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ON SOME METHODS OF CONSTRUCTION OF BALANCED TERNARY DESIGNS

By

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CHAPTER - I

INTRODUCTION

When the number of treatments to be tested is large, and the heterogeneity is in one direction, it is known that the adoption of a suitable incomplete block design instead of a randomized block design results in smaller error variation which consequently increases the precision of the experiment. Among the incomplete block designs, the Balanced Incomplete Block (BIB) designs, introduced by Yates (1936) form the most important class of designs. These designs have the property that the variance of any elementary contrast of the form $(\hat{t}_i - \hat{t}_j)$, where $\hat{t}_i$ and $\hat{t}_j$ are any two estimated treatment effects, is a constant.

Bose and Nair (1939), introduced the concept of Partially Balanced Incomplete Block (PBIB) designs. Those designs, though do not allow the estimation of every elementary contrast with the same precision, are usually available with smaller number of replications than the corresponding BIB designs.

The Balanced and the Partially Balanced Incomplete Block designs are binary designs i.e. their incidence matrices contain only two elements, 0 and 1. Tocher (1952) introduced the concept of n-ary designs. Tocher defined a design to be n-ary if its incidence matrix contains elements 0, 1, ..., n-1. The concept of n-ary design was later generalized by Das and Rao (1968). According to them, an n-ary design is one whose incidence matrix contains any n integral elements, $p_1, p_2, \ldots, p_n$ (not necessarily 0, 1, ... , n-1).

Tocher constructed some balanced ternary designs (incidence matrix containing elements 0, 1 and 2) through trial and error. He also observed that the construction of Balanced n-ary designs is in general more complicated than that of Balanced Incomplete Block (BIB) designs.

Das and Rao (1968) gave a general method of construction of Balanced n-ary designs using Balanced Incomplete Block designs. The principle result of Das and Rao is the following: -

If N_1 and N_2 are the incidence matrices of two Balanced Incomplete Block designs such that the matrix multiplication $N_1 N_2$ is compatible, then $N = N_1 N_2$ is the incidence matrix of a Balanced n-ary design.

Dey (1970) used affine α-resolvable, Balanced Incomplete Block design for constructing a class of Balanced Ternary designs. Infact, he proved a stronger result which is as follows: -

If N_1 is the incidence matrix of an affine α-resolvable BIB design and N_2 that of a two associate group divisible design such that the multiplication $N_1 N_2$ is compatible than $N = N_1 N_2$ is the incidence matrix of a Balanced n-ary design.

The method of differences for the construction of incomplete block designs was introduced by Bose (1939). He used this method for the construction of BIB designs. Subsequently, the method of differences was used by various other authors for constructing BIB and PBIB designs. The method of differences has been used for the construction of Balanced Ternary (BT) designs also. Saha (1972) obtained two series of BT design using the method of differences. Saha and Dey (1973) also obtained a series of BT designs and a series of Balanced n-ary design using the method of differences.

SCOPE OF THE PRESENT THESIS

The present thesis is divided into three chapters. In Chapter - II, two methods of Construction of Balanced Ternary Designs are discussed. These designs have been obtained by using the Balanced Incomplete Block (BIB) Designs with $\lambda = 1, 2$.

In Chapter - III, three new series of Balanced Ternary Designs are obtained through the method of differences. The first series is for even number of treatments, while the remaining two series of designs are obtained with the minimum possible block size, viz. 3. Some other difference set solutions have also been obtained through trial and error. A table of all such designs for $V \leqslant 15$ with their difference - set solutions have been appended.

In the last Chapter some Balanced Ternary Designs with equal block sizes but unequal replications of treatments are obtained. The actual layouts (non randomised) of these designs with $V \leq 14$ have been appended.

The efficiency factor of these Balanced Ternary (BT) Designs as compared to Randomised Block Designs (RBD) is also obtained.

CHAPTER - II

CONSTRUCTION OF BALANCED TERNARY DESIGNS THROUGH BALANCED INCOMPLETE BLOCK DESIGNS.

2.1. INTRODUCTION

A ternary design, defined by Tocher (1952), is one whose incidence matrix contains three elements 0, 1 and 2. An equireplicate, proper ternary design with V treatments and B blocks is said to be balanced if the inner product of any two rows of its incidence matrix is a constant say Λ. The definition of ternary designs was slightly modified by Das and Rao (1968). According to them, a ternary design is one whose incidence matrix contains three integral elements (including 0) p_1, p_2, p_3 where p_i's are not necessarily 0, 1 and 2. Systematic methods of construction of Balanced Ternary (BT) designs are given by Das and Rao (1968), Dey (1970), Saha (1972) and Saha and Dey (1973).

In this chapter, we have constructed two series of Balanced Ternary Designs (BTD). These designs have been constructed through Balanced Incomplete Block (BIB) Designs with $\lambda = 1$ and $\lambda = 2$.

2.2. METHOD OF CONSTRUCTION OF BALANCED TERNARY DESIGNS THROUGH BIB DESIGNS WITH $\lambda = 1$.

In a Balanced Incomplete Block (BIB) Design with parameters v, b, r, k, $\lambda = 1$, it is well known that any two blocks are either disjoint or intersect precisely in one treatment. In fact, every block intersects exactly $k (r - 1)$ blocks in one treatment. We may define the following association rule among the block numbers of the BIB design. Any two blocks are first associates if they intersect in one treatment, otherwise they are second associates.

Shrikhande (1952) has shown that the above association rule is an association scheme with two classes. The association scheme is called the Singly Linked Block (SLB) association scheme.

For getting the ternary design, we ''collapse'' each block with its first associates, two by two. This amounts to saying that the incidence matrix, N of the ternary design is obtained as $N = N_1 N_2$ where N_1 is the incidence matrix of the BIB design with $\lambda = 1$ and N_2 is the incidence matrix of the Partially Balanced Incomplete Block (PBIB) design based on the Singly Linked Block association scheme with the following parameters: -

$$v_2 = b, \quad b_2 = \frac{bk(r-1)}{2}, \quad r_2 = k(r-1),$$

$$k_2 = 2, \quad \lambda_1 = 1, \quad \lambda_2 = 0.$$

We now prove the following.

<u>THEOREM 2.1</u>:

The matrix N is the incidence matrix of a Balanced Ternary Design with the following parameters.

$$V = v, \quad B = \frac{bk(r-1)}{2}, \quad R = rk(r-1), \quad K = 2k$$

$$\wedge = (r-1)(2k+1). \qquad (2.2.1)$$

PROOF :- The fact that the design is ternary is obvious. Also, since N is a matrix of order $v \times b_2$ the values of V and B follow. Further, it is easy to see that every column sum of N is 2k and every row sum is $rk(r-1)$ and hence the values of K and R.

We shall now determine, the value of $\wedge$. In every row of N, there are exactly of $k(r-1)$, 2's and $k(r-1)(r-2)$ unities.

Let $f(i, j)'$ denote the frequency of the ordered column vector $(i, j)'$, $i, j = 0, 1, 2$ in any $2 \times B$ sub matrix of N. Then, for any $2 \times B$ sub matrix of N, the following relations are seen to hold:-

$$f(1, 2)' = f(2, 1)' = r - 1,$$

$$f(1, 1)' = (r-1)(k-1) + (r-1)(k-2)$$

$$= (r-1)\left[(k-1) + (k-2)\right]$$

Since $\wedge$ is the inner product of any two rows of N, we obviously have,

$$\wedge = 2(r-1) + 2(r-1) + (r-1)(k-1) + (r-1)(k-2)$$

$$= (r-1)(2k+1).$$

Hence the theorem.

