from testing the fertilizers under field conditions and the demonstration of the use of fertilizer the main aim of fertilizer trials on cultivator's fields is/to set up optimum standards of manuring for different soil - climatic regions of the country, so that the differential response or the interaction of fertilizers and fields within a given soil - climatic region is small, if any. If this interaction is not small the attempt should be made to further subdivide a given region into sub-regions by the physical or chemical properties of the soil, or the climatic factors including rainfall, temperature etc. so that the interaction of fertilizer and field is small withing the subrégions, which will make the further optimal economic use of fertilizers. Thus mainly we have to test the hypothesis that the component of variation due to the interaction of treatment and location within a region is negligible. We shall consider a model for such experiments which has been suggested by Scheffe in another context.

Scheffe (1956) considered an experiment involving I machines and J workers regarded as a sample from a large population of workers. Each worker is put on each machine for K days during the experiment and y_{ijk} is a measurement of the output of the jth worker, the kth day he is on the ith machine. The model considered was

(1) $$y_{ijk} = \mu + \alpha_i + b_j + c_{ij} + e_{ijk}$$

where the general mean μ and row effects $\{\alpha_i\}$ are constants and where the column effects $\{b_j\}$, interactions $\{c_{ij}\}$ and errors $\{e_{ijk}\}$ are random variables about whose joint distribution certain assumptions are made.

To make the assumption, he writes

(2) $$y_{ijk} = m_{ij} + e_{ijk}$$

where the set of errors $\{e_{ijk}\}$ are statistically independent of the set $\{m_{ij}\}$ of 'true' cell means. The basic assumption on the rectangular array of the $\{m_{ij}\}$ is that the J columns are distributed independently like a vector random variable m with I components $m_1, m_2, \ldots, m_I$. Then he assumes that $\{m_i\}$ have a joint normal distribution and $\{e_{ijk}\}$ are also jointly normal.

Under these assumption he shows that the expectation of mean squares ordinarily calculated turn out, with the suitable definitions of the variance components, to have the same values as those usually found in more restrictive models, and some of the customary tests and confidence intervals are justified but the over all test for the fixed main effects and the associated multiple comparison method require Hotelling's T^2.

M.N. Ghosh (I.A.R.S) (in a seminar talk) justified the same model for the use of a series of experiments distributed over different regions, in the case where y_{ijk} is the yield per plot of the treatment i in the jth region and kth plot. Scheffe (1956) and Imhof (1960) have considered simple two - way and three - way classifications with equal number of observation in each cell. In the case of experiments in cultivator's fields, the number of experimental plots in any cultivator's field must be small so that the cultivator may conduct these properly and may be induced to do so. For this and other reasons enunciated in Panse and Abraham (1953) the experiments are conducted in incomplete block layouts within the villages.

Uttam Chand and Abraham (1957) have given a few designs for the comparison of different sources of fertilizers in experiments in cultivator's fields. They have considered the analysis of such designs under the usual mixed model. We shall see how these experiments can be analysed in the light of Scheffe's theory.

2. The Model: The comparison of three sources of nitrogen fertilizer may be made by carrying out an equal number of experiments for each of the following three five - plot sets.

$$(3) \quad \begin{array}{llllll} (i) & 0, & n_1, & n_2, & n_1', & n_2' \\ (ii) & 0, & n_1, & n_2, & n_1'', & n_2'' \\ (iii) & 0, & n_1', & n_2', & n_1'', & n_2'', \end{array}$$

where suffixes denote the levels and the primes denote the source of the fertilizer.

Suppose that three sets are arranged in 3r fields in a given region such that r experiments are allocated to each of the sets. To distribute a total of 3r experiments in an administrative unit one might select r villages at random and then select three fields at random in each village. The mathematical model will then be given by

$$(4) \qquad y_{ijk} = \mu + t_i + \nu_j + \lambda_{ij} + f_{jk} + e_{ijk}.$$

where μ is the average yield, t_i is the average response of the treatment i ; μ and t_i are fixed effects and

ν_j, λ_{ij}, f_{jk} and e_{ijk} are random variables; ν_j being the response of the jth village, λ_{ij} the interaction between treatment and village, and f_{jk} and e_{ijk} done to the experimental errors, e_{ijk} arising from plot to plot variation and f_{jk} arising from field to field variation. In the usual mixed model analysis ν_j and λ_{ij} are considered as independent random variables and the components of variation are calculated on this basis.

The assumption of independence of ν_j and λ_{ij} is not justifiable in all cases. In case of field experiments such an assumption seems to be unrealistic. Thus consider two different regions A and B so that A has a uniformly distributed rainfall and B is arid or rainfall is uneven. It is well known that the response of nitrogeneous fertilizers is much better in A than in B. Thus comparing two treatments (1) no fertilizer and (2) 80 lb. of nitrogen per acre, ν_1 will be higher than ν_2, and the interaction for treatment (2) will be larger than for treatment (1), thus showing that ν_j and λ_{ij} are correlated. Thus the assumption of independence of ν_j and λ_{ij} in the usual mixed model analysis of variance is not justified in the case of experiments in cultivator's fields.

As ν_j and λ_{ij} are not independent, we may write

(8) $$y_{ijk} = m_{ij} + f_{jk} + e_{ijk}$$

where m_{ij} is the true response per plot of the ith treatment in the jth village and f_{jk} and e_{ijk} are the experimental errors which are independent of m_{ij} and are independently and identically distributed with means zero and respective variances σ_f^2 and σ_e^2.

For each pair (i,j), m_{ij} are random variables and since the villages are selected at random we may consider m_{ij} as a random variable depending upon i. For different j, m_{ij} may be considered as independent random variables but for different i, m_{ij} need not be independent. Thus the response of the treatment depends upon the village j so that in some village the yield of all the treatments are likely to be high and in some village the yield of all treatments are likely to be low, or some treatments may give high yield and others low, depending upon the chemicals etc. already present in the soil. However, the effect is not often additive and so we may consider a more general assumption that for each i, m_{ij} has a multivariate normal distribution. A multivariate normal distribution, of course, implies that the possible number of villages is infinite, which is not strictly true in this case, but we may take it as an approximate description of the nature of variation.

