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परियोजना प्रतिवेदन
PROJECT REPORT

लॉजिस्टिक एवं गोम्पर्ट्ज त्रुटि वितरणों के साथ बहुउपादानी परीक्षणों के लिए विश्लेषणात्मक प्रक्रिया
Analytical Procedure for Factorial Experiments with Logistic and Gompertz Error Distributions

हर कदम, हर डगर
किसानों का हमसफर
भारतीय कृषि अनुसंधान परिषद
Agrisearch with a human touch

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आमुख

परीक्षणात्मक अभिकल्पनाओं में प्रसरण के विश्लेषण (ANOVA) प्रक्रिया पारंपरिक रूप से नोर्मलिटी पर आधारित होता है, परन्तु व्यवहार में नॉन-नॉर्मल वितरण नॉर्मल वितरण से अधिक उपयुक्त हो सकते हैं, इसलिए बहुउपादानी परीक्षणों में F-स्टेटिस्टिक्स पर नॉन-नॉर्मलिटी के प्रभाव का अध्ययन करना महत्वपूर्ण हो जाता है। जब कई कारकों के प्रभावों की एक साथ जांच की जाती है, तो कृषि और संबद्ध परीक्षणों में बहुउपादानी परीक्षण प्रायः उत्तम और सर्वाधिक उपयोग किए जाने वाले अभिकल्पना माने जाते हैं। बहुउपादानी परीक्षण, कारकों के बीच अंतःक्रिया के प्रभाव के आंकलन की क्षमता भी प्रदान करते हैं।

वर्तमान अध्ययन नॉन-नॉर्मल परिस्थितियों से निपटने के लिए बहुउपादानी परीक्षणों के विश्लेषणात्मक प्रक्रिया के विकास पर केंद्रित है। यहाँ, दो नॉन-नॉर्मल वितरणों पर विचार किया गया है जिनमें से एक सामान्यीकृत लॉजिस्टिक वितरण और दूसरा गोम्पर्ट्ज वितरण है। इस अध्ययन में संशोधित अधिकतम संभावना आंकलन के सिद्धांत का प्रयोग किया गया है और कुशल आंकलक विकसित किए गए हैं। दोनों परिस्थितियों के अन्तर्गत, जहां त्रुटि लॉजिस्टिक तथा गोम्पर्ट्ज वितरण का पालन करती हैं, के लिए SAS कोड भी विकसित किया गया है। यह उपयोगकर्ताओं/शोधकर्ताओं को एक तत्काल परिकलक प्रदान करने में सहायक होगा।

संस्थान के निदेशक डॉ. राजेंद्र प्रसाद को उनके प्रोत्साहन और अनुसंधान कार्य को सफलतापूर्वक पूरा करने में सभी आवश्यक सुविधायें उपलब्ध कराने के लिए लेखक हार्दिक आभार व्यक्त करता है। परीक्षण अभिकल्पना प्रभाग के अध्यक्ष डॉ. अनिल कुमार, पूर्व प्रभागाध्यक्ष डॉ. सीमा जग्गी एवं प्रभाग के अन्य सभी वैज्ञानिकों से प्राप्त सहयोग के लिए आभार व्यक्त किया जाता है। लेखक अभ्यांतरीन समीक्षक का अत्यंत आभारी है जिनके बहुमूल्य सुझावों ने इस परियोजना रिपोर्ट की विषयवस्तु और प्रस्तुति को बेहतर बनाने में सहायता प्रदान की।

[सुनील कुमार यादव]

PREFACE

Analysis of Variance (ANOVA) procedure in the framework of experimental designs has traditionally been based on assumptions of normality. However, in practical situations non-normal distributions may be more useful than the usual normal distributions. Therefore, it is of great interest to study the effect of non-normality on the F statistics used for testing main and interaction effects in factorial experiments. Factorial experiments are often considered as the best and most used designs in agricultural and allied experiments when the effects of multiple factors are investigated simultaneously. They also provide the estimates of interactions between the factorial effects.

The present study focuses on the development of analytical procedure for the factorial experiments in order to tackle the non-normal situations. Here, two non-normal distributions have been considered out of which one is generalized logistic distribution and another is Gompertz distribution. The theory of modified maximum likelihood estimation has been applied and efficient estimators have been developed. SAS codes have also been developed for analysis of the data under both the situations where error follows logistic and Gompertz distribution. These would be helpful for providing a ready reckoner to the users/researchers.

The author expresses his deep sense of gratitude to Dr. Rajender Parsad, Director, ICAR-IASRI for his support and providing all necessary facilities to carry out the research work successfully. The cooperation received from Dr. Anil Kumar, Head, Division of Design of Experiments, Dr. Seema Jaggi, former Head of Division and other Scientists of the division are thankfully acknowledged. Author is also thankful to the internal referee whose valuable suggestions helped in improving the content and presentation of the report.

[Sunil Kumar Yadav]

विषय
CONTENTS

CHAPTER 1

INTRODUCTION AND BACKGROUND

1.1 Introduction

Statistical methodology is used almost often in practice to analyze data of scientific experiments. For example, an agricultural scientist might wish to examine the effects of different fertilizer on the growth of plants. A geneticist might wish to assess the effects of various mutagens on bacterial cells. An engineer may be interested in tensile strength of different alloy used in bridge construction. In each of these cases, various treatments (fertilizer, mutagenic substance and type of alloy) are examined in a systematic way to see if their effects are same or not. For instance, in case of fertilizer, if the mean height of the plants grown with Fertilizer *A* is significantly larger than that of Fertilizer *B* or *C*, then this suggests that Fertilizer *A* is the best choice.

In the light of a particular set of data, how does one decide if an observed difference is significant, or merely due to sampling fluctuation. Statistical techniques for answering this question are collectively termed as *Experimental Design and Analysis*.

Factorial experiments that were introduced by Fisher (1935) and Yates (1937) are often the best and most used experiments in agricultural and industrial applications, when the effects of multiple factors are investigated simultaneously. Evaluation of equipments and materials, product designs, performance testing, process development, *etc.* are examples where factorial experiments are used.

1.2 Genesis and rationale of the project:

Analysis of variance (ANOVA) procedure in the framework of experimental designs has traditionally been based on assumptions of normality. However in practice non-normal distributions are more prevalent, that is seen in Pearson (1932), Geary (1947), Elveback *et al.* (1970), Huber (1981), Tiku *et al.* (1986) and Senoglu and Tiku (2001). It is therefore of great interest to study the effect of non-normality on the *F* statistics used for testing main and interaction effects in factorial experiments.

Senoglu and Tiku (2001) gave the analysis in the frame work of two way classification model in experimental design, when the error follows the generalized logistic distribution by adopting the procedure of modified maximum likelihood. Applying the modified likelihood estimation procedure he obtained efficient and robust estimators of the parameters, defined F statistics for testing main effects and interaction effects. Also analyzed the Box-Cox data and showed that the methodology developed gives accurate results besides being easy theoretically and computationally.

Large number experiments have been conducted in Agricultural Field Experiments Information System (AFEIS). In AFEIS more than 20% of the experiments conducted do not follow the assumptions of Analysis of Variance and in 10 to 12% of the experiments the assumption of normality is violated. Blindly following statistical procedures without understanding the underlying assumptions may result in misleading or incorrect inference from the statistical analysis.

Thus, it is desirable to extend or modify classical statistical procedures based on normality to include non-normal situations, and even to create entirely new approaches not related to the classical procedures. Some of the alternate methods are being used in an attempt to render the normality assumption less crucial. These are

i) Transformation of data

ii) Non-parametric methods

iii) Employ robust procedures.

One of the ways of handling non-normal data is to invoke Box and Cox transformation (1964) so that the transformed data is normal, at any rate close to it. This may not be the proper solution because according to Bickel and Doksum (1981) all non-normal data cannot be amenable to this transformation. Moreover, it is often difficult to interpret transformed data. So in general we can say that the procedure of transformation is not the appropriate procedure. Also non-parametric methods have been developed for some of the specific experimental situations. Alternate way is to develop the robust procedures for the analysis of data. When the data do not follow the normal distribution, the analysis of data becomes problematic because the normal equations obtained from the log-likelihood function are generally non-linear and so are not solvable as in case of normal distribution. One may use Modified Maximum Likelihood Method of estimation and then based upon this the analysis of variance can be performed.

Senoglu (2005) has investigated the robustness of 2^k factorial experiments when the error follows Weibull error distribution. He developed robust and efficient estimators for the parameters in 2^k factorial design and defined F-statistics based on modified maximum likelihood estimators (MMLE) for testing the main effects and interactions. He showed that these tests have high powers and better robustness properties as compared to the normal theory solutions.

The methodology for factorial experiments with 2 levels when error follows logistic error distribution are developed and it performs well over the usual least square estimator based procedures Yadav (2013).

Under the present investigation, factorial experiments have been considered when error follows non-normal distribution. More specifically, two non-normal distributions have been taken, one is generalized logistic distribution and other is Gompertz distribution. The theory of modified maximum likelihood estimation has been applied and efficient estimators have been developed.

An experimental situation has been considered where error follows the Generalized Logistic distribution. The *pdf* of GL distribution is

$$GL(\mu,\sigma,\theta) = \frac{\theta}{\sigma}\frac{\exp\{-(\frac{x-\mu}{\sigma})\}}{[1+\exp\{-(\frac{x-\mu}{\sigma})\}]^{\theta+1}}$$

where, $-\infty < x < \infty;\ -\infty < \mu < \infty;\quad \theta > 0;\quad \sigma > 0$.

Here, μ is the location parameter σ is the scale parameter and θ is the shape parameter of the distribution.

For θ<1, θ=1, and θ>1, represents negatively skewed, symmetric, and positively skewed distributions, respectively. In particular $GL(\mu,\ \sigma,\ 1) = L(\mu,\ \sigma)$. Indeed, the logistic and normal distributions have a quite similar shape. The logistic distribution and the *S*-shaped pattern of its cumulative distribution function (the logistic function) and quantile function (the logit function) have been extensively used in many different areas. One of the most common applications is in logistic regression, which is used for modelling categorical dependent variables (e.g. yes-no choices or a choice of 3 or 4 possibilities), much as standard linear regression is used for modelling continuous variables (e.g. income or population). Specifically, logistic regression models can be phrased as latent variable models with error variables following a logistic distribution. This phrasing is common in the theory of discrete choice models, where the logistic distribution plays the same role in logistic regression as the normal distribution does in probit regression.

We have also considered the experimental situation when error follows Gompertz distribution. The *pdf* of Gompertz distribution is

$$G(\eta,\alpha) = \eta \exp\left[\alpha x - \frac{\eta}{\alpha}\left\{\exp(\alpha x) - 1\right\}\right]$$

where, $x \geq 0; \quad \eta > 0; \quad \alpha > 0$.

Gompertz distribution is a continuous probability distribution, named after Benjamin Gompertz. The Gompertz distribution is often applied to describe the distribution of adult life spans by demographers and actuaries. The Gompertz distribution is important in describing the pattern of adult deaths Wetterstrand (1981) and Gavrilov and Gavrilova (1991). The Gompertz distribution has received considerable attention from demographers and actuaries. Pollard and Valkovics (1992) were the first to study the Gompertz distribution thoroughly. However, their results are true only in the case when the initial level of mortality is very close to zero. Kunimura (1998) arrived at similar conclusions. They defined the moment generating function of the Gompertz distribution in terms of the incomplete or complete gamma function and their results are either approximate or left in an integral form.

Here is an example in the industry where one can find the use of Gompertz distribution in estimation of the time of dysfunction of the pot required in the smelting process of Aluminum from Alumina using the Hall-Héroult Process, Butler (2011). The entire smelting process (during extraction of Aluminum from Alumina) requires rows of reduction pots, or potlines, be in production 24 hours a day, 365 days a year. It is difficult to stop and start the smelting process because the result is a loss of money, energy, and product. Furthermore, if the temperature of the pots decreases and the molten aluminum hardens, the repair and cleanup is costly and time consuming. Unfortunately, as difficult as it is to change or repair the reduction pots, these pots do not last forever. It is very difficult to estimate when a pot stops working. Being able to estimate when a pot stops performing efficiently would not only save a company time, money, and energy but also reduces the costs to consumers. Four models for the pot survival data were fitted and it was found that the best model for the data is the Gompertz survival distribution.

Similarly, there are many situations where this Gompertz distribution is more appropriate one, specifically in survival analysis. In that case there is a need to develop the analysis procedure for these experiments where the data obtained follows Gompertz distribution.