<u>EXAMPLE</u> : - Let us take a Balanced Incomplete Block (BIB) design with parameters as follows:

$$v = 5, \quad b = 10, \quad k = 2, \quad r = 4, \quad \lambda = 1.$$

Then by "collapsing" the blocks of the above design in the manner described above, one can easily show that the matrix N, given below is the incidence matrix of the Balanced Ternary design with parameters

$$V = 5, \quad B = 30, \quad R = 24, \quad K = 4, \quad \Lambda = 15.$$

$$N = \begin{bmatrix} 2&1&2&2&1&1&1&0&1&0&0&0&2&2&1&1&0&1&1&0&0&1&0&0&0&2&1&1&0&0 \\ 1&2&1&1&2&2&0&0&0&1&1&0&0&0&1&0&1&0&0&1&0&1&2&2&1&0&1&0&2&1 \\ 1&0&0&0&1&0&2&1&1&2&1&2&1&1&2&2&0&0&0&0&1&0&1&0&1&0&0&1&1&2 \\ 0&0&1&0&0&1&1&2&2&1&2&1&1&0&0&0&1&2&1&2&1&0&0&1&0&1&2&0&1&0 \\ 0&1&0&1&0&0&0&1&0&0&0&1&0&1&0&1&2&1&2&1&2&2&1&1&2&1&0&2&0&1 \end{bmatrix}$$

2.3 <u>EFFICIENCY FACTOR OF DESIGNS CONSTRUCTED THROUGH BIB DESIGN WITH $\lambda = 1$.</u>

We know that the efficiency factor of the Balanced Ternary (BT) designs as compared to randomised block design is given by

$$E = \frac{\Lambda V}{RK}$$

Now, by putting the values of these parameters from (2.2.1), the efficiency factor comes out to be,

$$E = \frac{v\,(2k+1)}{2k^2 r}$$

2.4. CONSTRUCTION OF BALANCED TERNARY DESIGNS THROUGH THE BALANCED INCOMPLETE BLOCK DESIGNS WITH $\lambda = 2$.

In this section we have constructed the Balanced Ternary designs with the help of the Balanced Incomplete Block Designs with the parameters $v', b', r', k', \lambda' = 2$.

Let there exist BIB design with the parameters $v', b', r', k', \lambda' = 2$. Any block of this design intersects other blocks in at most two treatments. It is easy to see that every block of this design interesects precisely $k(k-1)/2$ blocks in exactly two treatments.

Now, if we "collapse" each block of the BIB design with those blocks which have two treatments common with the given block, we evidently get a ternary design. In fact, we can prove the following result, proceeding on the basis similar to the proof of theorem 2.1.

THEOREM 2.2:

The ternary design so obtained is Balanced and has the following parameters:

$$V = v,\ B = \binom{v}{2},\ K = 2k,\ R = rk(k-1)/2,\ \Lambda = 2k^2 - k - 2 .$$

The efficiency factor of this series of designs is given by

$$E = v(2k^2 - k - 2) / [\, rk^2 (k-1) \,]$$

EXAMPLE: - Let us take a Balanced Incomplete Block design with parameters,

$$v = 6, \; b = 10, \; r = 5, \; k = 3, \; \lambda = 2 .$$

The block contents of the above design can be written as under.

Block No.			
1	1	2	4
2	2	3	5
3	1	3	4
4	2	4	5
5	1	3	5
6	1	2	6
7	2	3	6
8	3	4	6
9	4	5	6
10	1	5	6

Now, the number of blocks which intersects each other precisely in exactly two treatments are,

1	and	3
1	and	6
2	and	4
2	and	5
2	and	7
3	and	4
3	and	5
3	and	8
4	and	9
5	and	10
6	and	7
6	and	10
7	and	8
8	and	9
9	and	10

After "collapsing" these blocks, we get a design whose incidence matrix is shown below: -

2	1	2	0	1	0	0	2	1	0	2	1	2	0	1
1	2	2	2	1	2	1	0	0	1	0	2	1	0	0
1	0	0	1	2	2	2	2	2	0	1	1	0	1	0
2	2	1	1	0	0	1	1	2	2	0	0	0	2	1
0	1	0	2	2	1	0	1	0	2	2	0	1	1	2
0	0	1	0	0	1	2	0	1	1	1	2	2	2	2

This is the incidence matrix of a Balanced Ternary design with parameters

$V = 6,\ B = 15,\ K = 6,\ R = 15,\ \Lambda = 13.$

2.5 SOME REMARKS

Though a large number of Balanced Ternary designs can be constructed using these methods, the number of blocks required by these designs is usually large. One may, therefore, be inclined to ask the question, is it possible to construct Balanced Ternary designs with smaller number of experimental units ? We tackle this question in the next chapter wherein we have provided designs with smaller number of experimental units.

CHAPTER - III

CONSTRUCTION OF BALANCED TERNARY DESIGNS THROUGH THE METHOD OF DIFFERENCES.

3.1 INTRODUCTION

The method of construction of incomplete block designs using the method of differences was developed by Bose (1939). He used this method for the construction of Balanced Incomplete Block (BIB) designs. Subsequently, many others used the method of differences for the construction of Balanced and Partially Balanced Incomplete Block designs.

The method of differences can be profitably used for the construction of Balanced Ternary Designs (BTD) also. Using the method of differences, Saha (1972) and Saha and Dey (1973) obtained some BT designs.

The present chapter is concerned with the construction of some new series of BT designs through the method of differences. Specifically, we construct three new series of BT designs. Some other difference-set solutions have also been obtained through trial and error. A table of all designs for $V \leq 15$ with their difference-set solutions has been appended.

3.2. A SERIES OF BT DESIGNS WITH EVEN NUMBER OF TREATMENTS.

Although, Saha (1972) and Saha and Dey (1973) have given some methods of construction of BT designs using the method of differences, all these designs are available with odd number of treatments only. Difference-set solutions for BT designs for even number of treatments have not been reported anywhere in literature. In this section, we present a method of construction of BT designs with even number of treatments. The solution of this series of designs is based on the method of differences.

For the sake of completeness, we first quote the following result from Saha and Dey (1973), as this result will be required in the sequel.

THEOREM 3.1. :

Let there exist a set of t initial blocks, each consisting of K elements of an additive abelian group of order V, not necessarily all distinct, such that (i) the number of times each element of the group occurs in any of the initial blocks is less than n in each of these initial blocks, and (ii) in the set of all possible $tK(K-1)$ differences (reduced mod V) arising from the t initial blocks, each of the non-zero elements of the group occurs a constant number of times, say μ. Then, by developing these t initial blocks mod V, we get a Balanced n-ary design with the following parameters

$$V,\ B = tV,\ R = tK,\ K,\ \Lambda = \mu .$$

We shall use this result for construction of Balanced Ternary designs.

Let V be of the form $2m+4$, m being a positive integer. We then prove the following.

THEOREM 3.2:

The initial block $(0, 0, 1, 2, \ldots, m, m+3, m+4, \ldots, 2m+2)$ when developed mod $V\ (= 2m+4)$ gives rise to a Balanced Ternary design with the following parameters:

$$V = 2m+4 = B, \quad R = 2m+2 = K, \quad \Lambda = 2m.$$

PROOF: The expressions for V, B, R and K are obvious and need no proof. We therefore, show only the value of Λ.

The non-zero differences arising from the block $(0, 0, 1, 2, \ldots, m, m+3, m+4, \ldots, 2m+2)$ are the same as those obtained from the following sub-initial blocks:

i) $(0, 0, 1, 2, \ldots, m)$

ii) $(0, 0, m+3, m+4, \ldots, 2m+2)$

iii) (i, j), $i = 1, 2, \ldots, m$; $j = m+3, m+4, \ldots, 2m+2$.

The non-zero differences from the sub-initial block (i) can be written as:

$$\pm 1, \pm 2, \pm 3, \pm 4, \ldots, \pm(m-2), \pm(m-1), \pm m$$

$$\pm 1, \pm 2, \pm 3, \pm 4, \ldots, \pm(m-2), \pm(m-1), \pm m$$

$$\pm 1, \pm 2, \pm 3, \ldots, \pm(m-3), \pm(m-2), \pm(m-1)$$

$$\pm 1, \pm 2, \ldots, \pm(m-4), \pm(m-3), \pm(m-2)$$

$$\ldots \quad \pm 1, \quad \pm 2$$

$$\ldots \pm 1.$$

Similarly, the non-zero differences from the sub-initial block (ii) are as under:

$$\begin{array}{l} \pm(m+1),\ \pm m,\ \pm(m-1),\ \ldots,\ \pm 4,\ \pm 3,\ \pm 2 \\ \pm(m+1),\ \pm m,\ \pm(m-1),\ \ldots,\ \pm 4,\ \pm 3,\ \pm 2 \\ \qquad\qquad \pm 1,\ \pm 2,\ \ldots,\ \pm(m-3),\ \pm(m-2),\ \pm(m-1) \\ \qquad\qquad\qquad \pm 1,\ \ldots,\ \pm(m-4),\ \pm(m-3),\ \pm(m-2) \\ \qquad\qquad\qquad\qquad \ddots \qquad \pm 1,\ \pm 2 \\ \qquad\qquad\qquad\qquad\qquad\qquad \pm 1. \end{array}$$

Further, the non-zero differences arising from the sub-initial block (iii) can be written as a symmetric matrix given by

$$\begin{bmatrix} \pm(m+2), & \pm(m+1), & \pm m, & \ldots, & \pm 3 \\ & \pm(m+2), & \pm(m+1), & \ldots, & \pm 4 \\ & & \pm(m+2), & \ldots, & \pm 5 \\ & & & \ddots & \\ & & & & \pm(m+2) \end{bmatrix}$$

One can easily count that among the differences coming from the sub-initial block (i), the elements ± 1 occurs (m+1)-times, ± 2 occurs m-times, ... , $\pm(m-1)$ occurs 3-times, $\pm m$ occurs twice and $\pm(m+1)$ does not occur.