3. <u>Definition of Effects and Variance Components.</u> Labeling the village in the population by an index u with the population distribution γ_u, we shall denote the true output of

the ith treatment in the village labeled u by m(i,u).

We define the true mean for the ith treatment to be

(6) $\mu_i = m(i,.) = E\{m(i,u)\}$

where the expected value of m (i,u) has been taken with respect to ρ_u. The general mean is defined as the arithmetic average of μ_i over all the treatments,

(7) $\mu = \mu_. = m(.,.) = E\left\{\dfrac{m(0,u) + 2\sum_{i=0}^{6} m(i,u)}{15}\right\}$

The amount by which this is exceeded by the true mean of the ith treatment is called the main effect of the ith treatment.

(8) $t_i = \mu_i - \mu_. = m(i,.) - m(.,.)$

The true mean for the village labeled u is, then

(9) $\nu(u) = m(.,u) - m(.,.)$

where $m(.,u) = \frac{1}{15}\left\{m(0,u) + 2\sum_{i=0}^{6} m(i,u)\right\}$

and is called the main effect of the village labeled u in the population.

The interaction of the ith treatment and the village labeled u, is defined as

(10) $\lambda_i(u) = m(i,u) - m(i,.) - m(.,u) + m(.,.)$

From (7), (8), (9), and (10), it is easily seen that

$$t_0 + 2\sum_{i=0}^{6} t_i = m(0,.) + 2\sum_{i=0}^{6} m(i,.) - 15m(.,.) = 0$$

$$E(\nu(u)) = 0$$

(11) $$\lambda_0(u) + 2\sum_{i=0}^{6}\lambda_i(u) = 0 \text{ for all } u \text{ and}$$

$$E\{\lambda_i(u)\} = 0 \text{ for all } i.$$

With the above definitions of t_i, ν_j, and λ_{ij}, we may again write

$$y_{ijk} = \mu + t_i + \nu_j + \lambda_{ij} + f_{jk} + e_{ijk}$$

The random effects $\{\nu(u), \lambda_0(u), \ldots, \lambda_6(u)\}$ are not independent, their variances and covariances are the functions of the covariance matrix of the random variables $\{m(i,u)\}$. If the elements of this covariance matrix are

$$\sigma_{ii'} = \mathrm{Cov}\{m(i,u), m(i',u)\},$$

then we may calculate from the definitions of the random effects that

$$\nu(u) = \frac{1}{15}\left\{m(0,u) + 2\sum_{i=0}^{6} m(i,u)\right\},$$

$$\mathrm{Var}(\nu(u)) = \frac{1}{15^2} E\left\{m(0,u) + 2\sum_{i=0}^{6} m(i,u)\right\}^2$$

$$= \frac{1}{15^2}\left\{\sigma_{00} + 4\sum_{i=0}^{6}\sigma_{0i} + 4\sum_{i=0}^{6}\sum_{i'=0}^{6}\sigma_{ii'}\right\}$$

or

(12) $$\mathrm{Var}(\nu(u)) = \sigma_{..}$$

where $\sigma_{..} = \frac{1}{15^2}\left\{\sigma_{00} + 4\sum_{i=0}^{6}\sigma_{0i} + 4\sum_{i=0}^{6}\sum_{i'=0}^{6}\sigma_{ii'}\right.$

Also since $\lambda_i(u) = m(i,u) - m(.,u) - \mu_i + \mu_.$

the value of $\mathrm{Cov}\{\lambda_i(u), \lambda_{i'}(u)\}$ will not depend on the $\{\mu_i\}$, and so in its calculation we may assume that all

$\mu_i = 0$. Then its value is the expectation of

$$[m(i,u)-m(\cdot,u)][m(i',u)-m(\cdot,u)]$$
$$= m(i,u)\ m(i',u)-m(i,u)\ \ m(\cdot,u)-m(\cdot,u)m(i',u)+[m(\cdot,u)]^2$$
$$= m(i,u)\ m(i',u)-m(i,u)\frac{1}{15}\Big[m(o,u)+2\sum_{i''=0}^{6}m(i'',u)\Big]$$
$$-\frac{1}{15}\Big[m(o,u)+2\sum_{i''=0}^{6}m(i'',u)\Big]m(i',u)+\frac{1}{15^2}\Big[m(o,u)+2\sum_{i''=0}^{6}m(i'',u)\Big]^2$$

hence

$$\mathrm{Cov}\{\lambda_i(u),\lambda_{i'}(u)\} = \sigma_{ii'}-\frac{1}{15}\Big(\sigma_{io}+2\sum_{i''=0}^{6}\sigma_{ii''}\Big)-$$
$$\frac{1}{15}\Big(\sigma_{oi'}+2\sum_{i''=0}^{6}\sigma_{i''i'}\Big)+\frac{1}{15^2}\Big(\sigma_{oo}+4\sum_{i''=0}^{6}\sigma_{i''o}+4\sum_{i''=0}^{6}\sum_{i'''=0}^{6}\sigma_{i''i'''}\Big)$$

or

(13) $$\mathrm{Cov}\{\lambda_i(u),\lambda_{i'}(u)\} = \sigma_{ii'}-\sigma_{i\cdot}-\sigma_{\cdot i'}+\sigma_{\cdot\cdot}\ \text{say}$$

where $\sigma_{i\cdot} = \frac{1}{15}\Big(\sigma_{io}+2\sum_{i''=0}^{6}\sigma_{ii''}\Big)$

and $\sigma_{\cdot i'} = \frac{1}{15}\Big(\sigma_{oi'}+2\sum_{i''=0}^{6}\sigma_{i''i'}\Big)$

We note that because of the symmetry of the matrix $(\sigma_{ii'})$,

$\sigma_{ii'} = \sigma_{i'i}$ and $\sigma_{i\cdot} = \sigma_{\cdot i}$

In a similar way we may calculate

$$\mathrm{Cov}\{\nu(u),\lambda_i(u)\} = E\{m(\cdot,u)[m(i,u)-m(\cdot,u)]\}$$
$$= E\{m(\cdot,u)\cdot m(i,u)-[m(\cdot,u)]^2\}$$
$$= \sigma_{i\cdot}-\sigma_{\cdot\cdot}$$