In literature, robust procedures are available when error follows different forms of non-normal distribution in designed experiments. But no work could be traced for the analysis of factorial experiments in general set-up (like asymmetrical factorial) when error follows the Logistic distribution and Gompertz distributions.

Considering the above mentioned research gap, following objectives were framed:

1.3 Objectives

- To develop analytical procedure for factorial experiments when error follows generalized logistic distribution.
- To develop analytical procedure for factorial experiments when error follows Gompertz distribution.

1.4 Critical review of the technology at national and international levels

Box and Cox (1964) made inferences about the transformation and about the parameters of the linear model by computing the likelihood function and the relevant posterior distribution. The contributions of normality, homoscedasticity and additivity to the transformation are separated. They discussed the relation of the present methods to earlier procedures for finding transformations.

Tiku (1967) and Tiku and Stewart (1977) have developed the theory of modified maximum likelihood estimation (MMLE). The theory of MMLE has an explicit solution of these equations and is asymptotically identical with MLE. It has been shown (Tiku *et al.* (1986)) that modified maximum likelihood estimates (MMLEs) are almost as efficient as maximum likelihood estimates (MLEs).

Tiku (1967, 1968a, 1968b) have examined the estimation of parameters of mean and standard deviation for the censored normal and log non-normal distributions. The estimation of coefficients in a simple regression model with auto-correlated errors is considered. The underlying distribution is assumed to be symmetric, one of Student's t family for illustration. Closed form estimators are obtained and shown to be remarkably efficient and robust. They assumed normality but based on their estimators on censored samples, they showed that the resulting estimators are robust to plausible deviations from normality.

Andrews *et al*. (1972) discussed various robust procedures for the estimation of location, the theory of robust estimation is based on specified properties of specified estimators under specified conditions, he showed the result of a study undertaken to establish the interaction of these three components while Schrader and McKean (1977) studied some robust methods of analysis.

Tiku (1980) obtained the modified maximum likelihood estimates for the censored data and developed the robust procedures.

Tiku, *et al.* (1986) made a detail study of robust estimation when error follows non-normal distribution such as log-normal distribution, logistic distribution, *etc.* or for the censored data.

Tiku and Suresh (1992) have pointed out that the maximum likelihood equations are (under some very general regularity conditions) asymptotically equivalent to the likelihood equations.

Tiku and Kambo (1992) gave the method of estimation for a new family of bivariate non-normal distribution.

Milosevic-Hill (1995) obtained the procedure of analysis of one-way and two-way classified data with equal number of observations per cell by using MML procedure when the errors follow symmetric *t*-family of distributions. The MML estimators Produced are found to be very similar in form to results from classical procedures, and not much more computationally intensive. From the simulation studies he showed that MML technique is remarkably efficient and powerful, even for small sample from a decidedly non-normal distribution.

Vaughan and Tiku (2000) observed that in numerous situations, one deals with a random vector (X, Y)**,** where Y is a consequence of X but not so much the other way round. Often in such situations, X has a non-normal distribution while the conditional distribution of Y given $X = x$ may or may not be normal. So, they assumed the distribution of X to be the extreme value distribution and the conditional distribution of Y to be normal. They derived the MML (modified maximum likelihood) estimators and showed that they are highly efficient. They also developed hypothesis testing procedures.

Senoglu and Tiku (2001) gave the analysis in the frame work of two way classification model in experimental design, when the error follows the generalized logistic distribution by adopting the procedure of modified maximum likelihood. From an application of the modified likelihood estimation he obtained efficient and robust estimators of the parameters, defined F statistics for testing main effects and interaction. Also analyzed the Box-Cox data and showed that the method developed gives accurate results besides being easy theoretically and computationally.

Suresh (2004) considered the estimation of scale and location parameters in the two-parameter exponential distribution using type-II censored sample. They have derived the MML estimators using the approach of Tiku and Suresh (1992). They have compared these estimators with the existing estimators and studied their properties.

Wong and Bian (2005) develop the modified maximum likelihood (MML) estimators for the multiple regression coefficients in linear model with the underlying distribution assumed to be symmetric, one of Student's t family. Their empirical study reveals that the modified maximum likelihood (MML) estimators are more efficient than the Least Square Estimator (LSE) in terms of relative efficiency of one-step-ahead forecast mean square error for small samples.

Senoglu (2005) has investigated the robustness of 2^k factorial designs when the error follows Weibull error distribution. From the methodology of modified likelihood, he has developed robust and efficient estimators for the parameters in 2^k factorial design and defined F statistics based on modified maximum likelihood estimators (MMLE) for testing the main effects and interaction. He showed that they have high powers and better robustness properties as compared to the normal theory solutions.

Ayesen *et al.* (2008) derived modified maximum likelihood estimators and showed that they are robust and considerably more efficient than the least square estimators besides being insensitive to moderate design anomalies.

Kantar and Senoglu (2008) made a comparative study for the location and scale parameters of the Weibull distribution with given shape parameter. Nine parametric estimators of the location and scale parameters of a two-parameter Weibull distribution have been compared in terms of their bias and efficiency in a simulation study. The estimators considered were the maximum likelihood estimators (MLE), moment estimators (ME), generalized spacing estimators (GSE), modified maximum likelihood estimators I (MMLE-I), modified maximum likelihood estimators II (MMLE-II), Tiku's modified maximum likelihood estimators (TMMLE), least-squares estimators (LSE), weighted least-squares estimators (WLSE) and percentile estimators (PCE). The aim of the comparisons was to identify the most efficient estimators among these nine estimators for different shape parameters and sample sizes.

Tiku *et al.* (2008) have developed the procedure for estimation of non-normal bivariate distributions of stochastic variance functions. They showed that data sets in numerous areas of application can be modelled by symmetric bi-variate non-normal distributions. Estimation of parameters in such situations is considered when the mean and variance of one variable is a linear and a positive function of the other variable. This is typically true of bi-variate *t*-distribution. They found the resulting estimators remarkably efficient. Hypothesis testing procedures are developed and shown to be robust and powerful.

Tiku and Akkaya (2010) derived Modified maximum likelihood estimators of the parameters in a second order polynomial regression model. These estimators are shown to be considerably more efficient and robust than the commonly used least squares estimators.

Lal *et al.* (2012) developed the MMLE procedures for the analysis of designs of one-way elimination of heterogeneity, two-way elimination of heterogeneity and for factorial experiments when error follows *t*-family of symmetric distribution. They have developed the estimation and testing procedures using the modified maximum likelihood estimation for all the three kinds of designs.

Yadav (2013) obtained the estimators of the model parameters by using the modified maximum likelihood methodology; proposed new test statistics based on these parameters in case of 2^k factorial experiments when error follows logistic distribution.

1.5 Scope of Present Study

The present study focuses on the development of analytical procedure for the factorial experiments with Generalized logistic distribution and Gompertz error distribution in order to tackle the situations where the error term violated the normality assumptions.

The present investigation would help agricultural scientist, research scholars and students under NARES dealing with factorial experiments where error term follows a non normal distribution specifically generalized logistic distribution and Gomperts error distribution. For easy accessibility by the users, the SAS codes were developed for the analytical procedure which provides a readymade solution to the users.

CHAPTER 2

FACTORIAL EXPERIMENTS WITH LOGISTIC ERROR DITRIBUTION

2.1 Introduction:

Factorial experiments that were introduced by Fisher (1935) and Yates (1937) are often the best and most used designs in agricultural experiments when the effects of multiple factors are investigated simultaneously. They also provide the ability to detect and estimate interactions between the factors. It indicates major trends to determine a direction for further experimentation Box *et al.* (1978), Montgomery (1984) and Hinkelmann & Kempthorne (1994).

Under the assumption of normality and independence of observations, the normal equations obtained from maximum likelihood function are linear and hence solvable. On the other hand when the data do not follow the normal distribution, the equations obtained from maximum likelihood estimation are not linear and so these equations are not easy to handle. In this chapter, it is assumed that the error follows generalized logistic distribution.

2.2 Modified Maximum Likelihood Procedures for 2×3×3 Factorial Experiments

A case has been considered where three factors (say *A*, *B* and *C*), in which factor *A* has 2 levels and factors *B* and *C* have 3 levels (2×3×3 factorial experiments) in unblocked situation. The statistical model for such experiment is

$$y_{ijul} = \mu + \tau_i + \beta_j + \lambda_u + (\tau\beta)_{ij} + (\tau\lambda)_{iu} + (\beta\lambda)_{ju} + (\tau\beta\lambda)_{iju} + e_{ijul} \qquad (2.1)$$
$$(i = 1,\ 2;\ j = 1,\ 2,\ 3\ ;\ u = 1,\ 2,\ 3\ ;\ l = 1, 2, ...,\ n)$$

where y_{ijul} denotes the observation for of i^{th} level of factor *A*, j^{th} level of factor *B*, u^{th} level of factor *C*, $l = 1, \ldots, n$, μ is the overall mean, τ_i is the effect of the i^{th} level of the factor *A*, β_j is the j^{th} effect of factor *B*, λ_u is the effect of the u^{th} level of factor *C*, $(\tau\beta)_{ij}$ is the effect of the interaction between τ_i and β_j and e_{ijul} ~ Logistic Distribution and is a random error component. Without loss of generality, we assume that

$$\sum_i \tau_i = \sum_j \beta_j = \sum_i (\tau\beta)_{ij} = \sum_j (\tau\beta)_{ij} = \sum_j (\tau\lambda)_{iu} = \sum_u (\tau\lambda)_{iu} = \sum_j (\beta\lambda)_{ju} = \sum_u (\beta\lambda)_{ju} =$$
$$\sum_i (\tau\beta\lambda)_{iju} = \sum_j (\tau\beta\lambda)_{iju} = \sum_u (\tau\beta\lambda)_{iju} = 0$$

Furthermore, the factors are considered as fixed and the design is assumed to be completely randomized design. Since the error follows generalized logistic distribution, its functional form is

$$f(e_{ijul}) \propto \frac{\theta}{\sigma}\left[\frac{\exp\left[-\left(\frac{y_{ijul}-\tau^*_{iju}}{\sigma}\right)\right]}{\left[1+\exp\left[-\left(\frac{y_{ijul}-\tau^*_{iju}}{\sigma}\right)\right]\right]^{\theta+1}}\right] \qquad (2.2)$$

where, $\tau^*_{iju} = \mu + \tau_i + \beta_j + \lambda_u + (\tau\beta)_{ij} + (\tau\lambda)_{iu} + (\beta\lambda)_{ju} + (\tau\beta\lambda)_{iju}$

The likelihood function L is

$$L = \left[\frac{\theta}{\sigma}\right]^N \prod_{i=1}^{2}\prod_{j=1}^{3}\prod_{u=1}^{3}\prod_{l=1}^{n}\frac{\exp\left[-\left(\frac{y_{ijul}-\tau^*_{iju}}{\sigma}\right)\right]}{\left[1+\exp\left[-\left(\frac{y_{ijul}-\tau^*_{iju}}{\sigma}\right)\right]\right]^{\theta+1}} \text{ ; where, } N = 18n. \qquad (2.3)$$

Log likelihood function is

$$\log L = \text{Constant} - N\log\sigma + \sum_{i=1}^{2}\sum_{j=1}^{3}\sum_{u=1}^{3}\sum_{l=1}^{n}\left[-\left(\frac{y_{ijul}-\tau^*_{iju}}{\sigma}\right)\right]$$
$$-(\theta+1)\sum_{i=1}^{2}\sum_{j=1}^{3}\sum_{u=1}^{3}\sum_{l=1}^{n}\log\left[1+exp\left[-\left(\frac{y_{ijul}-\tau^*_{iju}}{\sigma}\right)\right]\right] \qquad (2.4)$$

Let $z_{iju(l)} = \frac{y_{iju(l)}-\tau^*_{iju}}{\sigma}$, $\quad \frac{\partial z_{iju(l)}}{\partial\mu} = -\frac{1}{\sigma}$ and $\frac{\partial z_{iju(l)}}{\partial\sigma} = -\frac{z_{iju(l)}}{\sigma}$

$$\log L = \text{Constant} - N\log\sigma + \sum_{i=1}^{2}\sum_{j=1}^{3}\sum_{u=1}^{3}\sum_{l=1}^{n}\left(-z_{ijul}\right) -$$
$$(\theta+1)\sum_{i=1}^{2}\sum_{j=1}^{3}\sum_{u=1}^{3}\sum_{l=1}^{n}\log\left[1+exp\left(-z_{ijul}\right)\right] \qquad (2.5)$$