Similarly, among the differences arising from sub-initial block (ii) the element ± 1 occurs $(m-1)$-times, ± 2 occurs m-times, ... , $\pm(m-1)$ occurs 3-times, $\pm m$ occurs twice and $\pm(m+1)$ also occurs twice.

It is further easy to see that among the differences arising from sub-initial block (iii), the elements ± 1 and ± 2 do not occur, while, ± 3 occurs twice, ... , $\pm(m-1)$, occurs $(2m-6)$-times, $\pm m$ occurs $(2m-4)$-times and $\pm(m+1)$ occurs $(2m-2)$-times.

Adding the frequencies of the elements $\pm i$, $i=1, 2, \ldots, m+1$ among the differences arising from all the sub-initial blocks, we find that each such element occurs exactly $2m$-times. Thus, every non-zero element, except $m+2$, occurs among the differences exactly $2m$-times. Also, the elements $\pm(m+2)$ occur among the differences from sub-initial block (iii) exactly m-times. Since $m+2 = -(m+2)$ mod V, the element $m+2$ also occurs $2m$-times. This then completes the proof.

<u>EXAMPLE 1</u>: Let $m = 1$; then $V = 6$ and the initial block is $(0, 0, 1, 4)$. The blocks of the full design are

0, 0, 1, 4
1, 1, 2, 5
2, 2, 3, 0
3, 3, 4, 1
4, 4, 5, 2
5, 5, 0, 3

The other parameters of the design are

$$B = 6, R = 4 = K, \Lambda = 2.$$

EXAMPLE 2 : Let $m = 2$, then $V = 8$ and the initial block is (0, 0, 1, 2, 5, 6). The full design is shown below: -

0, 0, 1, 2, 5, 6
1, 1, 2, 3, 6, 7
2, 2, 3, 4, 7, 0
3, 3, 4, 5, 0, 1
4, 4, 5, 6, 1, 2
5, 5, 6, 7, 2, 3
6, 6, 7, 0, 3, 4
7, 7, 0, 1, 4, 5

The other parameters of the design are

$$B = 8, \quad R = 6 = K, \Lambda = 4 .$$

3.3. BALANCED TERNARY DESIGNS WITH K = 3 .

It is well known that for a Balanced Ternary design, the minimum possible block size is three. It is, therefore, worthwhile to examine whether Balanced Ternary designs with $K = 3$ can be obtained in general and if so, whether solutions of such designs can be obtained through the method of differences. These questions are tackled in this section. In fact, we show that two series of Balanced Ternary designs with $K = 3$ are possible, one for odd number of treatments and the other for even number of treatments.

We prove the following.

THEOREM 3.3 :

If V is odd, then the following $(V-1)/2$ initial blocks (0, 0, i), $i = 1, 2, \ldots, (V-1)/2$ when developed mod V give rise to a Balanced Ternary design with the following parameters:

$$V,\ B = V(V-1)/2,\ R = 3(V-1)/2,\ K = 3,\ \Lambda = 2,\ V \text{ odd}.$$

PROOF : The expressions for V, B, R and K need no explanation. We need only show the value of Λ.

The non-zero differences arising from these $(V-1)/2$ initial blocks are

$$\pm 1,\ \pm 1,\ \pm 2,\ \pm 2,\ \pm 3,\ \pm 3,\ \ldots,\ \pm(V-1)/2,\ \pm(V-1)/2.$$

Now, since $-i = V - i \mod V$, it is clear that among the non-zero differences, every non-zero element of the group occurs exactly twice, giving $\Lambda = 2$.

EXAMPLE 3 : Let $V = 5$. For this case, the initial blocks are (0, 0, 1) and (0, 0, 2). The full design is shown below :

0, 0, 1
1, 1, 2
2, 2, 3
3, 3, 4
4, 4, 0
0, 0, 2
1, 1, 3
2, 2, 4
3, 3, 0
4, 4, 1

The other parameters of the design are

$$B = 10,\ K = 3,\ R = 6,\ \Lambda = 2.$$

For even number of treatments, we can similarly prove the following.

THEOREM 3.4 :

If V is even, then the following (V - 1) initial blocks (0, 0, i), i = 1, 2, ... , V - 1 when developed mod V, give rise to a Balanced Ternary design with the following parameters

$$V, \; B = V(V-1), \; R = 3(V-1), \; K = 3, \; \Lambda = 4, \; V \text{ even.}$$

EXAMPLE 4 : Let V = 6. The initial blocks for this case are (0, 0, 1), (0, 0, 2), (0, 0, 3), (0, 0, 4) and (0, 0, 5). The full design is shown below :

0, 0, 1	0, 0, 2	0, 0, 3	0, 0, 4	0, 0, 5
1, 1, 2	1, 1, 3	1, 1, 4	1, 1, 5	1, 1, 0
2, 2, 3	2, 2, 4	2, 2, 5	2, 2, 0	2, 2, 1
3, 3, 4	3, 3, 5	3, 3, 0	3, 3, 1	3, 3, 2
4, 4, 5	4, 4, 0	4, 4, 1	4, 4, 2	4, 4, 3
5, 5, 0	5, 5, 1	5, 5, 2	5, 5, 3	5, 5, 4

The other parameters of the design are

$$B = 30, \; R = 15, \; K = 3, \; \Lambda = 4 .$$

3.4 LIST OF BALANCED TERNARY DESIGNS OBTAINED THROUGH INITIAL BLOCKS.

In the following table, we give a table of Balanced Ternary designs for $4 \leqslant V \leqslant 15$. In all the cases, a difference - set solution is provided. Though only one solution for each case has been presented, no claim for the uniqueness of the solution is made. Some of the designs presented in the table have been obtained by trial and error (indicated as T.S.). For the sake of completeness, designs reported by Saha (1972) have also been included. The efficiency-factor of each design, computed as ($\Lambda V/RK$) has also been tabulated for each design.

TABLE - 3.1

LIST OF BALANCED TERNARY DESIGNS

V	B	R	K	Λ	E	Solution	Reference
4	12	9	3	4	.59	(0,0,1)(0,0,2)(0,0,3)mod 4	Th. 3.4
5	10	6	3	2	.55	(0,0,1)(0,0,2) mod 5	Th. 3.3
5	10	8	4	5	.78	(0,0,1,2)(0,0,1,3) mod 5	Saha (1972)
6	30	15	3	4	.53	(0,0,1)(0,0,2)(0,0,3) (0,0,4)(0,0,5) mod 6	Th. 3.4
6	6	4	4	2	.75	(0,0,1,4) mod 6	Th. 3.2
7	21	9	3	2	.51	(0,0,1)(0,0,2)(0,0,3)mod 7	Th. 3.3
7	21	12	4	5	.72	(0,0,1,3)(0,0,1,5) (0,0,2,3) mod 7	T.S.
7	7	5	5	3	.84	(0,0,1,2,4) mod 7	Saha (1972)
8	56	21	3	4	.51	(0,0,1)(0,0,2)(0,0,3)(0,0,4) (0,0,5)(0,0,6)(0,0,7) mod 8	Th..3.4
8	8	6	6	4	.88	(0,0,1,2,5,6) mod 8	Th. 3.2
9	36	12	3	2	.50	(0,0,1)(0,0,2)(0,0,3) (0,0,4) mod 9	Th. 3.3
9	36	16	4	5	.70	(0,0,1,8)(0,0,2,6)(0,0,3,4) (0,0,4,7) mod 9	T.S.

contd...