We define

$\sigma_v^{\ 2} = \mathrm{Var}(\nu(u))$ and

(14) $\sigma_\lambda^2 = \frac{1}{6}\sum_{i=0}^{6} \mathrm{Var}\,(\lambda_i(u))$

The quantities σ_ν^2 and σ_λ^2 may be expressed interms of the covariance matrix of $\{m(i,u)\}$ as

$$\sigma_\nu^2 = \sigma_{..}$$

$$\text{and } \sigma_\lambda^2 = \frac{1}{6}\sum_{i=0}^{6}(\sigma_{ii} - 2\,\sigma_{i.} + \sigma_{..})$$

We note that $\sigma_\nu^2 = 0$, if and only if $\nu(u) = 0$ for all u, that is, if and only if the basic vector $m(u) = \{m(o,u), \ldots, m(6,u)\}$ has a degenerate distribution satisfying $m(o,u) + 2\sum_{i=0}^{6} m(i,u) = \text{constant} = 15\mu$. Also $\sigma_\lambda^2 = 0$ if and only if $\mathrm{Var}\{\lambda_i(u)\} = 0$ for all i, or $m(i,u) = m(.,u) + t_i$, that is except for additive constants $\{t_i\}$, the random variables $m(i,u)$ are identical (not just identically distributed).

4. Expection of Mean Squares. Let the treatment effects be denoted as

$0 \rightarrow t_o$ $\quad n_1' \rightarrow t_3$ $\quad n_1'' \rightarrow t_5$

$n_1 \rightarrow t_1$ $\quad n_2' \rightarrow t_4$ $\quad n_2'' \rightarrow t_6$

$n_2 \rightarrow t_2$

A reparameterization of the treatment parameters as given below will result in some simplification in the estimation and testing of treatment effects. Let

$$t_o = \theta_o; \quad t_1+t_2 = 2\theta_1\,, \quad t_3+t_4 = 2\,\theta_2, \quad t_5+t_6 = 2\,\theta_3,$$

$$t_1 - t_2 = 2\varphi_1 \quad ; \quad t_3 - t_4 = 2\varphi_2 \; ; \quad t_5 - t_6 = 2\varphi_3$$

and since $t_o + 2\sum_{i=0}^{6} t_i = 0$

(15) $$3\theta_o + 2(\theta_1 + \theta_2 + \theta_3) = 0.$$

Let T_{ij} denote the sum of yields of the plots in jth village in which the treatment t_i occurs and G be the grand total. Also let F_{jk} denote the total yield for kth field in the jth village and $\sum_{k=1}^{3} F_{jk} = V_j$, the total for the jth village. Further let

$$Q_{oj} = T_{oj} - \frac{1}{5} V_j$$

$$Q_{1j} = (T_{1j} + T_{2j}) - \frac{2}{5}(F_{j1} + F_{j2})$$

$$Q_{2j} = (T_{3j} + T_{4j}) - \frac{2}{5}(F_{j3} + F_{j4})$$

$$Q_{3j} = (T_{5j} + T_{6j}) - \frac{2}{5}(F_{j5} + F_{j6})$$

$$Q_s = \sum_{j=1}^{r} Q_{sj} \; ; \qquad s = 0, 1, 2, 3.$$

$$P_{1j} = T_{1j} - T_{2j}$$

$$P_{2j} = T_{3j} - T_{4j}$$

$$P_{3j} = T_{5j} - T_{6j}$$

and $$P_k = \sum_{j=1}^{r} P_{kj} \; ; \qquad k = 1, 2, 3.$$

The sum of squares due to treatments eliminating blocks and due to treatments x villages in the usual mixed model analysis are, as found by Uttam Chand and Abraham (1957), given respectively by

(16) $\frac{5}{16r} \sum_{s=0}^{3} Q_s^2 + \frac{1}{4r}\sum_{k=1}^{3} P_k^2$ with 6 d.f. and

(17) $\frac{5}{16} \sum_{j=1}^{r} \sum_{s=0}^{3} Q_{sj}^2 + \frac{1}{4}\sum_{j=1}^{r}\sum_{k=1}^{3} P_{kj}^2 - \frac{5}{16r} \sum_{s=0}^{3} Q_s^2 - \frac{1}{4r}\sum_{k=1}^{3} P_k^2$

with 6 (r-1) d.f.

To test for the significance of the interaction and treatment sum of squares under the assumptions made, we shall derive the expected values of the above sums of squares under the model considered.

$$Q_{oj} = T_{oj} - \frac{1}{5} V_j$$

$$= (3\mu + 3\,\theta_o + 3\,\nu_j + 3\,\lambda_{oj} + \sum_{k=1}^{3} f_{jk} + \sum_{k=1}^{3} e_{ojk})$$

$$- \frac{1}{5}(15\mu + 15\,\nu_j + 5 \sum_{k=1}^{3} f_{jk} + \sum_{k=1}^{3} \sum_{i(k)} e_{ijk})$$

$$= 3\,\theta_o + 3\,\lambda_{oj} + \frac{4}{5}\ (\text{sum of 3 e's}) - \frac{1}{5}(\text{sum of 12 e's})$$

$$Q_{1j} = (T_{1j} + T_{2j}) - \frac{2}{5}(F_{j1} + F_{j2})$$

$$= 4\,\theta_1 + 2(\lambda_{1j} + \lambda_{2j}) + (e_{1j1} + e_{1j2} + e_{2j1} + e_{2j2})$$

$$= \frac{2}{5}\Big\{(\theta_0+2\theta_1+2\theta_2) + (\theta_0+2\theta_1+2\theta_3) +$$

$$(\lambda_{0j}+\lambda_{1j}+\lambda_{2j}+\lambda_{3j}+\lambda_{4j}) + (\lambda_{0j}+\lambda_{1j}+\lambda_{2j}+\lambda_{5j}$$

$$+\lambda_{6j}) + (c_{0j1}+c_{1j1}+c_{2j1}+c_{3j1}+c_{4j1})$$

$$+ (c_{0j2}+c_{1j2}+c_{2j2}+c_{5j2}+c_{6j2})\Big\}$$

$$= \frac{1}{5}(16\theta_1 - \theta_0) + \frac{1}{5}(8\lambda_{1j}+8\lambda_{2j} - \lambda_{0j}) +$$

$$\frac{3}{5}(\text{sum of 4 c's}) - \frac{2}{5}(\text{sum of 6 c's})$$

and similar expressions for Q_{2j} and Q_{3j}. Also

$$P_{1j} = T_{1j} - T_{2j}$$
$$= 4\varphi_1 + 2(\lambda_{1j} - \lambda_{2j}) + (c_{1j1} + c_{1j2} - c_{2j1} - c_{2j2})$$

and similar expressions for P_{2j} and P_{3j}.