Let $y_{iju(1)} < y_{iju(2)} < y_{iju(3)} < \ldots < y_{iju(n)}$ be the order statistics of y_{ijul} $(1 < l < n)$. Since complete sum is invariant to ordering then $\sum_{l=1}^{n} h(y_l) = \sum_{l=1}^{n} h(y_{(l)})$ where, h(y) is any function of y. Now,

$$\log L = \text{Constant} - N\log\sigma + \sum_{i=1}^{2}\sum_{j=1}^{3}\sum_{u=1}^{3}\sum_{l=1}^{n}\left(-z_{iju(l)}\right)$$

$$-(\theta+1)\sum_{i=1}^{2}\sum_{j=1}^{3}\sum_{u=1}^{3}\sum_{l=1}^{n}\log\left[1+exp\left(-z_{iju(l)}\right)\right] \qquad (2.6)$$

Differentiating with respect to the parameters and equating to zero, we have

$$\frac{\partial \log L}{\partial \mu} = \frac{18n}{\sigma} - \frac{\theta+1}{\sigma}\sum_{i=1}^{2}\sum_{j-1}^{3}\sum_{u=1}^{3}\sum_{l=1}^{n}\frac{\exp(-z_{iju(l)})}{1+\exp(-z_{iju(l)})} = 0\,,$$

$$\frac{\partial \log L}{\partial \tau_i} = \frac{9n}{\sigma} - \frac{\theta+1}{\sigma}\sum_{j=1}^{3}\sum_{u=1}^{3}\sum_{l=1}^{n}\frac{\exp(-z_{iju(l)})}{1+\exp(-z_{iju(l)})} = 0\,,$$

$$\frac{\partial \log L}{\partial \beta_j} = \frac{6n}{\sigma} - \frac{\theta+1}{\sigma}\sum_{i=1}^{2}\sum_{u=1}^{3}\sum_{l=1}^{n}\frac{\exp(-z_{iju(l)})}{1+\exp(-z_{iju(l)})} = 0\,,$$

$$\frac{\partial \log L}{\partial \lambda_u} = \frac{6n}{\sigma} - \frac{\theta+1}{\sigma}\sum_{i=1}^{2}\sum_{j=1}^{3}\sum_{l=1}^{n}\frac{\exp(-z_{iju(l)})}{1+\exp(-z_{iju(l)})} = 0\,,$$

$$\frac{\partial \log L}{\partial (\tau\beta)_{ij}} = \frac{3n}{\sigma} - \frac{\theta+1}{\sigma}\sum_{u=1}^{3}\sum_{l=1}^{n}\frac{\exp(-z_{iju(l)})}{1+\exp(-z_{iju(l)})} = 0\,,$$

$$\frac{\partial \log L}{\partial (\tau\lambda)_{iu}} = \frac{3n}{\sigma} - \frac{\theta+1}{\sigma}\sum_{j=1}^{3}\sum_{l=1}^{n}\frac{\exp(-z_{iju(l)})}{1+\exp(-z_{iju(l)})} = 0\,,$$

$$\frac{\partial \log L}{\partial (\beta\lambda)_{ju}} = \frac{2n}{\sigma} - \frac{\theta+1}{\sigma}\sum_{i=1}^{2}\sum_{l=1}^{n}\frac{\exp(-z_{iju(l)})}{1+\exp(-z_{iju(l)})} = 0\,,$$

$$\frac{\partial \log L}{\partial (\alpha\beta\lambda)_{iju}} = \frac{n}{\sigma} - \frac{\theta+1}{\sigma}\sum_{l=1}^{n}\frac{\exp(-z_{iju(l)})}{1+\exp(-z_{iju(l)})} = 0\,,$$

$$\frac{\partial \log L}{\partial \sigma} = -\frac{18n}{\sigma} - \frac{1}{\sigma}\sum_{i=1}^{2}\sum_{j-1}^{3}\sum_{u=1}^{3}\sum_{l=1}^{n} z_{iju(l)} - \frac{\theta+1}{\sigma}\sum_{i=1}^{2}\sum_{j-1}^{3}\sum_{u=1}^{3}\sum_{l=1}^{n} z_{iju(l)}\frac{\exp(-z_{iju(l)})}{1+\exp(-z_{iju(l)})} = 0 \qquad (2.7)$$

The equations obtained from the first derivative of log likelihood function with respect to parameters do not yield the explicit solutions for the estimates due to non-linearity of the function. Solving them by iterations is indeed problematic for reasons of (i) multiple roots, (ii) non-convergence of iterations, and (iii) convergence to wrong values; see, for example, Smith (1985), Puthenpura and Sinha (1986) and Vaughan (1992, 2002). To alleviate these problems modified likelihood equations are used by linearizing the intractable terms (in likelihood equations). The resulting equations have explicit solutions called MML estimators. The MMLE are known to be asymptotically fully efficient under regularity conditions Bhattacharyya, (1985); Vaughan and Tiku, (2000).

2.3 Parameter Estimation

For obtaining the solutions, the method given by Tiku and Suresh (1992) by expanding log-likelihood function using Taylor series is applied for attaining the estimates of parameters. These solutions are called modified maximum likelihood estimates.

$$\text{Let} \quad \frac{\exp(-z_{iju(l)})}{1+\exp(-z_{iju(l)})} = g(z_{iju(l)})$$

$$\frac{\partial \log L}{\partial \mu} = \frac{18n}{\sigma} - \frac{\theta+1}{\sigma}\sum_{i=1}^{2}\sum_{j-1}^{3}\sum_{u=1}^{3}\sum_{l=1}^{n} g(z_{iju(l)}) = 0 \tag{2.8}$$

$$\frac{\partial \log L}{\partial \tau_i} = \frac{9n}{\sigma} - \frac{\theta+1}{\sigma}\sum_{j=1}^{3}\sum_{u=1}^{3}\sum_{l=1}^{n} g(z_{iju(l)}) = 0 \tag{2.9}$$

$$\frac{\partial \log L}{\partial \beta_j} = \frac{6n}{\sigma} - \frac{\theta+1}{\sigma}\sum_{i=1}^{2}\sum_{u=1}^{3}\sum_{l=1}^{n} g(z_{iju(l)}) = 0 \tag{2.10}$$

$$\frac{\partial \log L}{\partial \lambda_u} = \frac{6n}{\sigma} - \frac{\theta+1}{\sigma}\sum_{i=1}^{2}\sum_{j=1}^{3}\sum_{l=1}^{n} g(z_{iju(l)}) = 0 \tag{2.11}$$

$$\frac{\partial \log L}{\partial (\tau\beta)_{ij}} = \frac{3n}{\sigma} - \frac{\theta+1}{\sigma}\sum_{u=1}^{3}\sum_{l=1}^{n} g(z_{iju(l)}) = 0 \tag{2.12}$$

$$\frac{\partial \log L}{\partial (\tau\lambda)_{iu}} = \frac{3n}{\sigma} - \frac{\theta+1}{\sigma}\sum_{j=1}^{3}\sum_{l=1}^{n} g(z_{iju(l)}) = 0 \tag{2.13}$$

$$\frac{\partial \log L}{\partial (\beta\lambda)_{ju}} = \frac{2n}{\sigma} - \frac{\theta+1}{\sigma}\sum_{i=1}^{2}\sum_{l=1}^{n} g(z_{iju(l)}) = 0 \tag{2.14}$$

$$\frac{\partial \log L}{\partial(\alpha\beta\lambda)_{iju}} = \frac{n}{\sigma} - \frac{\theta+1}{\sigma}\sum_{l=1}^{n} g(z_{iju(l)}) = 0 \tag{2.15}$$

$$\frac{\partial \log L}{\partial \sigma} = -\frac{18n}{\sigma} - \frac{1}{\sigma}\sum_{i=1}^{2}\sum_{j-1}^{3}\sum_{u=1}^{3}\sum_{l=1}^{n} z_{iju(l)} - \frac{\theta+1}{\sigma}\sum_{i=1}^{2}\sum_{j-1}^{3}\sum_{u=1}^{3}\sum_{l=1}^{n} z_{iju(l)}\, g(z_{iju(l)}) = 0 \tag{2.16}$$

For large n, $z_{iju(l)}$ is close to its expected value. Let $t_{(l)}=E(z_{iju(l)})$; $1\leq l \leq n$. The first two terms of the Taylor Series expansion namely,

$$g(z_{iju(l)}) = \gamma_{(l)} + \delta_{(l)} z_{iju(l)}$$

$$g(z_{iju(l)}) = g(t_{(l)}) + [z_{iju(l)} - t_{(l)}]\; g'(t_{(l)})$$

$$= \frac{\exp(-t_{(l)})}{1+\exp(-t_{(l)})} + (z_{iju(l)} - t_{(l)})\frac{-\exp(-t_{(l)})}{\left[1+\exp(-t_{(l)})\right]^2}$$

$$= \frac{\exp(-t_{(l)})}{1+\exp(-t_{(l)})} + \frac{t_{(l)}\exp(-t_{(l)})}{\left[1+\exp(-t_{(l)})\right]^2} - \frac{\exp(-t_{(l)})}{\left[1+\exp(-t_{(l)})\right]^2} z_{iju(l)}$$

$$= \frac{[1+t_{(l)}+\exp(-t_{(l)})]\exp(-t_{(l)})}{\left[1+\exp(-t_{(l)})\right]^2} - \frac{\exp(-t_{(l)})}{\left[1+\exp(-t_{(l)})\right]^2} z_{iju(l)}$$

Multiplying $\exp(2t_{(l)})$ in numerator and denominator, we have

$$g(z_{iju(l)}) = \frac{[1+t_{(l)}\exp(t_{(l)})+\exp(t_{(l)})]}{\left[1+\exp(t_{(l)})\right]^2} - \frac{\exp(t_{(l)})}{\left[1+\exp(t_{(l)})\right]^2} z_{iju(l)} \tag{2.17}$$

Although table of the expected values of $t_{iju(l)}$, $1\leq l \leq n$ are available for n <15 [Balakrishnan and Leung(1988)] but, for convenience, we use their approximate values of $t_{ijk(l)}$ obtained from the equations

$$\int_{-\infty}^{t_{ijk(l)}} \frac{\theta\exp(-z)}{(1+\exp(-z))^{\theta+1}}dz = \frac{l}{n+1},\; t_{iju(l)} = -\ln\left(\left(\frac{l}{n+1}\right)^{-1/\theta} - 1\right) \tag{2.18}$$

Using approximate values instead of exact values does not adversely affect the efficiency of the MML estimators. [Senoglu and Tiku (2001)]

Putting the approximate value of g($z_{iju(l)}$) from equation in the normal equations (2.8) – (2.16) we have,

Thus,

$$g(z_{iju(l)}) \cong \gamma_{(l)} + \delta_{(l)} z_{iju(l)}$$

Hence,

$$\frac{\partial \log L}{\partial \mu} = \frac{18n}{\sigma} - \frac{\theta+1}{\sigma} \sum_{i=1}^{2} \sum_{j-1}^{3} \sum_{u=1}^{3} \sum_{l=1}^{n} \left[\gamma_{(l)} + \delta_{(l)} \left(\frac{y_{iju(l)} - \tau_{iju}^{*}}{\sigma} \right) \right] = 0,$$

$$\frac{\partial \log L}{\partial \tau_i} = \frac{9n}{\sigma} - \frac{\theta+1}{\sigma} \sum_{j=1}^{3} \sum_{u=1}^{3} \sum_{l=1}^{n} \left[\gamma_{(l)} + \delta_{(l)} \left(\frac{y_{iju(l)} - \tau_{iju}^{*}}{\sigma} \right) \right] = 0,$$

$$\frac{\partial \log L}{\partial \beta_j} = \frac{6n}{\sigma} - \frac{\theta+1}{\sigma} \sum_{i=1}^{2} \sum_{u=1}^{3} \sum_{l=1}^{n} \left[\gamma_{(l)} + \delta_{(l)} \left(\frac{y_{iju(l)} - \tau_{iju}^{*}}{\sigma} \right) \right] = 0,$$

$$\frac{\partial \log L}{\partial \lambda_u} = \frac{6n}{\sigma} - \frac{\theta+1}{\sigma} \sum_{i=1}^{2} \sum_{j=1}^{3} \sum_{l=1}^{n} \left[\gamma_{(l)} + \delta_{(l)} \left(\frac{y_{iju(l)} - \tau_{iju}^{*}}{\sigma} \right) \right] = 0,$$