V	B	R	K	$\wedge$	E	Solution	Reference
9	18	12	6	9	.87	$(0,0,x^0,x^2,x^4,x^6)$ $(0,0,x,x^3,x^5,x^7)$ $x \in GF(3^2)$	Saha (1972)
10	90	27	3	4	.50	(0,0,i) mod 10; i = 1, 2, 3, 4, 5, 6, 7, 8, 9	Th. 3.4
10	10	5	5	2	-	non exist.	
10	10	8	8	6	.93	(0,0,1,2,3,6,7,8) mod 10	Th. 3.2
11	55	15	3	2	.50	(0,0,i) mod 11; i=1,2,3,4,5	Th. 3.3
11	22	8	4	2	.68	(0,0,1,3)(0,0,4,6) mod 11	T.S.
11	11	7	7	4	.90	(0,0,1,3,4,5,9) mod 11	Saha (1972)
12	132	33	3	4	.49	(0,0,i) mod 12 ; i = 1,2,3,4,5,6,7,8,9,10,11	Th. 3.4
12	12	10	10	8	.96	(0,0,1,2,3,4,7,8,9,10) mod 12	Th. 3.2
13	78	18	3	2	.48	(0,0,i) mod 13 ; i=1, 2, 3, 4, 5, 6	Th. 3.3
13	26	10	5	3	.78	(0,0,1,3,9)(0,0,2,5,6) mod 13	T.S.
13	26	16	8	9	.91	(0,0,1,3,4,9,10,12) (0,0,2,5,6,7,8,11) mod 13	Saha (1972)
14	182	39	3	4	.47	(0,0,i) mod 14 ; i=1,2,3,4,5,6,7,8,9,10,11,12,13	Th. 3.4
14	14	12	12	10	.97	(0,0,1,2,3,4,5,8,9,10,11,12) mod 14	Th. 3.2
15	105	21	3	2	.47	(0,0,i) mod 15; i=1,2,3,4,5,6,7	Th. 3.3
15	30	12	6	4	.83	(0,0,1,3,5,7)(0,0,4,5,6,12) mod 15	T.S.

T.S. = Trial and Error Solution.

* New Design.

E = Efficiency Factor.

CHAPTER - IV

CONSTRUCTION OF BALANCED TERNARY DESIGNS WITH VARIABLE REPLICATIONS.

4.1. INTRODUCTION

In the previous two chapters, we have given some methods of construction of Balanced Ternary Designs (BTD). These designs are all equireplicate and proper (with equal block sizes). However, in many situations in practice, designs with variable replications of treatments and/or unequal block sizes are desirable as these designs are more flexible in nature. Designs with unequal block sizes are, of course, not suitable for agricultural experiments, as in such experiments, the intrablock variance is dependent on the block size. As such, in the following section we present methods of construction of Balanced designs with variable replications but equal block sizes.

It was proved by Rao (1958) that a necessary and sufficient condition for a design to be balanced i.e. Var ($\hat{t}_i - \hat{t}_j$) to be same for all pairs $i \neq j$ is that all the diagonal elements of matrix C of the reduced intra block normal equations are equal and all the off diagonal elements of C are equal. It has been recently shown by Chakravarty and Dey (1974) that the constancy of C_{ij} ($i \neq j$) is enough to ensure balance, where C_{ij} is the $(i, j)^{th}$ element of the matrix C. Using this result, we have constructed in this chapter, several series of Balanced Ternary Designs with variable replications. The non-randomised layouts of designs upto 14 treatments have been appended.

4.2. SOME PRELIMINARY RESULTS

It is known that the reduced normal equations for estimating treatment effects after eliminating block effects from a two way classification are,

$C\underline{t} = \underline{Q}$ where $\underline{t} = (t_1, \ldots, t_v)'$ denote the column vector of treatment effects, $\underline{Q} = (Q_1, \ldots, Q_v)'$ denotes the column vector of adjusted totals, i.e.,

$$Q_i = T_i - \sum_j \frac{n_{ij} B_j}{n_{.j}}$$, T_i denoting the total of the i-th treatment, B_j the total of the j-th block, n_{ij} is the $(i, j)^{th}$ element of the incidence matrix of the design and $n_{.j}$ is the block size of the j-th block.

C is a matrix of known coefficients given by

$$C_{ii} = n_{i.} - \sum_j \frac{n_{ij}^2}{n_{.j}}, \quad C_{im} = -\sum_j \frac{n_{ij}\, n_{mj}}{n_{.j}}, \quad i \neq m$$

where $n_{i.}$ is the replication of the i-th treatment.

Rao (1958) has shown that a necessary and sufficient condition for a design to be balanced is that,

$C_{ii} = a$ for all $i = 1, \ldots, v$ and

$C_{im} = b$ for all $i \neq m$

Chakravarty and Dey (1974) have proved the following result, which we include here for completeness.

LEMMA 4.1:

A necessary and sufficient condition for balanced is that C_{im} is a constant $\forall\ i \neq m$

PROOF: For proving the above lemma it is sufficient to show that the constancy of C_{im} $(i \neq m)$ implies the constancy of C_{ii}. Now, since the sum of every row of the matrix C is equal to zero, obviously the constancy of C_{im} implies that of C_{ii}. This completes the proof.

A design is said to be Binary if its incidence matrix contains only two elements 0 and 1. We get the following results as corollary to the above lemma.

COROLLARY 4.1:

A Balanced Binary design with equal block sizes is necessarily equireplicate.

PROOF: Let $C_{im} = -b$ for all $i \neq m$. Then from lemma 4.1, it follows that, $C_{ii} = (v-1)\,b$. Also from definition, $C_{ii} = r_i - \frac{r_i}{k}$ where r_i is the number of times the i-th treatment is replicated. Equating the two expressions for C_{ii} we get the required result.

This result was also proved by Rao (1958), using a different technique.

COROLLARY 4.2 :

A proper design is Balanced if and only if the inner product of any two rows of its incidence matrix is constant.

Since proper, Balanced Binary designs with variable replications are not possible, we attempt to construct Balanced Ternary designs with unequal replications.

4.3. CONSTRUCTION OF BALANCED TERNARY DESIGNS WITH VARIABLE REPLICATIONS IN BLOCK SIZE OF 3.

4.3.1. Let there exist a Balanced Incomplete Block (BIB) design with parameters v, b, r, $k = 3$ and $\lambda = 2$ and let N_1 be its $v \times b$ incidence matrix. Construct the matrix N as follows:

$$N = \left[\begin{array}{c:c} N_1 & 2 I_v \\ \hdashline O_{1b} & J_{1v} \end{array}\right], \qquad (4.3.1)$$

where I_v is a unit matrix of order v, $O_{1b} = (0, \ldots, 0)$, and $J_{1v} = (1, 1, \ldots, 1)$. Then we have the following

THEOREM 4.1 :

The matrix N in (4.3.1) is the incidence matrix of a Balanced Ternary design with the following parameters:

$$V = v + 1, \quad B = v + b, \quad k = 3, \quad R_1 = r + 2, \quad R_2 = v$$

$\Lambda = 2$. (R_1 denotes the replication of the first v treatments and R_2 that of the $(v+1)^{th}$ treatment).

PROOF : The result is immediate from corollary 4.2.

It is known that the following BIB designs satisfy $k = 3$, $\lambda = 2$ $(v \leq 15)$

i)	$v = 4$,	$b = 4$,	$r = 3$,	$k = 3$,	$\lambda = 2$
ii)	$v = 6$,	$b = 10$,	$r = 5$,	$k = 3$,	$\lambda = 2$
iii)	$v = 7$,	$b = 14$,	$r = 6$,	$k = 3$,	$\lambda = 2$
iv)	$v = 9$,	$b = 24$,	$r = 8$,	$k = 3$,	$\lambda = 2$
v)	$v = 10$,	$b = 30$,	$r = 9$,	$k = 3$,	$\lambda = 2$
vi)	$v = 12$,	$b = 44$,	$r = 11$,	$k = 3$,	$\lambda = 2$
vii)	$v = 13$,	$b = 52$,	$r = 12$,	$k = 3$,	$\lambda = 2$
viii)	$v = 15$,	$b = 70$,	$r = 14$,	$k = 3$,	$\lambda = 2$

Using the above BIB designs and Theorem 4.1, we get Balanced Ternary designs with two unequal replications of treatments for $V = 5, 7, 8, 10, 11, 13, 14$ and 16 in blocks of size 3.