The term containing θ's and φ's in the expected value of treatment s.s. (16) is

$$\frac{5}{16r}\left(9r^2\theta_0^2 + \frac{r^2}{25}\sum_{i=1}^{3}(16\theta_i - \theta_0)^2\right) + \frac{1}{4r}\cdot 16r^2\sum_{i=1}^{3}\varphi_i^2$$

$$= \frac{5r}{16}\left(9\theta_0^2 + \frac{16^2}{25}\sum_{i=1}^{3}\left(\theta_i - \frac{\theta_0}{16}\right)^2\right) + 4r\sum_{1}^{3}\varphi_i^2$$

$$= \frac{5r}{16}\left(9\theta_0^2 + \frac{16^2}{25}\sum_{i=1}^{3}(\theta_i - \bar{\theta})^2\right) + 4r\sum_{1}^{3}\varphi_i^2$$

$$= \frac{16r}{5}\sum_{i=0}^{3}(\theta_i - \bar{\theta})^2 + 4r\sum_{i=1}^{3}\varphi_i^2$$

The term containing λ in the expected value of (16) is

$$= E\,\frac{5}{16r}\left[9\left(\sum_{j=1}^{r}\lambda_{oj}\right)^2 + \frac{1}{25}\left\{\sum_{j=1}^{r}(8\lambda_{1j}+8\lambda_{2j}-\lambda_{oj})\right\}^2\right.$$

$$+\frac{1}{25}\left\{\sum_{j=1}^{r}(8\lambda_{3j}+8\lambda_{4j}-\lambda_{oj})\right\}^2 + \frac{1}{25}\left\{\sum_{j=1}^{r}(8\lambda_{5j}+$$

$$8\lambda_{6j}-\lambda_{oj})\right\}^2 + \frac{1}{r}\cdot 4\left[\left\{\sum_{j=1}^{r}(\lambda_{1j}-\lambda_{2j})\right\}^2\right.$$

$$\left.\left.\left\{\sum_{j=1}^{r}(\lambda_{3j}-\lambda_{4j})\right\}^2 + \left\{\sum_{j=1}^{r}(\lambda_{5j}-\lambda_{6j})\right\}^2\right]\right.$$

$$= E\,\frac{5}{16r}\left[9(r\lambda_{o.})^2 + \frac{r^2}{25}\left\{(8\lambda_{1.}+8\lambda_{2.}-\lambda_{o.})^2+\right.\right.$$

$$\left.\left.(8\lambda_{3.}+8\lambda_{4.}-\lambda_{o.})^2 + (8\lambda_{5.}+8\lambda_{6.}-\lambda_{o.})^2\right\}\right]+$$

$$\frac{1}{r}\cdot r^2\left[(\lambda_{1.}-\lambda_{2.})^2+(\lambda_{3.}-\lambda_{4.})^2+(\lambda_{5.}-\lambda_{6.})^2\right]$$

$$= E\,\frac{5r}{16}\left[\frac{252}{25}\lambda_{o.}^2 + \frac{64}{25}\sum_{i=1}^{6}\lambda_{i.}^2 + \frac{128}{25}(\lambda_{1.}\lambda_{2.}+\right.$$

$$\left.\lambda_{3.}\lambda_{4.}+\lambda_{5.}\lambda_{6.})\right] + r\left[\sum_{i=1}^{6}\lambda_{i.}^2 - 2(\lambda_{1.}\lambda_{2.}+\right.$$

$$\left.\lambda_{3.}\lambda_{4.}+\lambda_{5.}\lambda_{6.})\right]$$

$$= \frac{63r}{20}E(\lambda_{o.}^2) + \frac{9r}{5}E\sum_{i=1}^{6}(\lambda_{i.}^2) - \frac{2r}{5}E(\lambda_{1.}\lambda_{2.}+$$

$$\lambda_{3.}\lambda_{4.}+\lambda_{5.}\lambda_{6.})$$

$$= \frac{9r}{5}\sum_{i=0}^{6} r^{-1} \quad Var(\lambda_i(u)) + \frac{27r}{20} r^{-1} \; Var(\lambda_o(u))$$

$$+ \frac{2r}{5} r^{-1}\{Cov(\lambda_1(u), \lambda_2(u)) + Cov.(\lambda_3(u), \lambda_4(u)) +$$

$$Cov(\lambda_5(u), \lambda_6(u))\}$$

$$= \frac{9}{5}(6\ \sigma_\lambda^2) + \frac{27}{20} Var(\lambda_o(u)) - \frac{2}{5}\{Cov(\lambda_1(u), \lambda_2(u))$$

$$+ Cov(\lambda_3(u), \lambda_4(u)) + Cov(\lambda_5(u), \lambda_6(u))\}$$

and the error term will give

$$\frac{5}{16r}\left\{\frac{16r}{25}.\ 3\sigma_e^2 + \frac{r}{25}.\ 12\sigma_e^2 + 3\left(\frac{9r}{25}.\ 4\sigma_e^2 +\right.\right.$$

$$\left.\left.\frac{4r}{25}.\ 6\sigma_e^2\right) + \frac{1}{4r}(4r\ \sigma_e^2 \times 3)\right\}$$

$$= 6\ \sigma_e^2 \ .$$

Thus the expectation of mean squares due to treatments

$$\sigma_e^2 + \frac{9}{5}\sigma_\lambda^2 + \frac{9}{40} Var(\lambda_o(u)) - \frac{1}{15}\{Cov(\lambda_1(u), \lambda_2(u)) +$$

$$(18)\quad Cov(\lambda_3(u), \lambda_4(u)) + Cov(\lambda_5(u), \lambda_6(u))\} +$$

$$\frac{6r}{5}\sum_{0}^{3}\frac{(\theta_i - \bar{\theta})^2}{3} + \frac{2r}{3}\sum_{1}^{3}\varphi_i^2$$