$$\frac{\partial \log L}{\partial (\tau\beta)_{ij}} = \frac{3n}{\sigma} - \frac{\theta+1}{\sigma} \sum_{u=1}^{3} \sum_{l=1}^{n} \left[\gamma_{(l)} + \delta_{(l)} \left(\frac{y_{iju(l)} - \tau_{iju}^{*}}{\sigma} \right) \right] = 0,$$

$$\frac{\partial \log L}{\partial (\tau\lambda)_{iu}} = \frac{3n}{\sigma} - \frac{\theta+1}{\sigma} \sum_{j=1}^{3} \sum_{l=1}^{n} \left[\gamma_{(l)} + \delta_{(l)} \left(\frac{y_{iju(l)} - \tau_{iju}^{*}}{\sigma} \right) \right] = 0,$$

$$\frac{\partial \log L}{\partial (\beta\lambda)_{ju}} = \frac{2n}{\sigma} - \frac{\theta+1}{\sigma} \sum_{i=1}^{2} \sum_{l=1}^{n} \left[\gamma_{(l)} + \delta_{(l)} \left(\frac{y_{iju(l)} - \tau_{iju}^{*}}{\sigma} \right) \right] = 0,$$

$$\frac{\partial \log L}{\partial (\alpha\beta\lambda)_{iju}} = \frac{n}{\sigma} - \frac{\theta+1}{\sigma} \sum_{l=1}^{n} \left[\gamma_{(l)} + \delta_{(l)} \left(\frac{y_{iju(l)} - \tau_{iju}^{*}}{\sigma} \right) \right] = 0,$$

$$\frac{\partial \log L}{\partial \sigma} = -\frac{18n}{\sigma} - \frac{1}{\sigma} \sum_{i=1}^{2} \sum_{j-1}^{3} \sum_{u=1}^{3} \sum_{l=1}^{n} \left(\frac{y_{iju(l)} - \tau_{iju}^{*}}{\sigma} \right) - \frac{\theta+1}{\sigma} \sum_{i=1}^{2} \sum_{j-1}^{3} \sum_{u=1}^{3} \sum_{l=1}^{n} \left(\frac{y_{iju(l)} - \tau_{iju}^{*}}{\sigma} \right) \left[\gamma_{(l)} + \delta_{(l)} \left(\frac{y_{iju(l)} - \tau_{iju}^{*}}{\sigma} \right) \right] = 0 \tag{2.19}$$

Now solving the normal equations one by one, we have:

$$\frac{18n}{\sigma}-\frac{\theta+1}{\sigma}\left[18k+\left(\frac{18m\hat{\mu}....-18m\hat{\mu}-0-0-...-0}{\sigma}\right)\right]=0; \text{ Here } k=\sum_{l=1}^{n}\gamma_{(l)} \quad \text{and} \quad m=\sum_{l=1}^{n}\delta_{(l)}$$

$$\Rightarrow\frac{18n}{\sigma}=\frac{\theta+1}{\sigma}\left[18k+\left(\frac{18m\hat{\mu}....-18m\hat{\mu}}{\sigma}\right)\right]$$

$$\Rightarrow\frac{18n}{\theta+1}=\left[18k+\left(\frac{18m\hat{\mu}....-18m\hat{\mu}}{\sigma}\right)\right]$$

$$\Rightarrow\frac{n}{\theta+1}-k=\left(\frac{m\hat{\mu}....-m\hat{\mu}}{\sigma}\right)$$

$$\Rightarrow\left(\frac{n}{\theta+1}-k\right)\sigma=m\hat{\mu}....-m\hat{\mu}$$

$$\Rightarrow\hat{\mu}=\hat{\mu}....-\left(\frac{n}{\theta+1}-k\right)\frac{\sigma}{m} \tag{2.20}$$

$$\frac{9n}{\sigma}-\frac{\theta+1}{\sigma}\sum_{j=1}^{3}\sum_{u=1}^{3}\sum_{l=1}^{n}\left[\gamma_{(l)}+\delta_{(l)}\left(\frac{y_{iju(l)}-\tau_{iju}^{*}}{\sigma}\right)\right]=0$$

$$\sum_{j=1}^{3}\sum_{u=1}^{3}\sum_{l=1}^{n}\left[\gamma_{(l)}+\delta_{(l)}\left(\frac{y_{iju(l)}-\tau_{iju}^{*}}{\sigma}\right)\right]=\frac{9n}{b+1}$$

$$\Rightarrow 9k+\frac{9m\hat{\mu}_i...-9m\hat{\mu}-9m\hat{\tau}_i}{\sigma}=\frac{9n}{\theta+1}$$

$$\Rightarrow\frac{\sigma}{m}\left(\frac{n}{\theta+1}-k\right)=\left(\hat{\mu}_i...-\hat{\mu}-\hat{\tau}_i\right)$$

$$\Rightarrow\hat{\tau}_i=\hat{\mu}_i...-\hat{\mu}-\frac{\sigma}{m}\left(\frac{n}{\theta+1}-k\right)$$

$$\Rightarrow\hat{\tau}_i=\hat{\mu}_i...-\hat{\mu}....+\left(\frac{n}{\theta+1}-k\right)\frac{\sigma}{m}-\frac{\sigma}{m}\left(\frac{n}{\theta+1}-k\right)$$; (substituting the value of $\hat{\mu}$)

$$\Rightarrow\hat{\tau}_i=\hat{\mu}_i...-\hat{\mu}.... \tag{2.21}$$

$$\frac{6n}{\sigma}-\frac{\theta+1}{\sigma}\sum_{i=1}^{2}\sum_{u=1}^{3}\sum_{l=1}^{n}\left[\gamma_{(l)}+\delta_{(l)}\left(\frac{y_{iju(l)}-\tau_{iju}^{*}}{\sigma}\right)\right]=0$$

$$\Rightarrow\frac{6n}{\sigma}=\frac{\theta+1}{\sigma}\left[6k+\frac{6m\hat{\mu}._{j}..-6m\hat{\mu}-6m\hat{\beta}_{j}}{\sigma}\right]$$

$$\Rightarrow\frac{n}{\theta+1}-k=\frac{m\left(\hat{\mu}._{j}..-\hat{\mu}-\hat{\beta}_{j}\right)}{\sigma}$$

$$\Rightarrow\left(\frac{n}{\theta+1}-k\right)\frac{\sigma}{m}=\hat{\mu}._{j}..-\hat{\mu}-\hat{\beta}_{j}$$

$$\Rightarrow\hat{\beta}_{j}=\hat{\mu}._{j}..-\hat{\mu}-\left(\frac{n}{\theta+1}-k\right)\frac{\sigma}{m}$$

$$\Rightarrow\hat{\beta}_{j}=\hat{\mu}._{j}..-\hat{\mu}....+\left(\frac{n}{\theta+1}-k\right)\frac{\sigma}{m}-\left(\frac{n}{\theta+1}-k\right)\frac{\sigma}{m}$$

$$\Rightarrow\hat{\beta}_{j}=\hat{\mu}._{j}..-\hat{\mu}.... \qquad (2.22)$$

Similarly, we can have

$$\hat{\lambda}_{u}=\hat{\mu}.._{u}.-\hat{\mu}.... \qquad (2.23)$$

Further,

$$\frac{3n}{\sigma}-\frac{\theta+1}{\sigma}\sum_{u=1}^{3}\sum_{l=1}^{n}\left[\gamma_{(l)}+\delta_{(l)}\left(\frac{y_{iju(l)}-\tau_{iju}^{*}}{\sigma}\right)\right]=0$$

$$\frac{3n}{\sigma}-\frac{\theta+1}{\sigma}\left[3k+\frac{3m}{\sigma}\left(\hat{\mu}_{ij}..-\hat{\mu}+\hat{\tau}+\hat{\beta}_{j}+(\tau\beta)_{ij}\right)\right]=0$$

$$\Rightarrow est(\tau\beta)_{ij}=\hat{\mu}_{ij}..-\hat{\mu}_{i}...-\hat{\mu}._{j}..+\hat{\mu}.... \qquad (2.24)$$

Similarly, we can have

$$est(\tau\lambda)_{iu}=\hat{\mu}_{i}._{u}.-\hat{\mu}_{i}...-\hat{\mu}.._{u}.+\hat{\mu}.... \qquad (2.25)$$

$$est(\beta\lambda)_{ju}=\hat{\mu}._{ju}.-\hat{\mu}._{j}..-\hat{\mu}.._{u}.+\hat{\mu}.... \qquad (2.26)$$

$$est(\tau\beta\lambda)_{iju}=\hat{\mu}_{iju}.+\hat{\mu}_{i}...+\hat{\mu}._{j}..+\hat{\mu}.._{u}.-\hat{\mu}_{ij}..-\hat{\mu}_{i}._{u}.-\hat{\mu}._{ju}.-\hat{\mu}.... \qquad (2.27)$$

$$\frac{\partial \log L}{\partial \sigma} = -\frac{18n}{\sigma} - \frac{1}{\sigma}\sum_{i=1}^{2}\sum_{j-1}^{3}\sum_{u=1}^{3}\sum_{l=1}^{n}\left(\frac{y_{iju(l)} - \tau_{iju}^{*}}{\sigma}\right) - \frac{\theta+1}{\sigma}\sum_{i=1}^{2}\sum_{j-1}^{3}\sum_{u=1}^{3}\sum_{l=1}^{n}\left(\frac{y_{iju(l)} - \tau_{iju}^{*}}{\sigma}\right)\left[\gamma_{(l)} + \delta_{(l)}\left(\frac{y_{iju(l)} - \tau_{iju}^{*}}{\sigma}\right)\right] = 0$$

$$\text{or } -18n\sigma^2 + \sigma\left[\sum_{i=1}^{2}\sum_{j=1}^{3}\sum_{u=1}^{3}\sum_{l=1}^{n}\left(y_{iju(l)} - \tau_{iju}^{*}\right) - (\theta+1)\sum_{i=1}^{2}\sum_{j=1}^{3}\sum_{l=1}^{n}\gamma_{(l)}\left(y_{iju(l)} - \tau_{iju}^{*}\right)\right] - (\theta+1)\sum_{i=1}^{2}\sum_{j=1}^{3}\sum_{l=1}^{n}\delta_{(l)}\left(y_{iju(l)} - \tau_{iju}^{*}\right)^2 = 0$$

$$\text{or } -18n\sigma^2 + \sigma(\theta+1)\left[\sum_{i=1}^{2}\sum_{j=1}^{3}\sum_{u=1}^{3}\sum_{l=1}^{n}\left(\frac{1}{(\theta+1)} - \gamma_{(l)}\right)\left(y_{iju(l)} - \tau_{iju}^{*}\right)\right] - (\theta+1)\sum_{i=1}^{2}\sum_{j=1}^{3}\sum_{l=1}^{n}\delta_{(l)}\left(y_{iju(l)} - \tau_{iju}^{*}\right)^2 = 0$$

Thus

$$\hat{\sigma} = \frac{B + \sqrt{B^2 + 4NC}}{2\sqrt{N(N-18)}}, \qquad \text{here N=18n}$$

$$\text{where, } B = (\theta+1)\left[\sum_{i=1}^{2}\sum_{j=1}^{3}\sum_{u=1}^{3}\sum_{l=1}^{n}\left(\frac{1}{(\theta+1)} - \gamma_{(l)}\right)\left(y_{iju(l)} - \tau_{iju}^{*}\right)\right] \qquad (2.28)$$

$$C = (\theta+1)\sum_{i=1}^{2}\sum_{j=1}^{3}\sum_{l=1}^{n}\delta_{(l)}\left(y_{iju(l)} - \tau_{iju}^{*}\right)^2$$

It may be noted that, unlike the maximum likelihood (ML) estimator of σ, the Modified Maximum Likelihood estimator $\hat{\sigma}$ is always real and positive. The MML estimators have closed forms and have the similar structure with the classical procedures irrespective of the underlying distribution. They can also be easily represented in matrix form. For these reasons, the MML procedures become very attractive for practitioners. The divisor *N* in the expression for $\hat{\sigma}$ has been replaced by $\sqrt{N(N-18)}$, as a bias correction.