EXAMPLE 1 : Consider the BIB design with parameters, $v = 4$, $b = 4$, $r = 3$, $k = 3$, $\lambda = 2$. Then the matrix N is the incidence matrix of a Balanced Ternary design with the following parameters,

$$V = 5,\ K = 3,\ B = 8,\ R_1 = 5,\ R_2 = 4,\ \Lambda = 2.$$

where $N = \begin{bmatrix} 1 & 1 & 1 & 0 & 2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 2 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$

4.3.2. In Chapter - III we have constructed two series of equi-replicate Balanced Ternary designs in blocks of size 3. Using both these series of designs we can construct Balanced Ternary design with two unequal replications. The following two theorems can be proved easily.

THEOREM 4.2.:

The matrix N given in (4.3.2) is the incidence matrix of a Balanced Ternary design with the following parameters.

$V = V'+1,\ B = V'(V'+1)/2,\ K = 3,\ R_1 = (3V'+1)/2,$

$R_2 = V',\ \Lambda = 2,\ V'$ Odd ;

$$N = \begin{bmatrix} N_2 & \vdots & 2I_{V'} \\ \cdots & \cdots & \cdots \\ O_{1B'} & \vdots & J_{1V'} \end{bmatrix} \qquad (4.3.2)$$

where N_2 is the incidence matrix of the Balanced Ternary design (BTD) in Theorem 3.3 of Chapter - III and $B' = V'(V'-1)/2$.

EXAMPLE 11. The incidence matrix for the design

$V = 4,\ K = 3,\ B = 6,\ R_1 = 5,\ R_2 = 3,\ \Lambda = 2$ is given below.

$$N = \begin{bmatrix} 2 & 0 & 1 & 2 & 0 & 0 \\ 1 & 2 & 0 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

EXAMPLE - 2 : The incidence matrix of the design $V = 6$, $B = 15$, $R_1 = 8$,, $R_2 = 5$, $K = 3$, $\wedge = 2$ is given below :

$$N = \begin{bmatrix} 2 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 1 & 0 & 2 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 2 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 & 0 & 2 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 1 & 0 & 2 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 & 0 & 1 & 0 & 2 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

THEOREM 4.3 :

The matrix N given in (4.3.3) is the incidence matrix of a Balanced Ternary design with the following parameters,

$V = V'' + 1$, $B = V''(V''+1)$, $K = 3$, $R_1 = 3V''+1$, $R_2 = 2V''$,

$\wedge = 4$, V'' even ;

$$N = \left[\begin{array}{c:c:c} N_3 & 2I_{V''} & 2I_{V''} \\ \hdashline O_{1B''} & J_{1V''} & J_{1V''} \end{array}\right] \qquad (4.3.3)$$

where N_3 is the incidence matrix of the Balanced Ternary design (BTD) in Theorem 3.4 of Chapter III and $B'' = V''(V''-1)$.

4.3.4. Let there exist a BIB design with parameters $v, b, r, k=4$, $\lambda=4$ and let N_4 be the incidence matrix of the design. Then we have the following,

THEOREM 4.4:

The matrix N given by,

$$N = \left[\begin{array}{c:c} N_4 & 2 I_v \\ \hdashline O_{1b} & 2 J_{1v} \end{array}\right] \qquad (4.3.4)$$

is the incidence matrix of a Balanced Ternary design with the following parameters.

$V = v+1$, $B = v+b$, $R_1 = r+2$, $R_2 = 2v$, $K=4$, $\Lambda = 4$.

EXAMPLE: The incidence matrix of the Balanced Ternary design $V=8$, $B=21$, $R_1=10$, $R_2=14$, $K=4$, $\Lambda=4$, is given below:

$$N = \begin{bmatrix}
1&1&1&0&1&0&0&1&1&1&0&1&0&0&2&0&0&0&0&0&0\\
0&1&1&1&0&1&0&0&1&1&1&0&1&0&0&2&0&0&0&0&0\\
0&0&1&1&1&0&1&0&0&1&1&1&0&1&0&0&2&0&0&0&0\\
1&0&0&1&1&1&0&1&0&0&1&1&1&0&0&0&0&2&0&0&0\\
0&1&0&0&1&1&1&0&1&0&0&1&1&1&0&0&0&0&2&0&0\\
1&0&1&0&0&1&1&1&0&1&0&0&1&1&0&0&0&0&0&2&0\\
1&1&0&1&0&0&1&1&1&0&1&0&0&1&0&0&0&0&0&0&2\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&2&2&2&2&2&2&2
\end{bmatrix}$$

4.4. EFFICIENCY FACTOR OF DESIGNS WITH TWO UNEQUAL REPLICATIONS

In this section, we derive the efficiency factor of a balanced design with two unequal replications and constant block size.

Let n_1 treatments be replicated R_1 times each and n_2 treatments be replicated R_2 times each such that $n_1 + n_2 = V$.

If t_i denotes the i-th treatment effect, then the normal equations for estimating the treatment effects are known as:

$$C_{i1} t_1 + C_{i2} t_2 + \dots + C_{ii} t_i + \dots + C_{iv} t_v = Q_i , \qquad i = 1, 2, \dots, v \tag{4.4.1}$$

where C_{ij}'s and Q_i are defined in 4.2.

For the designs under consideration, we have

$$C_{ij} = -\Lambda/K , \quad \text{for all } i \neq j .$$

where is the inner-product of any two rows of the incidence matrix of the design and K, the block-size.

Also, $C_{ii} = -(V-1)\, C_{ij}$

$$= (V-1)\Lambda/K , \text{ for all } i = 1, \dots, v .$$

Thus (4.4.1) reduces to

$$-\frac{\Lambda}{K} \sum_{m \neq i} t_m + \frac{(V-1)\Lambda}{K} t_i = Q_i . \tag{4.4.2}$$

Using the restriction $\sum_i t_i = 0$, we get an estimate of t_i as

$$\hat{t}_i = K\, Q_i /\Lambda V . \tag{4.4.3}$$

Also, $\operatorname{Var}(\hat{t}_i - \hat{t}_j) = 2 K \sigma^2/\Lambda V$, (4.4.4)

where σ^2 is the per observation variance.

Let $r = (n_1 R_1 + n_2 R_2)/V$. i.e. r is the weighted average of R_1 and R_2. Then, if a Randomised Block Design (RBD) with r replications is used, the variance of $\hat{t}_i - \hat{t}_j$ is given by

$$\text{Var}\,(\hat{t}_i - \hat{t}_j)_{RBD} = 2\,\sigma'^2 / r, \tag{4.4.5}$$

where σ'^2 is the per-plot variance in the case of RBD.