The term containing λ is

$$(19)\quad E\left\{\frac{5}{16}\sum_{j=1}^{r}\sum_{s=0}^{3} Q_{sj}^2 + \frac{1}{4}\sum_{j=1}^{r}\sum_{k=1}^{3} P_{kj}^2\right\}$$

is equal to

$$E\,\frac{6}{16}\left[9\sum_{j=1}^{r}\lambda_{oj}^{2}+\frac{1}{25}\sum_{j=1}^{r}\left\{(3\lambda_{1j}+3\lambda_{2j}-\lambda_{oj})^{2}+\right.\right.$$

$$\left.\left.(3\lambda_{3j}+3\lambda_{4j}-\lambda_{oj})^{2}+(3\lambda_{5j}+3\lambda_{6j}-\lambda_{oj})^{2}\right\}\right]+$$

$$\frac{1}{4}\sum_{j=1}^{r}4\left\{(\lambda_{1j}-\lambda_{2j})^{2}+(\lambda_{3j}-\lambda_{4j})^{2}+(\lambda_{5j}-\lambda_{6j})^{2}\right\}$$

$$=\frac{9}{5}\,E\sum_{j=1}^{r}\sum_{i=0}^{6}\lambda_{ij}^{2}+\frac{27}{20}\,E\sum_{j=1}^{r}\lambda_{oj}^{2}-\frac{2}{5}\,E\sum_{j=1}^{r}(\lambda_{1j}\lambda_{2j}+$$

$$\lambda_{3j}\lambda_{4j}+\lambda_{5j}\lambda_{6j})$$

$$=\frac{9r}{5}\cdot 6\sigma_{\lambda}^{2}+\frac{27r}{20}\operatorname{Var}(\lambda_{o}(u))-\frac{2r}{5}\left\{\operatorname{Cov}(\lambda_{1}(u),\lambda_{2}(u))\right.$$

$$\left.+\operatorname{Cov}(\lambda_{3}(u),\lambda_{4}(u))+\operatorname{Cov}(\lambda_{5}(u),\lambda_{6}(u))\right\}$$

Also the term containing θ's and φ's in (19) will be

$$\frac{16r}{5}\sum_{i=0}^{3}(\theta_{i}-\bar{\theta})^{2}+4r\sum_{i=1}^{3}\varphi_{i}^{2}$$

Hence the expectation of the sum of squares due to treatments x villages is

$$\frac{9}{5}(r-1)\,6\,\sigma_{\lambda}^{2}+\frac{27}{20}(r-1)\operatorname{Var}(\lambda_{o}(u))-$$

$$\frac{2}{5}(r-1)\left\{\operatorname{Cov}(\lambda_{1}(u),\lambda_{2}(u))+\operatorname{Cov}(\lambda_{3}(u),\lambda_{4}(u))\right.$$

$$\left.+\operatorname{Cov}(\lambda_{5}(u),\lambda_{6}(u))\right\}+6(r-1)\,\sigma_{o}^{2}$$

and the expectation of mean squares due to the interaction of treatments and villages is

$$(20)\quad \sigma_e^2 + \frac{9}{5}\sigma_\lambda^2 + \frac{9}{40}\,\mathrm{Var}\,(\lambda_0(u)) -$$

$$\frac{1}{15}\left\{ \mathrm{Cov}\,(\lambda_1(u), \lambda_2(u)) + \mathrm{Cov}\,(\lambda_3(u), \lambda_4(u)) + \mathrm{Cov}\,(\lambda_5(u), \lambda_6(u))\right\}$$

The error sum of squares is

$$\sum\sum\sum y_{ijk}^2 - \left\{ \frac{5}{16}\sum_{j=1}^{r}\sum_{s=0}^{3} C_{sj}^2 + \frac{1}{3}\sum_{j=1}^{r}\sum_{k=1}^{3} P_{kj}^2 - \frac{1}{6}\sum_{j=1}^{r}\sum_{k=1}^{3} F_{jk}^2 \right.$$

and its expected value is given by

$$E\Big[\sum_{j=1}^{r}\Big\{18\mu^2 + 3t_0^2 + 2\sum_{i=1}^{6} t_i^2 + 18\nu_j^2 + 3\lambda_{0j}^2 + 2\sum_{i=1}^{6}\lambda_{ij}^2 +$$

$$6\sum_{k=1}^{3} f_{jk}^2 + 18\,e^2\,s\Big\} - \Big\{\frac{16r}{5}\sum_{i=0}^{3}(\theta_i - \bar{\theta})^2 +$$

$$4r\sum_{i=1}^{3}\varphi_i^2 + \frac{9}{5}\sum_{j=1}^{r}\sum_{i=0}^{6}\lambda_{ij}^2 + \frac{27}{20}\sum_{j=1}^{r}\lambda_{0j}^2 -$$

$$\frac{2}{5}\sum_{j=1}^{r}(\lambda_{1j}\lambda_{2j} + \lambda_{3j}\lambda_{4j} + \lambda_{5j}\lambda_{6j})\Big\} - 6r\,\sigma_e^2$$

$$-\frac{1}{6}\sum_{j=1}^{r}\Big\{[6\mu + (\theta_0 + 2\theta_1 + 2\theta_2) + 6\nu_j + (\lambda_{0j} + \lambda_{1j} +$$

$$\lambda_{2j} + \lambda_{3j} + \lambda_{4j}) + 6f_{j1} + 6e'\,s]^2 + [6\mu + (\theta_0 + 2\theta_1 + 2\theta_3)$$

$$+ 6\nu_j + (\lambda_{0j} + \lambda_{1j} + \lambda_{2j} + \lambda_{5j} + \lambda_{6j}) + 6f_{j2} + 6e'\,s]^2 +$$

$$+ \left[5\mu + (\theta_0 + 2\theta_1 + 2\theta_3) + 5\nu_j + (\lambda_{0j} + \lambda_{3j} + \lambda_{4j} + \lambda_{5j} + \lambda_{6j}) + 5f_{j3} + 5\ e's\right]^2 \Big\}\Big]$$

$$= E\Big[\sum_{j=1}^{r}\Big\{3\theta_0^2 + 4\sum_{i=1}^{3}\theta_i^2 + 4\sum_{i=1}^{3}\varphi_i^2 + \lambda_{0j}^2 + 2\sum_{i=0}^{6}\lambda_{ij}^2\Big\} -$$