2.4 Main and Interaction Effects

The treatments consisting of all combinations that can be formed from the different factors may be represented as A, B, AB, B^2, AB^2, C, AC, BC, ABC, B^2C, AB^2C, C^2, AC^2, BC^2, ABC^2, B^2C^2, AB^2C^2

Letters (1), a, b, ab, b^2, ab^2, c, ac, bc, abc. b^2c, ab^2c, c^2, ac^2, bc^2, abc^2, b^2c^2 and ab^2c^2 represent the weighted totals of n observations obtained from each treatment combination.

$$\begin{aligned}
&(1) = \sum_l \delta_l y_{111(l)},\ a = \sum_l \delta_l y_{211(l)},\ b = \sum_l \delta_l y_{121(l)},\ ab = \sum_l \delta_l y_{221(l)},\ b^2 = \sum_l \delta_l y_{131(l)},\\
&ab^2 = \sum_l \delta_l y_{231(l)},\ c = \sum_l \delta_l y_{112(l)},\ ac = \sum_l \delta_l y_{212(l)},\ bc = \sum_l \delta_l y_{122(l)},\ abc = \sum_l \delta_l y_{222(l)},\\
&b^2c = \sum_l \delta_l y_{132(l)},\ ab^2c = \sum_l \delta_l y_{232(l)},\ c^2 = \sum_l \delta_l y_{113(l)},\ ac^2 = \sum_l \delta_l y_{213(l)},\ bc^2 = \sum_l \delta_l y_{123(l)},\\
&abc^2 = \sum_l \delta_l y_{223(l)},\ b^2c^2 = \sum_l \delta_l y_{133(l)},\ ab^2c^2 = \sum_l \delta_l y_{233(l)}.
\end{aligned} \qquad (2.29)$$

The contrast are obtained using Yates algorithm and consequently sum of squares and F statistics for the main effects, two-factor and three-factor interactions have been computed.

$$SS_A = \frac{1}{18m}(Contrast_A)^2; \quad SS_B = \frac{1}{18m}(Contrast_B)^2; \quad SS_C = \frac{1}{12m}(Contrast_C)^2;$$

$$SS_{AB} = \frac{1}{12m}(Contrast_{AB})^2; \quad SS_{AC} = \frac{1}{12m}(Contrast_{AC})^2; \quad SS_{BC} = \frac{1}{8m}(Contrast_{BC})^2;$$

$$\ldots, \; SS_{AB^2C^2} = \frac{1}{72m}(Contrast_{AB^2C^2})^2.$$

The test statistics for testing the hypotheses are

$$F_A^* = \frac{SS_A}{\hat{\sigma}^2}, \; F_B^* = \frac{SS_B}{\hat{\sigma}^2}, \; F_C^* = \frac{SS_C}{\hat{\sigma}^2}, \; F_{AB}^* = \frac{SS_{AB}}{\hat{\sigma}^2}, \; F_{BC}^* = \frac{SS_{BC}}{\hat{\sigma}^2}, \; F_{AC}^* = \frac{SS_{AC}}{\hat{\sigma}^2},$$

$$F_{B^2}^* = \frac{SS_{B^2}}{\hat{\sigma}^2}, \ldots, \; F_{AB^2}^* = \frac{SS_{AB^2}}{\hat{\sigma}^2} \ldots, \; F_{AB^2C^2}^* = \frac{SS_{AB^2C^2}}{\hat{\sigma}^2}.$$

For large n, their null distributions are central F with degrees of freedom

$(\upsilon_1,\upsilon_{18}), \; (\upsilon_2,\upsilon_{18}), \; (\upsilon_3,\upsilon_{18}), \; (\upsilon_4,\upsilon_{18}), \; (\upsilon_5,\upsilon_{18}), \; (\upsilon_6,\upsilon_{18}),(\upsilon_7,\upsilon_{18}),(\upsilon_8,\upsilon_{18}),(\upsilon_9,\upsilon_{18}),$
$(\upsilon_{10},\upsilon_{18}),(\upsilon_{11},\upsilon_{18}),(\upsilon_{12},\upsilon_{18}),(\upsilon_{13},\upsilon_{18}),(\upsilon_{14},\upsilon_{18}),(\upsilon_{15},\upsilon_{18}),(\upsilon_{16},\upsilon_{18}),$ and $(\upsilon_{17},\upsilon_{18}),$ respectively;
where $\upsilon_1 = \upsilon_2 = \upsilon_3 = \upsilon_4 = \upsilon_5 = \upsilon_6 = \upsilon_7 = \upsilon_8 = \upsilon_9$
$= \upsilon_{10} = \upsilon_{11} = \upsilon_{12} = \upsilon_{13} = \upsilon_{14} = \upsilon_{15} = \upsilon_{16} = \upsilon_{17} = 1$ and $\upsilon_{18} = 18(n-1)$.

The results are valid for different values of θ, (θ <1, θ =1, θ >1) of the logistic distribution when the distribution is negatively skewed, symmetric, and positively skewed, respectively.

2.5 Discussion

The analytical procedure have been developed for the 2×3×3 factorial experiments when error follows generalized logistic distribution. The estimates are obtained through Modified Maximum Likelihood Estimation procedure (MMLE) for the main effect and interaction effects. SAS code has been developed for the developed procedure.

CHAPTER 3

FACTORIAL EXPERIMENTS WITH GOMPERTZ ERROR DITRIBUTION

3.1 Introduction:

Factorial experiments that were introduced by Fisher (1935) and Yates (1937) are often the best and most used designs in agricultural experiments when the effects of multiple factors are investigated simultaneously. They also provide the ability to detect and estimate interactions between the factors. It indicates major trends to determine a direction for further experimentation, see Box *et al.* (1978), Montgomery (1984) and Hinkelmann & Kempthorne (1994).

Under the assumption of normality and independence of observations, the normal equations obtained from maximum likelihood function are linear and hence solvable. On the other hand when the data do not follow the normal distribution, the equations obtained of MLE are not linear and so these equations are difficult to handle. In this chapter, it is assumed that error follows Gompertz distribution.

3.2 Modified Maximum Likelihood Procedures for 2×3×3 Factorial Experiments

A case has been considered where three factors (say *A*, *B* and *C*), in which factor *A* has 2 levels, factors *B* and *C* have 3 levels (2×3×3 factorial experiments) in unblocked situation. The statistical model for such experiments is

$$y_{ijul} = \mu + \tau_i + \beta_j + \lambda_u + (\tau\beta)_{ij} + (\tau\lambda)_{iu} + (\beta\lambda)_{ju} + (\tau\beta\lambda)_{iju} + e_{ijul}$$
$$(i = 1,\ 2;\ j = 1,\ 2,\ 3\ ;\ u = 1,\ 2,\ 3\ ;\ l = 1, 2, ...,\ n) \tag{3.1}$$

where y_{ijul} denotes the observation for of i^{th} level of factor *A*, j^{th} level of factor *B*, u^{th} level of factor *C*, $l = 1, \ldots, n$, μ is the overall mean, τ_i is the effect of the i^{th} level of the factor *A*, β_j is the j^{th} effect of factor *B*, λ_u is the effect of the u^{th} level of factor *C*, $(\tau\beta)_{ij}$ is the effect of the interaction between τ_i and β_j and e_{ijul} ~ Gompertz distribution and is a random error component. Without loss of generality, we assume that

$$\sum_i \tau_i = \sum_j \beta_j = \sum_i (\tau\beta)_{ij} = \sum_j (\tau\beta)_{ij} = \sum_j (\tau\lambda)_{iu} = \sum_u (\tau\lambda)_{iu} = \sum_j (\beta\lambda)_{ju} = \sum_u (\beta\lambda)_{ju} =$$
$$\sum_i (\tau\beta\lambda)_{iju} = \sum_j (\tau\beta\lambda)_{iju} = \sum_u (\tau\beta\lambda)_{iju} = 0$$

Furthermore, the factors are considered as fixed and the design is assumed to be completely randomized design. Since the error follows Gompertz distribution, its functional form is

$$G(\eta,\alpha)=\eta\exp\left[\alpha x-\frac{\eta}{\alpha}\left\{\exp(\alpha x)-1\right\}\right] \tag{3.2}$$
$$\text{where, } x\geq 0;\ \eta>0;\ \alpha>0$$

where, η and α are the shape and scale parameter of the distribution respectively.
The likelihood function L is

$$L=\eta^{N}\prod_{i=1}^{2}\prod_{j=1}^{3}\prod_{u=1}^{3}\prod_{l=1}^{n}\exp\left[\alpha(y_{ijul}-\tau^{*})-\frac{\eta}{\alpha}\{\exp(\alpha(y_{ijul}-\tau^{*}))-1\}\right];\text{ where, } N=18n. \tag{3.3}$$

Log likelihood function is

$$\log L=N\log\eta+\sum_{i=1}^{2}\sum_{j=1}^{3}\sum_{u=1}^{3}\sum_{l=1}^{n}\left[\alpha\left(y_{ijul}-\tau_{iju}^{*}\right)\right]$$
$$-\frac{\eta}{\alpha}\sum_{i=1}^{2}\sum_{j=1}^{3}\sum_{u=1}^{3}\sum_{l=1}^{n}\left[exp\ \{\alpha(y_{ijul}-\tau_{iju}^{*})\}-1\right] \tag{3.4}$$

For obtaining the solutions, the method given by Tiku and Suresh (1992) by expanding log-likelihood function using Taylor series is applied for obtaining the estimates of parameters.

Let $z_{ijul}=\alpha(y_{ijul}-\tau_{iju}^{*})$

$$\log L=N\log\eta+\sum_{i=1}^{2}\sum_{j=1}^{3}\sum_{u=1}^{3}\sum_{l=1}^{n}\left(z_{ijul}\right)-\frac{\eta}{\alpha}\sum_{i=1}^{2}\sum_{j=1}^{3}\sum_{u=1}^{3}\sum_{l=1}^{n}\left[\exp(z_{ijul})-1\right] \tag{3.5}$$

Let $y_{iju(1)} < y_{iju(2)} < y_{iju(3)} < \ldots < y_{iju(n)}$ be the order statistics of y_{ijul} $(1 < l < n)$. Since complete sum is invariant to ordering then $\sum_{l=1}^{n}h(y_l)=\sum_{l=1}^{n}h(y_{(l)})$ where, h(y) is any function of y. Now

$$\log L=N\log\eta+\sum_{i=1}^{2}\sum_{j=1}^{3}\sum_{u=1}^{3}\sum_{l=1}^{n}\left(z_{iju(l)}\right)-\frac{\eta}{\alpha}\sum_{i=1}^{2}\sum_{j=1}^{3}\sum_{u=1}^{3}\sum_{l=1}^{n}\left[\exp(z_{iju(l)})-1\right] \tag{3.6}$$

The equations obtained from the log likelihood function do not yield the explicit solutions for the estimates, due to non-linearity of the function $\exp(z_{iju(l)})$

$$\frac{\partial\log L}{\partial\mu}=-\alpha N+\eta\sum_{i=1}^{2}\sum_{j-1}^{3}\sum_{u=1}^{3}\sum_{l=1}^{n}\exp(z_{iju(l)})=0,$$

$$\frac{\partial\log L}{\partial\tau_i}=-9n\alpha+\eta\sum_{j=1}^{3}\sum_{u=1}^{3}\sum_{l=1}^{n}\exp(z_{iju(l)})=0,$$

$$\frac{\partial \log L}{\partial \beta_j} = -6n\alpha + \eta \sum_{i=1}^{2} \sum_{u=1}^{3} \sum_{l=1}^{n} \exp(z_{iju(l)}) = 0 ,$$

$$\frac{\partial \log L}{\partial \lambda_u} = -6n\alpha + \eta \sum_{i=1}^{2} \sum_{j=1}^{3} \sum_{l=1}^{n} \exp(z_{iju(l)}) = 0 ,$$

$$\frac{\partial \log L}{\partial (\tau\beta)_{ij}} = -3n\alpha + \eta \sum_{u=1}^{3} \sum_{l=1}^{n} \exp(z_{iju(l)}) = 0 ,$$

$$\frac{\partial \log L}{\partial (\tau\lambda)_{iu}} = -3n\alpha + \eta \sum_{j=1}^{3} \sum_{l=1}^{n} \exp(z_{iju(l)}) = 0 ,$$

$$\frac{\partial \log L}{\partial (\beta\lambda)_{ju}} = -2n\alpha + \eta \sum_{i=1}^{2} \sum_{l=1}^{n} \exp(z_{iju(l)}) = 0 ,$$

$$\frac{\partial \log L}{\partial (\alpha\beta\lambda)_{iju}} = -n\alpha + \eta \sum_{l=1}^{n} \exp(z_{iju(l)}) = 0 ,$$

$$\frac{\partial \log L}{\partial \alpha} = -N\frac{\eta}{\alpha} + \frac{1}{\alpha} \sum_{i=1}^{2} \sum_{j-1}^{3} \sum_{u=1}^{3} \sum_{l=1}^{n} z_{iju(l)} - \frac{\eta}{\alpha^2} \sum_{i=1}^{2} \sum_{j-1}^{3} \sum_{u=1}^{3} \sum_{l=1}^{n} \alpha \exp(z_{iju(l)}) \left\{ z_{iju(l)} - 1 \right\} = 0 \qquad (3.7)$$

The equations obtained from the first derivative of log likelihood function with respect to parameters do not yield the explicit solutions for the estimates due to non-linearity of the function. Solving them by iterations is indeed problematic for reasons of (i) multiple roots, (ii) non-convergence of iterations, and (iii) convergence to wrong values; see, for example, Smith (1985), Puthenpura and Sinha (1986) and Vaughan (1992, 2002). To alleviate these problems modified likelihood equations are used by linearizing the intractable terms (in likelihood equations). The resulting equations have explicit solutions called MML estimators. The MMLE are known to be asymptotically fully efficient under regularity conditions Bhattacharyya, (1985); Vaughan and Tiku, (2000).