Comparing (4.4.4) and (4.4.5), we have the efficiency of the design under consideration w.r.t. RBD as

$$\text{Eff.} = (\wedge V / r K)\;(\sigma'^2 / \sigma^2).$$

Hence the efficiency factor of the design, E is given by

$$E = \frac{\wedge V}{r K} = \frac{\wedge V^2}{K[\,n_1 R_1 + n_2 R_2\,]}.$$

APPENDIX

Balanced Ternary Designs with Variable Replications

1. $V = 4$, $B = 6$, $K = 3$, $R_1 = 5$, $R_2 = 3$, $\wedge = 2$, $E = .59$

2	0	1	2	0	0
1	2	0	0	2	0
0	1	2	0	0	2
0	0	0	1	1	1

2. $V = 5$, $B = 8$, $R_1 = 5$, $R_2 = 4$, $K = 3$, $\wedge = 2$, $E = .69$

1	0	1	1	2	0	0	0
1	1	0	1	0	2	0	0
1	1	1	0	0	0	2	0
0	1	1	1	0	0	0	2
0	0	0	0	1	1	1	1

3. $V = 6$, $B = 15$, $R_1 = 8$, $R_2 = 5$, $K = 3$, $\wedge = 2$, $E = .53$

2	0	0	0	1	2	0	0	1	0	2	0	0	0	0
1	2	0	0	0	0	2	0	0	1	0	2	0	0	0
0	1	2	0	0	1	0	2	0	0	0	0	2	0	0
0	0	1	2	0	0	1	0	2	0	0	0	0	2	0
0	0	0	1	2	0	0	1	0	2	0	0	0	0	2
0	0	0	0	0	0	0	0	0	0	1	1	1	1	1

4. $V = 7$, $B = 16$, $R_1 = 7$, $R_2 = 6$, $K = 3$, $\wedge = 2$, $E = .68$

1	0	0	0	1	1	0	1	0	1	2	0	0	0	0	0
1	1	0	0	0	1	1	0	1	0	0	2	0	0	0	0
0	1	1	0	0	0	1	1	0	1	0	0	2	0	0	0
0	0	1	1	0	1	0	1	1	0	0	0	0	2	0	0
0	0	0	1	1	0	1	0	1	1	0	0	0	0	2	0
1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	2
0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1

5. — $V = 8$, $B = 21$, $R_1 = 8$, $R_2 = 7$, $K = 3$, $\Lambda = 2$, $E = .68$

1	1	0	0	0	0	0	0	1	1	1	1	0	0	2	0	0	0	0	0	0
1	1	1	1	0	0	0	0	0	0	0	0	1	1	0	2	0	0	0	0	0
0	0	1	1	1	1	0	0	0	0	1	1	0	0	0	0	2	0	0	0	0
1	1	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	2	0	0	0
0	0	1	1	0	0	1	1	1	1	0	0	0	0	0	0	0	0	2	0	0
0	0	0	0	1	1	0	0	1	1	0	0	1	1	0	0	0	0	0	2	0
0	0	0	0	0	0	1	1	0	0	1	1	1	1	0	0	0	0	0	0	2
0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1

6. $V = 9$, $B = 72$, $R_1 = 25$, $R_2 = 16$, $K = 3$, $\Lambda = 4$, $E = .50$

2	0	0	0	0	0	0	1	2	0	0	0	0	0	1	0	2	0	0	0	0	1	0	0
1	2	0	0	0	0	0	0	0	2	0	0	0	0	0	1	0	2	0	0	0	0	1	0
0	1	2	0	0	0	0	0	1	0	2	0	0	0	0	0	0	0	2	0	0	0	0	1
0	0	1	2	0	0	0	0	0	1	0	2	0	0	0	0	1	0	0	2	0	0	0	0
0	0	0	1	2	0	0	0	0	0	1	0	2	0	0	0	0	1	0	0	2	0	0	0
0	0	0	0	1	2	0	0	0	0	0	1	0	2	0	0	0	0	1	0	0	2	0	0
0	0	0	0	0	1	2	0	0	0	0	0	1	0	2	0	0	0	0	1	0	0	2	0
0	0	0	0	0	0	1	2	0	0	0	0	0	1	0	2	0	0	0	0	1	0	0	2
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

2	0	0	0	1	0	0	0	2	0	0	1	0	0	0	0	2	0	1	0	0	0	0	0
0	2	0	0	0	1	0	0	0	2	0	0	1	0	0	0	0	2	0	1	0	0	0	0
0	0	2	0	0	0	1	0	0	0	2	0	0	1	0	0	0	0	2	0	1	0	0	0
0	0	0	2	0	0	0	1	0	0	0	2	0	0	1	0	0	0	0	2	0	1	0	0
1	0	0	0	2	0	0	0	0	0	0	0	2	0	0	1	0	0	0	0	2	0	1	0
0	1	0	0	0	2	0	0	1	0	0	0	0	2	0	0	0	0	0	0	0	2	0	1
0	0	1	0	0	0	2	0	0	1	0	0	0	0	2	0	1	0	0	0	0	0	2	0
0	0	0	1	0	0	0	2	0	0	1	0	0	0	0	2	0	1	0	0	0	0	0	2
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

2	1	0	0	0	0	0	0	2	0	0	0	0	0	0	0	2	0	0	0	0	0	0	0
0	2	1	0	0	0	0	0	0	2	0	0	0	0	0	0	0	2	0	0	0	0	0	0
0	0	2	1	0	0	0	0	0		2	0	0	0	0	0	0	0	2	0	0	0	0	0
0	0	0	2	1	0	0	0	0	0	0	2	0	0	0	0	0	0	0	2	0	0	0	0
0	0	0	0	2	1	0	0	0	0	0	0	2	0	0	0	0	0	0	0	2	0	0	0
0	0	0	0	0	2	1	0	0	0	0	0	0	2	0	0	0	0	0	0	0	2	0	0
0	0	0	0	0	0	2	1	0	0	0	0	0	0	2	0	0	0	0	0	0	0	2	0
1	0	0	0	0	0	0	2	0	0	0	0	0	0	0	2	0	0	0	0	0	0	0	2
0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

7. $V = 10$, $B = 33$, $R_1 = 10$, $R_2 = 9$, $K = 3$, $\wedge = 2$, $E = .67$

1	1	0	0	0	0	1	1	0	0	0	0	1	1	0	0	0	0	1	1	0	0	0	0
1	1	0	0	0	0	0	0	1	1	0	0	0	0	1	1	0	0	0	0	1	1	0	0
1	1	0	0	0	0	0	0	0	0	1	1	0	0	0	0	1	1	0	0	0	0	1	1
0	0	1	1	0	0	1	1	0	0	0	0	0	0	1	1	0	0	0	0	0	0	1	1
0	0	1	1	0	0	0	0	1	1	0	0	0	0	0	0	1	1	1	1	0	0	0	0
0	0	1	1	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	1	1	0	0
0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	1	1	0	0	1	1	0	0
0	0	0	0	1	1	0	0	1	1	0	0	1	1	0	0	0	0	0	0	0	0	1	1
0	0	0	0	1	1	0	0	0	0	1	1	0	0	1	1	0	0	1	1	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

2	0	0	0	0	0	0	0	0
0	2	0	0	0	0	0	0	0
0	0	2	0	0	0	0	0	0
0	0	0	2	0	0	0	0	0
0	0	0	0	2	0	0	0	0
0	0	0	0	0	2	0	0	0
0	0	0	0	0	0	2	0	0
0	0	0	0	0	0	0	2	0
0	0	0	0	0	0	0	0	2
1	1	1	1	1	1	1	1	1

8. $V = 11$, $B = 40$, $R_1 = 11$, $R_2 = 10$, $K = 3$, $\wedge = 2$, $E = .67$

1	0	0	1	0	0	0	0	0	0	1	0	1	0	0	0	0	0	1	0
0	1	0	0	1	0	0	0	0	0	0	1	1	1	0	0	0	0	0	1
0	0	1	1	0	1	0	0	0	0	0	0	0	1	1	0	0	0	0	0
1	0	0	0	1	0	1	0	0	0	0	0	1	0	1	1	0	0	0	0
0	1	0	0	0	1	0	1	0	0	0	0	0	1	0	1	1	0	0	0
0	0	1	0	0	0	1	0	1	0	0	0	0	0	1	0	1	1	0	0
1	0	0	0	0	0	0	1	0	1	0	0	0	0	0	1	0	1	1	0
0	1	0	0	0	0	0	0	1	0	1	0	0	0	0	0	1	0	1	1
0	0	1	0	0	0	0	0	0	1	0	1	0	0	0	0	0	1	0	1
0	0	0	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

contd. . .