$$\Big\{\frac{45r}{16}\theta_0^2 + \frac{16r}{5}\sum_{i=1}^{3}\Big(\theta_i - \frac{\theta_0}{16}\Big)^2 + 4r\sum_{i=1}^{3}\varphi_i^2\Big\} -$$

$$\Big\{\frac{9}{5}\sum_{j=1}^{r}\sum_{0}^{6}\lambda_{ij}^2 + \frac{27}{20}\sum_{j=1}^{r}\lambda_{0j}^2 - \frac{2}{5}\sum_{j=1}^{r}(\lambda_{1j}\lambda_{2j} + \lambda_{3j}\lambda_{4j} + \lambda_{5j}\lambda_{6j})\Big\}$$

$$- \frac{1}{5}\sum_{j=1}^{r}\sum_{i=1}^{3}\Big(2\theta_i + \tfrac{1}{2}\theta_0\Big)^2 - \frac{1}{5}\sum_{j=1}^{r}\Big\{\Big(\tfrac{1}{2}\lambda_{0j} + \lambda_{5j} + \lambda_{6j}\Big)^2 +$$

$$\Big(\tfrac{1}{2}\lambda_{0j} + \lambda_{3j} + \lambda_{4j}\Big)^2 + \Big(\tfrac{1}{2}\lambda_{0j} + \lambda_{1j} + \lambda_{2j}\Big)^2\Big\} -$$

$$\frac{1}{5}\sum_{j=1}^{r} 10\,\nu_j\Big(3\lambda_{0j} + 2\sum_{i=1}^{6}\lambda_{ij}\Big)\Big] + 15r\,\sigma_o^2 - 6r\,\sigma_e^2 - 3r\,\sigma_e^2$$

$$= 3r\,\theta_0^2 + 4r\sum_{1}^{3}\varphi_i^2 - \frac{45r}{16}\theta_0^2 - \frac{16r}{5}\sum_{1}^{3}\Big(\theta_i - \frac{\theta_0}{16}\Big)^2 -$$

$$\frac{r}{5}\sum_{1}^{3}\Big(\frac{4\theta_i + \theta_0}{2}\Big)^2 + E\Big[\sum_{j=1}^{r}\Big\{\Big(\lambda_{0j}^2 - \frac{27}{20}\lambda_{0j}^2 + \frac{7}{20}\lambda_{0j}^2\Big)$$

$$+ 2\sum_{i=0}^{6}\lambda_{ij}^2 - \frac{9}{5}\sum_{i=0}^{6}\lambda_{ij}^2 - \frac{1}{5}\sum_{i=0}^{6}\lambda_{ij}^2\Big\} + \sum_{j=1}^{r}\Big\{\frac{2}{5}(\lambda_{1j}\lambda_{2j} +$$

$$\lambda_{3j}\lambda_{4j} + \lambda_{5j}\lambda_{6j}) - \frac{2}{5}(\lambda_{1j}\lambda_{2j} + \lambda_{3j}\lambda_{4j} + \lambda_{5j}\lambda_{6j})\Big\} + 6r\,\sigma_e^2\Big]$$

$$= 6r\,\sigma_e^2 .$$

Hence the expectation of error mean square is σ_e^2.

The nalysis of variance table will thus be given as on the next page.

Analysis of Variance Table:

Source	d.f.	s.s.	Expected value of Mean Square.
Village	r-1	$\frac{1}{15}\sum_{j=1}^{r} V_j^2 - \frac{G^2}{15r}$	
Fields within villages (ignoring treatments)	2 r	$\sum_{j=1}^{r}\left(\sum_{k=1}^{3}\frac{1}{5}F_{jk}^2 - \frac{1}{15}V_j^2\right)$	
Treatments (eliminating blocks)	6	$\frac{5}{16r}\sum_{s=0}^{3} Q_s^2 + \frac{1}{3r}\sum_{k=1}^{3} P_k^2$	$\sigma_e^2 + \frac{2}{5}\sigma_\lambda^2 + \frac{n}{40}\,Var(\lambda_0(u)) - \frac{1}{15}\{Cov(\lambda_1(u),\lambda_2(u)) + Cov(\lambda_3(u),\lambda_4(u)) + Cov(\lambda_5(u),\lambda_6(u))\} + \frac{2r}{5}\sum_0^3\frac{(\theta_{i} - \bar{\theta})^2}{3} + \frac{2r}{3}\sum\varphi_i^2$
Treatments x villages	6(r-1)	$\left[\frac{5}{15}\sum_{j=1}^{r}\sum_{s=0}^{3} Q_{sj}^2 + \frac{1}{3}\sum_{j=1}^{r}\sum_{k=1}^{3} P_{kj}^2\right] - \frac{5}{16r}\sum_{s=0}^{3} Q_s^2 - \frac{1}{3r}\sum_{k=1}^{3} P_k^2$	$\sigma_e^2 + \frac{2}{5}\sigma_\lambda^2 + \frac{2}{40}\,Var(\lambda_0(u)) - \frac{1}{15}\{Cov(\lambda_1(u),\lambda_2(u)) + Cov(\lambda_3(u),\lambda_4(u)) + Cov(\lambda_5(u),\lambda_6(u))\}$
Error.	6r	By difference.	σ_e^2
Total	15r-1	$\sum\sum\sum y_{ijl}^2 - \frac{G^2}{15r}$	