3.3 Parameter Estimation

For obtaining the solutions, the method given by Tiku and Suresh (1992) by expanding log-likelihood function using Taylor series is applied for attaining the estimates of parameters. These solutions are called modified maximum likelihood estimates.

For large n, $z_{iju(l)}$ is close to its expected value. Let $t_{(l)}$=E($z_{iju(l)}$); $1 \leq l \leq n$. The first two terms of the Taylor Series expansion namely, Let $\exp(z_{iju(l)})=g(z_{iju(l)})$

$g(z_{iju(l)}) = \gamma_{(l)} + \delta_{(l)} z_{iju(l)}$

$g(z_{iju(l)}) = g(t_{(l)}) + [z_{iju(l)} - t_{(l)}]\; g'(t_{(l)})$

$g(z_{iju(l)}) = (1-t_{(l)})\exp(t_{(l)}) + \exp(t_{(l)}) z_{iju(l)}$

$\exp(z_{iju(l)})=g(z_{iju(l)}) = \gamma_{(l)} + \delta_{(l)} z_{iju(l)}$

Approximate values of $t_{iju(l)}$ are obtained from the equation

$$\int_{-\infty}^{t_{(l)}} \frac{\eta}{\alpha} \exp\left[z - \frac{\eta}{\alpha}(\exp(z)-1) \right] dz = \frac{l}{n+1};$$

$$\Rightarrow t_{(l)} = \ln\left[1 - \frac{\eta}{\alpha} \ln\left(1 + \frac{l}{n+1} \right) \right]$$

Thus $\gamma_{(l)} = (1-t_{(l)})\exp(t_{(l)})$ and $\delta_{(l)} = \exp(t_{(l)})$ is computed.

Using approximate values instead of exact values does not adversely affect the efficiency of the MML estimators. [Senoglu and Tiku (2001)]

Putting the approximate value of $g(z_{iju(l)})$ from equation in the normal equations (3.7) we have,

Thus,

$$g(z_{iju(l)}) \cong \gamma_{(l)} + \delta_{(l)} z_{iju(l)}$$

$$\frac{\partial \log L}{\partial \mu} = -18n\alpha + \eta \sum_{i=1}^{2} \sum_{j-1}^{3} \sum_{u=1}^{3} \sum_{l=1}^{n} \gamma_{(l)} + \delta_{(l)} \alpha (y_{iju(l)} - \tau^{*}_{iju}) = 0 \tag{3.8}$$

$$\frac{\partial \log L}{\partial \tau_i} = -9n\alpha + \eta \sum_{j=1}^{3} \sum_{u=1}^{3} \sum_{l=1}^{n} \gamma_{(l)} + \delta_{(l)} \alpha (y_{iju(l)} - \tau^{*}_{iju}) = 0 \tag{3.9}$$

$$\frac{\partial \log L}{\partial \beta_j} = -6n\alpha + \eta \sum_{i=1}^{2} \sum_{u=1}^{3} \sum_{l=1}^{n} \gamma_{(l)} + \delta_{(l)} \alpha (y_{iju(l)} - \tau^{*}_{iju}) = 0 \tag{3.10}$$

$$\frac{\partial \log L}{\partial \lambda_u} = -6n\alpha + \eta \sum_{i=1}^{2} \sum_{j=1}^{3} \sum_{l=1}^{n} \gamma_{(l)} + \delta_{(l)} \alpha (y_{iju(l)} - \tau^{*}_{iju}) = 0 \tag{3.11}$$

$$\frac{\partial \log L}{\partial (\tau\beta)_{ij}} = -3n\alpha + \eta \sum_{u=1}^{3} \sum_{l=1}^{n} \gamma_{(l)} + \delta_{(l)} \alpha (y_{iju(l)} - \tau^{*}_{iju}) = 0 \tag{3.12}$$

$$\frac{\partial \log L}{\partial(\tau\lambda)_{iu}} = -3n\alpha + \eta\sum_{j=1}^{3}\sum_{l=1}^{n}\gamma_{(l)} + \delta_{(l)}\alpha(y_{iju(l)} - \tau^{*}_{iju}) = 0 \quad (3.13)$$

$$\frac{\partial \log L}{\partial(\beta\lambda)_{ju}} = -2n\alpha + \eta\sum_{i=1}^{2}\sum_{l=1}^{n}\gamma_{(l)} + \delta_{(l)}\alpha(y_{iju(l)} - \tau^{*}_{iju}) = 0 \quad (3.14)$$

$$\frac{\partial \log L}{\partial(\alpha\beta\lambda)_{iju}} = -n\alpha + \eta\sum_{l=1}^{n}\gamma_{(l)} + \delta_{(l)}\alpha(y_{iju(l)} - \tau^{*}_{iju}) = 0 \quad (3.15)$$

$$\frac{\partial \log L}{\partial \alpha} = -18n\frac{\eta}{\alpha} + \frac{1}{\alpha}\sum_{i=1}^{2}\sum_{j-1}^{3}\sum_{u=1}^{3}\sum_{l=1}^{n}\alpha(y_{iju(l)} - \tau^{*}_{iju})$$
$$-\frac{\eta}{\alpha^{2}}\sum_{i=1}^{2}\sum_{j-1}^{3}\sum_{u=1}^{3}\sum_{l=1}^{n}\alpha\left\{\gamma_{(l)} + \delta_{(l)}\alpha(y_{iju(l)} - \tau^{*}_{iju})\right\}\left\{\alpha(y_{iju(l)} - \tau^{*}_{iju}) - 1\right\} = 0 \quad (3.16)$$

Now solving the normal equations one by one (3.8 to 3.16), we have

$-18n\alpha + \eta\left[18k + 18m\alpha\hat{\mu}.... - 18m\alpha\hat{\mu} - 0\right] = 0$; Here $k = \sum_{l=1}^{n}\gamma_{(l)}$ and $m = \sum_{l=1}^{n}\delta_{(l)}$

$$n\frac{\alpha}{\eta} = k + m\alpha\hat{\mu}.... - m\alpha\hat{\mu}$$

$$\hat{\mu} = \frac{1}{m\alpha}\left[k + m\alpha\hat{\mu}.... - n\frac{\alpha}{\eta}\right]$$

$$\hat{\mu} = \frac{k}{m\alpha} + \hat{\mu}.... - \frac{n}{m\eta}$$

$$\hat{\mu} = \hat{\mu}.... - \frac{n}{m\eta} + \frac{k}{m\alpha} \quad (3.17)$$

$$-9n\alpha + \eta\left[9k + 9m\alpha\hat{\mu}_{i}... - 9m\alpha\hat{\mu} - 9m\alpha\hat{\tau}_{i}\right] = 0$$

$$\frac{9n\alpha}{\eta} = 9k + 9m\alpha\hat{\mu}_{i}... - 9m\alpha\hat{\mu} - 9m\alpha\hat{\tau}_{i}$$

$$\hat{\tau}_{i} = \frac{1}{m\alpha}\left[k + m\alpha\hat{\mu}_{i}... - m\alpha\hat{\mu} - \frac{n\alpha}{\eta}\right]$$

$$\hat{\tau}_{i} = \hat{\mu}_{i}... - \hat{\mu} - \frac{n}{m\eta} + \frac{k}{m\alpha}$$

$\hat{\tau}_{i} = \hat{\mu}_{i}... - \hat{\mu}.... + \frac{n}{m\eta} - \frac{k}{m\alpha} \quad - \frac{n}{m\eta} + \frac{k}{m\alpha}$; (substituting the value of $\hat{\mu}$)

$$\hat{\tau}_i = \hat{\mu}_i... - \hat{\mu}.... \tag{3.18}$$

$$-6n\alpha + \eta\left[6k + 6m\alpha\hat{\mu}._j.. - 6m\alpha\hat{\mu} - 6m\alpha\hat{\beta}_j\right] = 0$$

$$\frac{n\alpha}{\eta} = k + m\alpha\hat{\mu}._j.. - m\alpha\hat{\mu} - m\alpha\hat{\beta}_j$$

$$\hat{\beta}_j = \frac{k}{m\alpha} + \hat{\mu}._j.. - \hat{\mu} - \frac{n}{m\eta}$$

$$\hat{\beta}_j = \hat{\mu}._j.. - \hat{\mu} + \frac{k}{m\alpha} - \frac{n}{m\eta}$$

$$\hat{\beta}_j = \hat{\mu}._j.. - \hat{\mu}.... \tag{3.19}$$

Simillarily, we can have

$$\hat{\lambda}_u = \hat{\mu}.._u. - \hat{\mu}.... \tag{3.20}$$

$$-3n\alpha + \eta\left[3k + 3m\alpha\hat{\mu}_{ij}.. - 3m\alpha\hat{\mu} - 3m\alpha(\widehat{\tau\beta})_{ij} - 3m\alpha\hat{\tau}_i - 3m\alpha\hat{\beta}_j\right] = 0$$

$$\frac{3n\alpha}{\eta} = 3k + 3m\alpha\hat{\mu}_{ij}.. - 3m\alpha\hat{\mu} - 3m\alpha(\widehat{\tau\beta})_{ij} - 3m\alpha\hat{\tau}_i - 3m\alpha\hat{\beta}_j$$

$$(\widehat{\tau\beta})_{ij} = \frac{k}{m\alpha} + \hat{\mu}_{ij}.. - \hat{\mu} - \frac{n}{m\eta} - \hat{\tau}_i - \hat{\beta}_j \tag{3.21}$$

$$(\widehat{\tau\beta})_{ij} = \hat{\mu}_{ij}.. - \hat{\mu} + \frac{k}{m\alpha} - \frac{n}{m\eta} - \hat{\tau}_i - \hat{\beta}_j \tag{3.22}$$

$$(\widehat{\tau\beta})_{ij} = \hat{\mu}_{ij}.. - \hat{\mu}_i... - \hat{\mu}._j.. + \hat{\mu}.... \tag{3.23}$$

Similarily,

$$(\widehat{\tau\lambda})_{iu} = \hat{\mu}_i._u. - \hat{\mu}_i... - \hat{\mu}.._u. + \hat{\mu}.... \tag{3.24}$$

$$(\widehat{\beta\lambda})_{ju} = \hat{\mu}._{ju}. - \hat{\mu}._j.. - \hat{\mu}.._u. + \hat{\mu}.... \tag{3.25}$$

$$(\widehat{\alpha\beta\lambda})_{iju} = \hat{\mu}_{iju}. + \hat{\mu}_i... + \hat{\mu}._j.. + \hat{\mu}.._u. - \hat{\mu}_{ij}.. - \hat{\mu}_i._u. - \hat{\mu}._{ju}. - \hat{\mu}.... \tag{3.26}$$