1	1	0	0	0	1	0	0	0	1	2	0	0	0	0	0	0	0	0	0
0	1	1	0	0	0	1	0	0	0	0	2	0	0	0	0	0	0	0	0
1	0	1	1	0	0	0	1	0	0	0	0	2	0	0	0	0	0	0	0
0	0	0	1	1	0	0	0	1	0	0	0	0	2	0	0	0	0	0	0
0	0	0	0	1	1	0	0	0	1	0	0	0	0	2	0	0	0	0	0
0	1	0	0	0	1	1	0	0	0	0	0	0	0	0	2	0	0	0	0
0	0	1	0	0	0	1	1	0	0	0	0	0	0	0	0	2	0	0	0
0	0	0	1	0	0	0	1	1	0	0	0	0	0	0	0	0	2	0	0
1	0	0	0	1	0	0	0	1	1	0	0	0	0	0	0	0	0	2	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2
0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1

9. $V = 12$, $B = 66$, $R_1 = 17$, $R_2 = 11$, $K = 3$, $\wedge = 2$, $E = .48$

2	0	0	0	0	0	0	0	0	0	1	2	0	0	0	0	0	0	0	0	1	0
1	2	0	0	0	0	0	0	0	0	0	0	2	0	0	0	0	0	0	0	0	1
0	1	2	0	0	0	0	0	0	0	0	1	0	2	0	0	0	0	0	0	0	0
0	0	1	2	0	0	0	0	0	0	0	0	1	0	2	0	0	0	0	0	0	0
0	0	0	1	2	0	0	0	0	0	0	0	0	1	0	2	0	0	0	0	0	0
0	0	0	0	1	2	0	0	0	0	0	0	0	0	1	0	2	0	0	0	0	0
0	0	0	0	0	1	2	0	0	0	0	0	0	0	0	1	0	2	0	0	0	0
0	0	0	0	0	0	1	2	0	0	0	0	0	0	0	0	1	0	2	0	0	0
0	0	0	0	0	0	0	1	2	0	0	0	0	0	0	0	0	1	0	2	0	0
0	0	0	0	0	0	0	0	1	2	0	0	0	0	0	0	0	0	1	0	2	0
0	0	0	0	0	0	0	0	0	1	2	0	0	0	0	0	0	0	0	1	0	2
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

2	0	0	0	0	0	0	0	1	0	0	2	0	0	0	0	0	0	1	0	0	0
0	2	0	0	0	0	0	0	0	1	0	0	2	0	0	0	0	0	0	1	0	0
0	0	2	0	0	0	0	0	0	0	1	0	0	2	0	0	0	0	0	0	1	0
1	0	0	2	0	0	0	0	0	0	0	0	0	0	2	0	0	0	0	0	0	1
0	1	0	0	2	0	0	0	0	0	0	1	0	0	0	2	0	0	0	0	0	0
0	0	1	0	0	2	0	0	0	0	0	0	1	0	0	0	2	0	0	0	0	0
0	0	0	1	0	0	2	0	0	0	0	0	0	1	0	0	0	2	0	0	0	0
0	0	0	0	1	0	0	2	0	0	0	0	0	0	1	0	0	0	2	0	0	0
0	0	0	0	0	1	0	0	2	0	0	0	0	0	0	1	0	0	0	2	0	0
0	0	0	0	0	0	1	0	0	2	0	0	0	0	0	0	1	0	0	0	2	0
0	0	0	0	0	0	0	1	0	0	2	0	0	0	0	0	0	1	0	0	0	2
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

contd...

2	0	0	0	0	0	1	0	0	0	0	2	0	0	0	0	0	0	0	0	0	0
0	2	0	0	0	0	0	1	0	0	0	0	2	0	0	0	0	0	0	0	0	0
0	0	2	0	0	0	0	0	1	0	0	0	0	2	0	0	0	0	0	0	0	0
0	0	0	2	0	0	0	0	0	1	0	0	0	0	2	0	0	0	0	0	0	0
0	0	0	0	2	0	0	0	0	0	1	0	0	0	0	2	0	0	0	0	0	0
1	0	0	0	0	2	0	0	0	0	0	0	0	0	0	0	2	0	0	0	0	0
0	1	0	0	0	0	2	0	0	0	0	0	0	0	0	0	0	2	0	0	0	0
0	0	1	0	0	0	0	2	0	0	0	0	0	0	0	0	0	0	2	0	0	0
0	0	0	1	0	0	0	0	2	0	0	0	0	0	0	0	0	0	0	2	0	0
0	0	0	0	1	0	0	0	0	2	0	0	0	0	0	0	0	0	0	0	2	0
0	0	0	0	0	1	0	0	0	0	2	0	0	0	0	0	0	0	0	0	0	2
0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1

10. $V = 13$, $B = 56$, $R_1 = 13$, $R_2 = 12$, $K = 3$, $\Lambda = 2$, $E = .67$

1	0	0	0	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	1	0	1
0	1	0	0	0	0	0	0	1	0	0	1	1	0	0	0	0	0	0	0	1	0
0	0	1	0	0	0	0	0	0	1	0	0	1	1	0	0	0	0	0	0	0	1
0	0	0	1	0	0	0	0	0	0	1	1	0	1	1	0	0	0	0	0	0	0
1	0	0	0	1	0	0	0	0	0	0	0	1	0	1	1	0	0	0	0	0	0
0	1	0	0	0	1	0	0	0	0	0	0	0	1	0	1	1	0	0	0	0	0
0	0	1	0	0	0	1	0	0	0	0	0	0	0	1	0	1	1	0	0	0	0
0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	1	0	1	1	0	0	0
0	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	1	0	1	1	0	0
0	0	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	1	0	1	1	0
0	0	0	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	1	0	1	1
1	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

1	0	0	0	0	0	1	0	0	0	1	1	0	0	0	0	0	1	0	0	1	0
1	1	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0	1	0	0	1
0	1	1	0	0	0	0	0	1	0	0	1	0	1	0	0	0	0	0	1	0	0
0	0	1	1	0	0	0	0	0	1	0	0	1	0	1	0	0	0	0	0	1	0
0	0	0	1	1	0	0	0	0	0	1	0	0	1	0	1	0	0	0	0	0	1
1	0	0	0	1	1	0	0	0	0	0	1	0	0	1	0	1	0	0	0	0	0
0	1	0	0	0	1	1	0	0	0	0	0	1	0	0	1	0	1	0	0	0	0
0	0	1	0	0	0	1	1	0	0	0	0	0	1	0	0	1	0	1	0	0	0
0	0	0	1	0	0	0	1	1	0	0	0	0	0	1	0	0	1	0	1	0	0
0	0	0	0	1	0	0	0	1	1	0	0	0	0	0	1	0	0	1	0	1	0
0	0	0	0	0	1	0	0	0	1	1	0	0	0	0	0	1	0	0	1	0	1
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

contd...

2	0	0	0	0	0	0	0	0	0	0	0
0	2	0	0	0	0	0	0	0	0	0	0
0	0	2	0	0	0	0	0	0	0	0	0
0	0	0	2	0	0	0	0	0	0	0	0
0	0	0	0	2	0	0	0	0	0	0	0
0	0	0	0	0	2	0	0	0	0	0	0
0	0	0	0	0	0	2	0	0	0	0	0
0	0	0	0	0	0	0	2	0	0	0	0
0	0	0	0	0	0	0	0	2	0	0	0
0	0	0	0	0	0	0	0	0	2	0	0
0	0	0	0	0	0	0	0	0	0	2	0
0	0	0	0	0	0	0	0	0	0	0	2
1	1	1	1	1	1	1	1	1	1	1	1

11. $V = 14$, $B = 65$, $R_1 = 14$, $R_2 = 13$, $K = 3$, $\Lambda = 2$, $E = .67$

1	0	0	0	0	0	0	0	0	1	0	0	1	1	0	0	0	0	0	1	0	0	0	0	1	0
1	1	0	0	0	0	0	0	0	0	1	0	0	0	1	0	0	0	0	0	1	0	0	0	0	1
0	1	1	0	0	0	0	0	0	0	0	1	0	1	0	1	0	0	0	0	0	1	0	0	0	0
0	0	1	1	0	0	0	0	0	0	0	0	1	0	1	0	1	0	0	0	0	0	1	0	0	0
1	0	0	1	1	0	0	0	0	0	0	0	0	0	0	1	0	1	0	0	0	0	0	1	0	0
0	1	0	0	1	1	0	0	0	0	0	0	0	0	0	0	1	0	1	0	0	0	0	0	1	0
0	0	1	0	0	1	1	0	0	0	0	0	0	0	0	0	0	1	0	1	0	0	0	0	0	1
0	0	0	1	0	0	1	1	0	0	0	0	0	1	0	0	0	0	1	0	1	0	0	0	0	0
0	0	0	0	1	0	0	1	1	0	0	0	0	0	1	0	0	0	0	1	0	1	0	0	0	0
0	0	0	0	0	1	0	0	1	1	0	0	0	0	0	1	0	0	0	0	1	0	1	0	0	0
0	0	0	0	0	0	1	0	0	1	1	0	0	0	0	0	1	0	0	0	0	1	0	1	0	0
0	0	0	0	0	0	0	1	0	0	1	1	0	0	0	0	0	1	0	0	0	0	1	0	1	0
0	0	0	0	0	0	0	0	1	0	0	1	1	0	0	0	0	0	1	0	0	0	0	1	0	1
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