6. Tests of Significance: Now the quantity

$$E(MS)_I - E(MS)_e$$

$$= \frac{9}{5}\sigma_\lambda^2 + \frac{9}{40}Var(\lambda_0(u)) - \frac{1}{15}\{Cov(\lambda_1(u), \lambda_2(u)) + Cov(\lambda_3(u), \lambda_4(u)) + Cov(\lambda_5(u), \lambda_6(u))\}$$

$$\geq \frac{9}{5} \cdot \frac{1}{8}\sum_{i=0}^{6} Var(\lambda_i(u)) + \frac{9}{40}Var(\lambda_0(u)) - \frac{1}{15}\left\{\sqrt{Var(\lambda_1(u))\ Var(\lambda_2(u))} + \sqrt{Var(\lambda_3(u))\ Var(\lambda_4(u))} + \sqrt{Var(\lambda_5(u))\ Var(\lambda_6(u))}\right\}$$

$$= \frac{21}{40}Var(\lambda_0(u)) + \frac{4}{15}\sum_{i=1}^{6} Var(\lambda_i(u)) + \frac{1}{30}\left\{\left[\sqrt{Var(\lambda_1(u))} - \sqrt{Var(\lambda_2(u))}\right]^2 + \left[\sqrt{Var(\lambda_3(u))} - \sqrt{Var(\lambda_4(u))}\right]^2 + \left[\sqrt{Var(\lambda_5(u))} - \sqrt{Var(\lambda_6(u))}\right]^2\right\}$$

is not less than zero, it being zero only when $\sigma_\lambda^2 = 0$. Thus the hypothesis H_λ: $\sigma_\lambda^2 = 0$ may be tested with the statistic $(MS)_I/(MS)_e$ which has, under H_λ, the F - distribution with degrees of freedom 6(r-1) and 6r.

However, the power is not expressible in terms of the central or non-central F-distribution, since $(MS)_I$ is not distributed as a constant times a χ^2 variable when $\sigma_\lambda^2 \neq 0$. When H_λ is false, $(MS)_I$ is distributed as a linear function of χ^2 variables with unequal coefficients and $(MS)_e$ has a

χ^2 distribution and so the power may be calculated using Box's (1954) result. The distribution of $(MS)_I$ can be found and using the following theorem due to Box (1954).

Theorem. If z denotes a column vector of p random variables $z_1, z_2, \ldots, z_p$ having expectations zero and distributed in a multinormal distribution with variance-covariance matrix V, and if $Q = z' M z$ is any real quadratic from of rank $r \leq p$, then Q is distributed like a quantity

$$X = \sum_{j=1}^{r} \lambda_j \chi^2(1)$$

where each χ^2 variate is distributed independently of every other and the λ's are the r real non - zero latent roots of the matrix

$$U = VM$$

Now

$$Q_{oj} = 3m_{oj} + \sum_{k=1}^{3} f_{jk} + \sum_{k=1}^{3} e_{ojk} - \frac{1}{8}\left\{ m_{oj} + 8\sum_{i=0}^{6} m_{ij} + 8 \sum_{k=1}^{3} f_{jk} + 15 \quad e's \right\}$$

$$= \frac{14}{8}(m_{oj} - \bar{m}_{.j}) + \frac{4}{8} \sum_{k=1}^{3} e_{ojk} - \frac{1}{8}(\text{sum of } 18\ e's)$$

$$\text{where } \bar{m}_{.j} = \frac{1}{7} \sum_{i=0}^{6} m_{ij}$$

and since m_{oj} is normally distributed with mean $\bar{m}_{.j}$ and e_{ijk}'s are also normally distributed with means zero, Q_{oj} is distributed normally with mean zero. Also,

$$Q_s = \sum_{j=1}^{r} Q_{sj} \qquad (s = 0, 1, 2, 3,)$$

and
$$P_k = \sum_{j=1}^{r} P_{kj} \qquad (k = 1,2,3,)$$

are distributed normally with mean zero. Thus

$$(21) \quad \frac{1}{6(r-1)}\left[\frac{5}{16}\sum_{j=1}^{r}\sum_{s=0}^{3} Q_{sj}^2 + \frac{1}{4}\sum_{j=1}^{r}\sum_{k=1}^{3} P_{kj}^2 - \frac{5}{16r}\sum_{s=0}^{3} Q_s^2 - \frac{1}{4r}\sum_{k=1}^{3} P_k^2\right]$$

which can be written in the form $z'Nz$ where z' denotes the vector

$$(Q_{01}, \ldots, Q_{0r}, Q_{11}, \ldots, Q_{3r}, P_{11}, \ldots, P_{3r}, Q_0, \ldots, Q_3, P_1, P_2, P_3)$$

and N is a suitable diagonal matrix, is distributed as a linear function of independent χ^2 variates with coefficients in general unequal.

To test the hypothesis H_T that $t_0 = t_1 = \ldots = t_3$, the following two cases may be considered.

Case I. When $\sigma_\lambda^2 = 0$; when the interaction mean square is insignificant we can consider it as a case of Model I analysis of variance and pool it with error mean square and use the usual F - test with this new error mean square for testing H_T. Under the alternative hypothesis we then get a non-central F - distribution. The estimates of t_i and their standard errors can be calculated as in the fixed effect analysis of variance without interaction, it being taken as zero.

<u>Case II.</u> When $\sigma_\lambda^2 \neq 0$: Even though the mean square due to treatments and mean square due to interaction of treatments and villages are independent and have the same expected values under the hypothesis H_T : all $t_i = 0$, their quotient does not, in general, have the F distribution under H_T. A test of H_T based on Hotelling's T^2 may be used as given below.

In general, let I be the number of treatments and we assume now that $r \geq I$. To calculate Hotelling's T^2 statistic for H_T and to make multiple comparisons in case we find it significant, we construct a rectangular table with $(I-1)$ rows and r columns, the entry in the ith row and jth column being

(22) $$d_{ij} = y_{ij.} - y_{Ij.} \;, \quad i = 0, 1, \ldots, I-1.$$

where $y_{ij.}$ denotes the mean per plot of ith treatment in jth village. We compute then the $(I-1)$ means $\{d_{i.}\}$ and the $\frac{1}{2} I (I-1)$ sums of products (which divided by $r(r-1)$, are estimate of the covariances of the $\{d_{i.}\}$)

(23) $$a_{ii'} = \sum_{j=1} (d_{ij} - d_{i.})(d_{i'j} - d_{i'.})$$

The T^2 statistic is (except for a constant factor)

(24) $$I = r(r-I+1)(I-1)^{-1} Q,$$

where Q is the quadratic form

$$Q = \sum_i \sum_{i'} a^{ii'} d_{i.} d_{i'.},$$

where $(a^{ii'})$ is the matrix inverse to $(a_{ii'})$. It is not necessary actually to compute the inverse matrix, since ω may be written in a form given by Rao (1948) in terms of two determinants of order (I-1) calculated from $(c_{ii'})$,

$$Q = \frac{|c_{ii'} + d_{i.}\, d_{i'.}|}{|c_{ii'}|} - 1.$$

The statistic F in (24) has under H_T the F distribution with (I-1) and (r-I+1) d.f., so that if F_α denotes the upper α point of the F distribution with these numbers of d.f. H_T is rejected at the level of significance if and only if $F > F_\alpha$.