Where,

$$\hat{\mu}.... = \frac{1}{18m}\sum_{i=1}^{2}\sum_{j=1}^{3}\sum_{u=1}^{3}\sum_{l=1}^{n}\delta_{(l)}y_{iju(l)} \; ; \; \hat{\mu}_i... = \frac{1}{9m}\sum_{j=1}^{3}\sum_{u=1}^{3}\sum_{l=1}^{n}\delta_{(l)}y_{iju(l)} \; ; \; \hat{\mu}._j.. = \frac{1}{6m}\sum_{i=1}^{2}\sum_{u=1}^{3}\sum_{l=1}^{n}\delta_{(l)}y_{iju(l)}$$

$$\hat{\mu}.._u. = \frac{1}{6m}\sum_{i=1}^{2}\sum_{j=1}^{3}\sum_{l=1}^{n}\delta_{(l)}y_{iju(l)} \; ; \; \hat{\mu}_{ij}.. = \frac{1}{3m}\sum_{u=1}^{3}\sum_{l=1}^{n}\delta_{(l)}y_{iju(l)} \; ; \; \hat{\mu}_i ._u. = \frac{1}{3m}\sum_{j=1}^{3}\sum_{l=1}^{n}\delta_{(l)}y_{iju(l)}$$

$$\hat{\mu}._{ju}. = \frac{1}{2m}\sum_{i=1}^{2}\sum_{l=1}^{n}\delta_{(l)}y_{iju(l)} \; ; \; \hat{\mu}_{iju}. = \frac{1}{m}\sum_{l=1}^{n}\delta_{(l)}y_{iju(l)} \tag{3.27}$$

$$-18n\frac{\eta}{\alpha} + \frac{1}{\alpha}\sum_{i=1}^{2}\sum_{j-1}^{3}\sum_{u=1}^{3}\sum_{l=1}^{n}\alpha(y_{iju(l)} - \tau^*_{iju}) - \frac{\eta}{\alpha^2}\sum_{i=1}^{2}\sum_{j-1}^{3}\sum_{u=1}^{3}\sum_{l=1}^{n}\alpha\left\{\gamma_{(l)} + \delta_{(l)}\alpha(y_{iju(l)} - \tau^*_{iju})\right\}\left\{\alpha(y_{iju(l)} - \tau^*_{iju}) - 1\right\} = 0$$

Solving this equation we have,

$$\alpha^2\eta\sum_{i=1}^{2}\sum_{j-1}^{3}\sum_{u=1}^{3}\sum_{l=1}^{n}\delta_{(l)}(y_{iju(l)} - \tau^*_{iju})^2 + \alpha\left[\eta\sum_{i=1}^{2}\sum_{j-1}^{3}\sum_{u=1}^{3}\sum_{l=1}^{n}(y_{iju(l)} - \tau^*_{iju})(\gamma_{(l)} - \delta_{(l)})\right] + \eta\sum_{i=1}^{2}\sum_{j-1}^{3}\sum_{u=1}^{3}\sum_{l=1}^{n}\gamma_{(l)} - N\eta = 0$$

$$\hat{\alpha} = \frac{-B + \sqrt{B^2 - 4AC}}{2A},$$

where, $A = \eta\sum_{i=1}^{2}\sum_{j-1}^{3}\sum_{u=1}^{3}\sum_{l=1}^{n}\delta_{(l)}(y_{iju(l)} - \tau^*_{iju})^2$

$$B = \left[\eta\sum_{i=1}^{2}\sum_{j-1}^{3}\sum_{u=1}^{3}\sum_{l=1}^{n}(y_{iju(l)} - \tau^*_{iju})(\gamma_{(l)} - \delta_{(l)})\right] \tag{3.28}$$

$$C = \eta\sum_{i=1}^{2}\sum_{j-1}^{3}\sum_{u=1}^{3}\sum_{l=1}^{n}\gamma_{(l)} - N\eta$$

The MML estimators have closed forms and have the similar structure with the classical procedures irrespective of the underlying distribution. They can also be easily represented in matrix form. For these reasons, the MML procedures become very attractive for practitioners.

$$\hat{\mu}.... = \frac{1}{18m}\sum_{i=1}^{2}\sum_{j=1}^{3}\sum_{u=1}^{3}\sum_{l=1}^{n}\delta_{(l)}y_{iju(l)} \; ; \; \hat{\mu}_i... = \frac{1}{9m}\sum_{j=1}^{3}\sum_{u=1}^{3}\sum_{l=1}^{n}\delta_{(l)}y_{iju(l)} \; ; \; \hat{\mu}._j.. = \frac{1}{6m}\sum_{i=1}^{2}\sum_{u=1}^{3}\sum_{l=1}^{n}\delta_{(l)}y_{iju(l)}$$

$$\hat{\mu}.._u. = \frac{1}{6m}\sum_{i=1}^{2}\sum_{j=1}^{3}\sum_{l=1}^{n}\delta_{(l)}y_{iju(l)} \; ; \; \hat{\mu}_{ij}.. = \frac{1}{3m}\sum_{u=1}^{3}\sum_{l=1}^{n}\delta_{(l)}y_{iju(l)} \; ; \; \hat{\mu}_i ._u. = \frac{1}{3m}\sum_{j=1}^{3}\sum_{l=1}^{n}\delta_{(l)}y_{iju(l)}$$

$$\hat{\mu}._{ju}. = \frac{1}{2m}\sum_{i=1}^{2}\sum_{l=1}^{n}\delta_{(l)}y_{iju(l)} \; ; \; \hat{\mu}_{iju}. = \frac{1}{m}\sum_{l=1}^{n}\delta_{(l)}y_{iju(l)} \tag{3.29}$$

3.4 Main and Interaction Effects

The treatments consisting of all combinations that can be formed from the different factors may be represented as *A*, *B*, *AB*, B^2, AB^2, *C*, *AC*, *BC*, *ABC*, B^2C, AB^2C, C^2, AC^2, BC^2, ABC^2, B^2C^2, AB^2C^2

Letters (1), *a*, *b*, *ab*, b^2, ab^2, *c*, *ac*, *bc*, *abc*. b^2c, ab^2c, c^2, ac^2, bc^2, abc^2, b^2c^2 and ab^2c^2 represent the weighted totals of n observations obtained from each treatment combination.

$$(1)=\sum_l \delta_l y_{111(l)},\ a=\sum_l \delta_l y_{211(l)},\ b=\sum_l \delta_l y_{121(l)},\ ab=\sum_l \delta_l y_{221(l)},\ b^2=\sum_l \delta_l y_{131(l)},$$
$$ab^2=\sum_l \delta_l y_{231(l)},\ c=\sum_l \delta_l y_{112(l)}\ ,\ ac=\sum_l \delta_l y_{212(l)},\ bc=\sum_l \delta_l y_{122(l)},\ abc=\sum_l \delta_l y_{222(l)},$$
$$b^2c=\sum_l \delta_l y_{132(l)},\ ab^2c=\sum_l \delta_l y_{232(l)},\ c^2=\sum_l \delta_l y_{113(l)},\ ac^2=\sum_l \delta_l y_{213(l)},\ bc^2=\sum_l \delta_l y_{123(l)},$$
$$abc^2=\sum_l \delta_l y_{223(l)},\ b^2c^2=\sum_l \delta_l y_{133(l)},\ ab^2c^2=\sum_l \delta_l y_{233(l)}.$$

The contrasts are obtained using Yates algorithm and consequently sum of squares and *F* statistics for the main effects, two-factor and three-factor interactions have been computed.

$$SS_A=\frac{1}{18m}(Contrast_A)^2;\quad SS_B=\frac{1}{18m}(Contrast_B)^2;\quad SS_C=\frac{1}{12m}(Contrast_C)^2;$$

$$SS_{AB}=\frac{1}{12m}(Contrast_{AB})^2;\quad SS_{AC}=\frac{1}{12m}(Contrast_{AC})^2;\quad SS_{BC}=\frac{1}{8m}(Contrast_{BC})^2;$$

$$\ldots,\ SS_{AB^2C^2}=\frac{1}{72m}(Contrast_{AB^2C^2})^2.$$

The test statistics for testing the hypotheses are

$$F_A^*=\frac{SS_A}{\hat{\sigma}^2},\ F_B^*=\frac{SS_B}{\hat{\sigma}^2},\ F_C^*=\frac{SS_C}{\hat{\sigma}^2},\ F_{AB}^*=\frac{SS_{AB}}{\hat{\sigma}^2},\ F_{BC}^*=\frac{SS_{BC}}{\hat{\sigma}^2},\ F_{AC}^*=\frac{SS_{AC}}{\hat{\sigma}^2},$$
$$F_{B^2}^*=\frac{SS_{B^2}}{\hat{\sigma}^2},\ldots,\ F_{AB^2}^*=\frac{SS_{AB^2}}{\hat{\sigma}^2}\ldots,\ F_{AB^2C^2}^*=\frac{SS_{AB^2C^2}}{\hat{\sigma}^2}.$$

For large n, their null distributions are central *F* with degrees of freedom

$(\upsilon_1,\upsilon_{18})$, $(\upsilon_2,\upsilon_{18})$, $(\upsilon_3,\upsilon_{18})$, $(\upsilon_4,\upsilon_{18})$, $(\upsilon_5,\upsilon_{18})$, $(\upsilon_6,\upsilon_{18})$, $(\upsilon_7,\upsilon_{18})$, $(\upsilon_8,\upsilon_{18})$, $(\upsilon_9,\upsilon_{18})$, $(\upsilon_{10},\upsilon_{18})$, $(\upsilon_{11},\upsilon_{18})$, $(\upsilon_{12},\upsilon_{18})$, $(\upsilon_{13},\upsilon_{18})$, $(\upsilon_{14},\upsilon_{18})$, $(\upsilon_{15},\upsilon_{18})$, $(\upsilon_{16},\upsilon_{18})$, and $(\upsilon_{17},\upsilon_{18})$, respectively; where $\upsilon_1=\upsilon_2=\upsilon_3=\upsilon_4=\upsilon_5=\upsilon_6=\upsilon_7=\upsilon_8=\upsilon_9$ $=\upsilon_{10}=\upsilon_{11}=\upsilon_{12}=\upsilon_{13}=\upsilon_{14}=\upsilon_{15}=\upsilon_{16}=\upsilon_{17}=1$ and $\upsilon_{18}=18(n-1)$.

3.5 Discussion

The analytical Procedure have been developed for the 2×3×3 factorial experiments when error follows Gompertz distribution. The estimates are obtained through Modified Maximum Likelihood Estimation procedure (MMLE) for the main effect and interaction effects and *F* statistics have been obtained. SAS code has been developed for the developed procedure.

CHAPTER 4

APPLICATION OF THE MODIFIED MAXIMUM LIKELIHOOD ESTIMATION PROCEDURE ON SIMULATED DATA

4.1 Introduction:

In this Chapter, a factorial experiment has been considered in which there are three factors *A*, *B* and *C*. In which factor *A* has 2 levels, factor *B* and *C* have 3 levels. Data sets for 2×3×3 factorial experiments have been obtained for both the situations of non-normality. The developed procedure has been applied on the generated data and sum of squares and *F* statistics have been obtained for all the treatment combinations. The analysis of the datasets of 2×3×3 factorial experiments considering the two cases generalized logistic distribution and Gompertz distribution have been given.

Probability density function of generalized logistic distribution

$$GL(\mu,\sigma,\theta)=\frac{\theta}{\sigma}\frac{\exp\{-(\frac{x-\mu}{\sigma})\}}{[1+\exp\{-(\frac{x-\mu}{\sigma})\}]^{\theta+1}} \quad ; \quad -\infty<x<\infty;\ \sigma>0;\ \theta>0.$$

Probability density function of Gompertz distribution

$$G(\eta,\alpha)=\eta\exp\left[\alpha x-\frac{\eta}{\alpha}\left\{\exp(\alpha x)-1\right\}\right]$$

$$\text{where, } x\geq 0;\ \eta>0;\ \alpha>0$$

Procedure for calculation of sum of squares has been worked out for 2×3×3 factorial experiments.

4.2 Generation of data in case of logistic error distribution

In some situations, when the errors are non-normal, the distribution of test statistic is unknown. It is, therefore, necessary to generate empirical critical values in order to test H_0 against H_1. This is done by means of simulation studies. In this Section, data has been generated for which error follows the logistic distribution for different values of θ which gives different shape of the distribution. A real data set has been taken in 2×3×3 factorial set up with three observation per cell (54 observations0 and predicted values and residuals have been obtained then a sample of 54 observations of logistic error has been generated with the parameter values θ =0.5, 1 and 2 for which the data is positively skewed, symmetric and negatively skewed respectively. The predicted values are added with the logistic error and therefore we have three sets of data for the 2×3×3 Factorial experiments where error follows logistic distribution. The generated data for θ =0.5, 1 and 2 has been given in Table: 4.1, 4.2 and 4.3 respectively.