1	0	0	0	0	0	0	0	0	1	0	0	1	1	0	0	0	0	0	1	0	0	0	0	1	0
1	1	0	0	0	0	0	0	0	0	1	0	0	0	1	0	0	0	0	0	1	0	0	0	0	1
0	1	1	0	0	0	0	0	0	0	0	1	0	1	0	1	0	0	0	0	0	1	0	0	0	0
0	0	1	1	0	0	0	0	0	0	0	0	1	0	1	0	1	0	0	0	0	0	1	0	0	0
1	0	0	1	1	0	0	0	0	0	0	0	0	0	0	1	0	1	0	0	0	0	0	1	0	0
0	1	0	0	1	1	0	0	0	0	0	0	0	0	0	0	1	0	1	0	0	0	0	0	1	0
0	0	1	0	0	1	1	0	0	0	0	0	0	0	0	0	0	1	0	1	0	0	0	0	0	1
0	0	0	1	0	0	1	1	0	0	0	0	0	1	0	0	0	0	1	0	1	0	0	0	0	0
0	0	0	0	1	0	0	1	1	0	0	0	0	0	1	0	0	0	0	1	0	1	0	0	0	0
0	0	0	0	0	1	0	0	1	1	0	0	0	0	0	1	0	0	0	0	1	0	1	0	0	0
0	0	0	0	0	0	1	0	0	1	1	0	0	0	0	0	1	0	0	0	0	1	0	1	0	0
0	0	0	0	0	0	0	1	0	0	1	1	0	0	0	0	0	1	0	0	0	0	1	0	1	0
0	0	0	0	0	0	0	0	1	0	0	1	1	0	0	0	0	0	1	0	0	0	0	1	0	1
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

contd...

2	0	0	0	0	0	0	0	0	0	0	0	0
0	2	0	0	0	0	0	0	0	0	0	0	0
0	0	2	0	0	0	0	0	0	0	0	0	0
0	0	0	2	0	0	0	0	0	0	0	0	0
0	0	0	0	2	0	0	0	0	0	0	0	0
0	0	0	0	0	2	0	0	0	0	0	0	0
0	0	0	0	0	0	2	0	0	0	0	0	0
0	0	0	0	0	0	0	2	0	0	0	0	0
0	0	0	0	0	0	0	0	2	0	0	0	0
0	0	0	0	0	0	0	0	0	2	0	0	0
0	0	0	0	0	0	0	0	0	0	2	0	0
0	0	0	0	0	0	0	0	0	0	0	2	0
0	0	0	0	0	0	0	0	0	0	0	0	2
1	1	1	1	1	1	1	1	1	1	1	1	1

12. $V = 14$, $B = 65$, $R_1 = 18$, $R_2 = 26$, $K = 4$, $\Lambda = 4$, $E = .75$

1	0	0	0	0	1	0	0	0	0	1	0	1	1	0	0	0	0	1	0	0	0	0	1	0	1
1	1	0	0	0	0	1	0	0	0	0	1	0	1	1	0	0	0	0	1	0	0	0	0	1	0
0	1	1	0	0	0	0	1	0	0	0	0	1	0	1	1	0	0	0	0	1	0	0	0	0	1
1	0	1	1	0	0	0	0	1	0	0	0	0	1	0	1	1	0	0	0	0	1	0	0	0	0
0	1	0	1	1	0	0	0	0	1	0	0	0	0	1	0	1	1	0	0	0	0	1	0	0	0
0	0	1	0	1	1	0	0	0	0	1	0	0	0	0	1	0	1	1	0	0	0	0	1	0	0
0	0	0	1	0	1	1	0	0	0	0	1	0	0	0	0	1	0	1	1	0	0	0	0	1	0
0	0	0	0	1	0	1	1	0	0	0	0	1	0	0	0	0	1	0	1	1	0	0	0	0	1
1	0	0	0	0	1	0	1	1	0	0	0	0	1	0	0	0	0	1	0	1	1	0	0	0	0
0	1	0	0	0	0	1	0	1	1	0	0	0	0	1	0	0	0	0	1	0	1	1	0	0	0
0	0	1	0	0	0	0	1	0	1	1	0	0	0	0	1	0	0	0	0	1	0	1	1	0	0
0	0	0	1	0	0	0	0	1	0	1	1	0	0	0	0	1	0	0	0	0	1	0	1	1	0
0	0	0	0	1	0	0	0	0	1	0	1	1	0	0	0	0	1	0	0	0	0	1	0	1	1
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

1	0	0	0	0	1	0	0	0	0	1	0	1	1	0	0	0	0	1	0	0	0	0	1	0	1
1	1	0	0	0	0	1	0	0	0	0	1	0	1	1	0	0	0	0	1	0	0	0	0	1	0
0	1	1	0	0	0	0	1	0	0	0	0	1	0	1	1	0	0	0	0	1	0	0	0	0	1
1	0	1	1	0	0	0	0	1	0	0	0	0	1	0	1	1	0	0	0	0	1	0	0	0	0
0	1	0	1	1	0	0	0	0	1	0	0	0	0	1	0	1	1	0	0	0	0	1	0	0	0
0	0	1	0	1	1	0	0	0	0	1	0	0	0	0	1	0	1	1	0	0	0	0	1	0	0
0	0	0	1	0	1	1	0	0	0	0	1	0	0	0	0	1	0	1	1	0	0	0	0	1	0
0	0	0	0	1	0	1	1	0	0	0	0	1	0	0	0	0	1	0	1	1	0	0	0	0	1
1	0	0	0	0	1	0	1	1	0	0	0	0	1	0	0	0	0	1	0	1	1	0	0	0	0
0	1	0	0	0	0	1	0	1	1	0	0	0	0	1	0	0	0	0	1	0	1	1	0	0	0
0	0	1	0	0	0	0	1	0	1	1	0	0	0	0	1	0	0	0	0	1	0	1	1	0	0
0	0	0	1	0	0	0	0	1	0	1	1	0	0	0	0	1	0	0	0	0	1	0	1	1	0
0	0	0	0	1	0	0	0	0	1	0	1	1	0	0	0	0	1	0	0	0	0	1	0	1	1
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

contd...

2	0	0	0	0	0	0	0	0	0	0	0	0
0	2	0	0	0	0	0	0	0	0	0	0	0
0	0	2	0	0	0	0	0	0	0	0	0	0
0	0	0	2	0	0	0	0	0	0	0	0	0
0	0	0	0	2	0	0	0	0	0	0	0	0
0	0	0	0	0	2	0	0	0	0	0	0	0
0	0	0	0	0	0	2	0	0	0	0	0	0
0	0	0	0	0	0	0	2	0	0	0	0	0
0	0	0	0	0	0	0	0	2	0	0	0	0
0	0	0	0	0	0	0	0	0	2	0	0	0
0	0	0	0	0	0	0	0	0	0	2	0	0
0	0	0	0	0	0	0	0	0	0	0	2	0
0	0	0	0	0	0	0	0	0	0	0	0	2
2	2	2	2	2	2	2	2	2	2	2	2	2

SUMMARY

The present thesis is concerned with some systematic methods of construction of Balanced Ternary designs.

In Chapter II of the thesis, two methods of construction of Balanced Ternary designs are discussed. These designs have been constructed through Balanced Incomplete Block (BIB) designs with $\lambda = 1, 2$.

In Chapter III, three new series of Balanced Ternary designs are obtained, using the method of differences. The first of these series is for even number of treatments and has special significance, as difference-set solution for even number of varieties are not available in literature. The other two series of designs are obtained with the minimum block size, viz. 3. Some other designs are obtained through trial and error also. A list of designs having 15 or less treatments is appended alongwith their solutions.

In the last chapter, some Balanced Ternary designs with equal block sizes but unequal replications of treatments are obtained. Actual lay-outs (non-randomised) of such designs with 14 or less treatments have been appended.

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