The above form of T^2 test appears to lack symmetry, since Ith row plays a distinguished role. It has been shown by Hsu (1938) that if instead of the $\{d_{i.}\}$ any other basis is used for the (I-1) dimensional space spanned by the differences $\{y_{i..} - y_{i'..}\}$, we would obtain the same test.

The power of the T^2 test of H_T may be expressed, as shown by Hsu (1938), in terms of the non-central F - distribution. The statistic F in (24) is distributed as non-central F with (I-1) and (r-I+1) d.f. and non-centrality parameter δ^2, whose value will be given below, that is as

$$(25) \qquad \frac{(r-I+1)\left[(x_1+\delta)^2 + \sum_{\nu=2}^{I-1} x_\nu^2\right]}{(I-1) \sum_{\nu=1}^{r} x_\nu^2}$$

where the x_ν are independently N (0,1). The non-centrality parameter δ^2 has the value

$$\delta^2 = \sum_l \sum_{l'} \alpha^{ll'} \delta_l \delta_{l'}$$

where $\alpha_{ll'} = Cov(d_{l.}, d_{l'.})$

and $\delta_l = \mu_l - \mu_I$

6 Estimates and their standard errors: Now we can find the estimates of treatment effects and the estimate of any treatment differences and estimate their variances. An estimate of the t_o will be given by

$$\hat{t}_o = y_{o..} - y_{...}$$

and an estimate of its variance will be

$$\frac{1}{r(r-1)} \sum_{j=1}^{r} (y_{oj.} - y_{o..} - y_{.j.} + y_{...})^2$$

An estimate of t_i ($i = 1,2,\ldots,6$) will be given by

$$\hat{t}_i = \frac{1}{r} \sum_{j=1}^{r} (y'_{ij.} - y'_{.j.}) = y'_{i..} - y'_{...}$$

where $y'_{ij.}$ denotes the mean taken over the two plots in which the ith treatment occurs in the jth village. An estimate of the variance of $\hat{t}_i$ will be

$$\frac{1}{r(r-1)} \sum_{j=1}^{r} (y'_{ij.} - y'_{i..} - y'_{.j.} + y'_{...})$$

An estimate of the difference $t_i - t_{i'}$ is

$$y'_{i..} - y'_{i'..}, \quad i \neq i' = 1,2,\ldots,6$$

and the estimate of its variance is

$$\frac{1}{r(r-1)} \sum_{j=1}^{r} (y'_{ij.} - y'_{i..} - y'_{i'j.} + y'_{i'..})^2$$

Also an estimate of $t_i - t_0$, $(i = 1,2,...6)$ is

$$(y'_{i'..} - y'_{...}) - (y_{0..} - y_{...})$$

and the estimate of its variance will be

$$\frac{1}{r(r-1)} \sum_{j=1}^{r} (y'_{ij.} - y'_{i..} - y'_{.j.} + y'_{...} - y_{0j.} + y_{0..} + y_{.j.} - y_{...})^2$$

It is thus possible to find the standard error of the difference of the main effects t_i and $t_{i'}$ without calculating the parameters of the multivariate normal distribution. However, when $\sigma_\lambda^2 = 0$, the parameters $\{t_i\}$ are of no value, and we are interested in finding the differences of yields for treatments within the village only. This can be done by considering the three fields withing the village and doing the analysis of variance withing each village separately.

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SUMMARY

In the first part of the thesis, a test, based on the approximate distribution of the quadratic form, for testing the sphericity of a normal population, has been suggested. The distribution of the ration of two quadratic forms is approximately Fisher - Snedecor's F, on which the test is based. Using G.E.P. Box's (1954) results, the exact power of the test may be calculated. Also using the approximate distribution of the ratio of two quadratic forms, the power of the test can be calculated. A table showing the exact and approximate powers is given which shows that they are not very different and the approximation gives sufficiently good results.

Such a test is useful for small sample sizes only, because it is not consistent. A modification of the test has been made for large samples. Several F's are calculated from the sample and the arithmetic mean of their square roots is found, which for large samples, follows a normal distribution with given mean and standard deviation, the values of which are different under the null and alternative hypotheses. The calculation of the power of such a test is shown.

The distribution of the likelihood ratio test criterion, due to Mauchly (1940) for testing the sphericity of a population

has been approximated in a particular case of bivariate normal population when the correlation is zero and the sample size is large. By which we can get the lower bound of the power for any given alternative hypothesis. The power of this test is compared with the power of the proposed normal test in this particular case. The power of the likelihood ratio test is very large. Thus the proposed test for large samples, is not very powerful but it can be easily calculated and no separate tables for the percentage points are necessary, unlike the case of the likelihood ratio test. Also nothing is known about the distribution of the likelihood ratio test criterion under the alternative hypothesis and so its power cannot be calculated.

In the second part of the thesis, the analysis of fertilizer experiments in cultivator's fields has been considered under a mixed model, used by Scheffe (1956) in connection with an industrial experiment. In such experiments the usual assumption of independence of place s effect and the effect of interaction between treatments and places does not seem to be realistic. An so the analysis has been considered under the model in which the affect due to places and the effect due to interaction of treatments and places have not been taken as independent, but their joint distribution, has been assumed to follow a multivariate normal distribution, which seems to be more realistic.

A design given by Uttam Chand and Abraham (1957) for comparing three sources of nitrogen fertilizer has been considered and analysed under the Scheffes' theory. Expectations of different mean squares have been found under the assumed model, which are not same as those given by Uttam Chand & Abraham (1957) under the usual mixed model. The usual F test, for testing the component of variance due to interaction to be zero, is valid but the power of this test is not expressible in terms of the non-central F - distribution, but it can be calculated using Box's (1954) results.

The F test for testing the equality of fixed treatment effects is valid when the variance component due to interaction is zero. If this variance component is not zero, F test can no longer be used, and the over all test for the fixed treatment effects requires Hotelling's T^2. The standard errors of the estimates of treatment effects and of the difference of any two treatment effects has been found without calculating the parameters of the multivariate normal distribution.

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