Table: 4.1 Data generated for 2×3×3 factorial experiment when error follows logistic distribution for θ =0.5

Treatment effects	mean	A	B	AB	B^2	AB^2	C	AC	BC
	000	**100**	**010**	**011**	**020**	**120**	**001**	**101**	**011**
Observations Generated Per cell	46.600	85.510	39.367	78.188	91.575	75.483	79.415	77.445	88.546
	46.996	86.546	57.851	95.894	91.920	79.452	81.132	78.751	89.907
	47.965	91.348	58.888	97.297	97.552	81.559	87.255	80.108	101.604
	ABC	B^2C	AB^2C	C^2	AC^2	BC^2	ABC^2	B^2C^2	AB^2C^2
	111	**021**	**121**	**002**	**102**	**012**	**112**	**022**	**122**
Observations Generated Per cell	82.349	66.439	71.914	67.579	78.601	55.315	74.496	66.271	82.349
	84.240	68.450	73.483	85.913	78.681	74.976	78.088	69.679	84.240
	91.306	73.036	73.517	92.569	84.751	78.099	82.332	72.187	91.306

Table: 4.2 Data generated for 2×3×3 factorial experiment when error follows logistic distribution for θ =1

Treatment effects	mean	A	B	AB	B^2	AB^2	C	AC	BC
	000	**100**	**010**	**011**	**020**	**120**	**001**	**101**	**011**
Observations Generated Per cell	49.076	93.326	63.210	97.339	98.980	78.881	82.652	82.655	76.091
	54.074	96.483	66.496	98.388	99.516	84.450	86.161	84.171	77.578
	58.041	98.360	69.754	101.581	102.059	89.242	91.532	87.465	79.757
	ABC	B^2C	AB^2C	C^2	AC^2	BC^2	ABC^2	B^2C^2	AB^2C^2
	111	**021**	**121**	**002**	**102**	**012**	**112**	**022**	**122**
Observations Generated Per cell	81.430	81.502	69.237	78.890	79.159	76.033	72.382	79.171	65.790
	88.182	83.077	69.254	79.646	82.162	80.455	76.224	81.099	68.286
	93.432	87.729	73.805	80.923	94.030	82.140	79.452	85.431	72.087

Table: 4.3 Data generated for 2×3×3 factorial experiment when error follows logistic distribution for θ =2

Treatment effects	**mean**	***A***	***B***	***AB***	***B^2***	***AB^2***	***C***	***AC***	***BC***
	000	**100**	**010**	**011**	**020**	**120**	**001**	**101**	**011**
Observations Generated Per cell	51.268	95.302	65.432	99.452	100.865	82.320	85.302	84.911	78.385
	55.885	98.292	68.389	100.388	101.374	86.591	88.217	86.226	79.662
	59.790	100.130	71.533	103.397	103.838	91.048	93.309	89.297	81.655
	ABC	***B^2C***	***AB^2C***	***C^2***	***AC^2***	***BC^2***	***ABC^2***	***B^2C^2***	***AB^2C^2***
	111	**021**	**121**	**002**	**102**	**012**	**112**	**022**	**122**
Observations Generated Per cell	85.257	84.742	71.756	80.939	82.300	79.883	75.766	81.398	69.981
	90.315	85.868	71.769	81.623	84.562	82.999	78.612	83.095	71.594
	95.225	89.728	75.721	82.810	95.772	84.372	81.418	87.216	74.439

4.3 Analysis 2×3×3 factorial experiment when error follows logistic distribution

A dataset for 2×3×3 factorial experiments with three observations per cell is taken with total of 54 observations for which error follows the logistic distribution which have been generated in above section 4.2. This data has been analyzed with the methodology developed and presented in Table: 4.4, 4.5 and 4.6 for the different values of the distribution parameter θ (0.5, 1 and 2). SAS code has been developed for the analysis with modified maximum likelihood procedure developed and given in ANNEXURE.

Table: 4.4 Analysis of 2×3×3 factorial experiments for θ = 0.5

Source	**DF**	**Mean Square**	**F^* Value**
A	1	96.9584	5.52785*
B	1	29.0356	1.65539
AB	1	387.597	22.0979*
B^2	1	0.26204	0.01494
AB^2	1	132.118	7.53237*
C	1	0.39498	0.02252
AC	1	287.006	16.3629*

BC	1	160.285	9.13824*
ABC	1	106.141	6.05139*
B^2C	1	8.31018	0.47378
AB^2C	1	136.191	7.7646*
C^2	1	67.7792	3.86426
AC^2	1	57.0143	3.25053
BC^2	1	46.4333	2.64728
ABC^2	1	59.0471	3.36643
B^2C^2	1	29.8689	1.7029
AB^2C^2	1	64.4351	3.67361
Error	36	3.107	

Note: F^* Value is obtained from the procedure of MMLE methodology.

Table: 4.5 Analysis of 2×3×3 factorial experiments for θ = 1

Source	DF	Mean Square	F^* Value
A	1	60.682	9.989*
B	1	0.651	0.107
AB	1	401.210	66.042*
B^2	1	0.065	0.011
AB^2	1	97.348	16.024*
C	1	43.566	7.171*
AC	1	260.292	42.846*
BC	1	184.508	30.371*
ABC	1	125.465	20.652*
B^2C	1	0.010	0.002
AB^2C	1	37.050	6.099*
C^2	1	2.755	0.454
AC^2	1	58.745	9.670*
BC^2	1	75.838	12.483*
ABC^2	1	74.974	12.341*

B^2C^2	1	2.487	0.409
AB^2C^2	1	8.341	1.373
Error	16	2.465	

Table: 4.6 Analysis of 2×3×3 factorial experiments for θ = 2

Source	DF	Mean Square	F^* Value
A	1	57.2572	9.61416*
B	1	2.32479	0.39036
AB	1	358.294	60.1618*
B^2	1	0.00912	0.00153
AB^2	1	84.8404	14.2457*
C	1	31.558	5.29897*
AC	1	240.451	40.3745*
BC	1	165.154	27.7313*
ABC	1	130.723	21.95*
B^2C	1	0.00219	0.00037
AB^2C	1	37.34	6.26983*
C^2	1	2.95179	0.49564
AC^2	1	56.3878	9.46819*
BC^2	1	68.5124	11.504*
ABC^2	1	61.956	10.4031*
B^2C^2	1	1.45181	0.24378
AB^2C^2	1	7.38491	1.24001
Error	36	2.440	

4.4 Computation of Size of the Test

The probability P (F^*>$F_{0.05}$) (v_1, v_{18}) is calculated empirically. The computation of size of the test is done by using a program in SAS-IML. Further, the size of the test has been obtained by re-sampling technique to obtain the distribution using Monte Carlo simulation in 5000 runs and SAS code has been developed for the purpose which is given in ANNEXURE-2 and the results are given in the Table: 4.7.

Table: 4.7 Size of the Test from Monte Carlo simulation of 5000 runs

For n=3			
Source	θ**=1**	θ **=0.5**	θ **=2**
A	0.0742	0.303	0.1334
B	0.1392	0.2842	0.4098
AB	0.0028	0.1052	0.0432
B^2	0.2006	0.1092	0.0594
AB^2	0.0278	0.2682	0.1572
C	0.0758	0.1402	0.2706
AC	0.0052	0.1076	0.1146
BC	0.0286	0.2122	0.0624
ABC	0.0286	0.1592	0.1958
B^2C	0.0222	0.3482	0.0242
AB^2C	0.1162	0.2648	0.1718
C^2	0.3422	0.161	0.3978
AC^2	0.0956	0.2946	0.1462
BC^2	0.0592	0.2268	0.1794
ABC^2	0.0904	0.2358	0.1126
B^2C^2	0.3156	0.3202	0.4116
AB^2C^2	0.1696	0.2768	0.3234

From the above it can be seen that size of the test is approximate 0.05 for some of the main effects and some of the interactions for the parameter value θ = 1. For other parameter values it considerably deviates from the value 0.05. This also validates the procedure of modified maximum likelihood estimation.

4.5 Generation of data in case of Gompertz error distribution

In this Section, data has been generated for which error follows the Gompertz distribution for different values of η. A real data set has been taken in 2×3×3 Factorial set up with three observation per cell (54 observations) and predicted values and residuals have been obtained then a sample of 54 observations of error has been generated which follows Gompertz distribution with the parameter values η =1 and 2. The predicted values are added with these errors and therefore we have two data sets for the 2×3×3 Factorial experiments where error follows Gompertz distribution. The data generated for η =1 and 2 has been given in Table: 4.8 and 4.9 respectively.

Table: 4.8 Data generated for 2×3×3 factorial experiment when error follows Gompertz distribution for η =1

Treatment effects	mean	A	B	AB	B^2	AB^2	C	AC	BC
	000	**100**	**010**	**011**	**020**	**120**	**001**	**101**	**011**
Observations Generated Per cell	49.248	91.689	63.407	96.926	95.670	84.135	85.087	83.281	76.732
	49.466	91.943	63.666	96.959	95.976	84.362	85.753	83.451	76.991
	49.747	92.071	63.799	97.487	96.397	84.480	85.784	83.535	77.274
	ABC	B^2C	AB^2C	C^2	AC^2	BC^2	ABC^2	B^2C^2	AB^2C^2
	111	**021**	**121**	**002**	**102**	**012**	**112**	**022**	**122**
Observations Generated Per cell	87.673	86.224	71.128	78.322	83.513	82.835	77.505	79.334	73.249
	87.896	86.273	71.246	78.379	83.583	82.864	77.627	79.449	73.569
	88.190	86.743	71.307	78.911	83.789	82.893	77.696	79.766	73.791

Table: 4.9 Data generated for 2×3×3 factorial experiment when error follows Gompertz distribution for η =2

Treatment effects	mean	A	B	AB	B^2	AB^2	C	AC	BC
	000	**100**	**010**	**011**	**020**	**120**	**001**	**101**	**011**
Observations Generated Per cell	49.086	91.475	63.398	96.670	95.704	84.057	85.151	83.131	76.698
	49.182	91.636	63.748	96.944	95.804	84.286	85.299	83.204	76.838
	49.280	91.737	64.061	97.139	95.844	84.406	85.358	83.540	76.886
	ABC	B^2C	AB^2C	C^2	AC^2	BC^2	ABC^2	B^2C^2	AB^2C^2
	111	**021**	**121**	**002**	**102**	**012**	**112**	**022**	**122**
Observations Generated Per cell	87.949	86.249	71.027	77.667	83.542	82.345	77.357	79.382	73.211
	88.060	86.469	71.281	77.980	83.568	82.915	77.456	79.394	73.317
	88.347	86.566	71.439	78.016	83.947	83.217	77.921	79.442	73.351

4.6 Analysis of 2×3×3 factorial experiment when error follows Gompertz distribution

A SAS code has been developed (which in given in ANNEXURE) to analyse the data sets which follows Gompertz error distribution for the above mentioned factorial setup. The data sets generated in the above section have been analysed and presented in tables 4.10 and 4.11.

Table: 4.10 Analysis of 2×3×3 factorial experiments for η = 1

Source	**DF**	**Mean Square**	**F^* Value**
A	1	270.157	113.823*
B	1	51.747	21.802*
AB	1	924.881	389.672*
B^2	1	3.912	1.648
AB^2	1	216.694	91.298*
C	1	6.792	2.862
AC	1	748.180	315.224*
BC	1	523.071	220.381*
ABC	1	418.124	176.165*
B^2C	1	2.787	1.174
AB^2C	1	158.786	66.900*
C^2	1	31.040	13.078*
AC^2	1	251.824	106.099*
BC^2	1	207.316	87.347*
ABC^2	1	114.996	48.450*
B^2C^2	1	0.014	0.006
AB^2C^2	1	67.047	28.248*
Error	36	1.541	

Table: 4.11 Analysis of 2×3×3 factorial experiments for η = 2

Source	**DF**	**Mean Square**	**F^* Value**
A	1	278.895	106.155*
B	1	57.489	21.882*

AB	1	944.362	359.449*
B^2	1	5.381	2.048
AB^2	1	213.847	81.395*
C	1	7.059	2.687
AC	1	731.705	278.506*
BC	1	522.991	199.064*
ABC	1	400.743	152.533*
B^2C	1	2.722	1.036
AB^2C	1	157.216	59.840*
C^2	1	34.607	13.172*
AC^2	1	248.788	94.695*
BC^2	1	205.700	78.295*
ABC^2	1	117.856	44.859*
B^2C^2	1	0.008	0.003
AB^2C^2	1	73.705	28.054*
Error	36	1.621	

4.7 Discussion

Error has been generated which follows logistic distribution for different values of θ which gives different shapes of the distribution. The distribution is negatively skewed, positively skewed or symmetric for the values of θ = 0.5, 1 and 2, respectively. Using these generated logistic error 3 sets of 54 observations in 2×3×3 factorial setup has been obtained. The model is assumed to be fixed effect model and design considered is completely randomized design for equal number of observations per cell. These sets are analysed through the developed modified maximum likelihood estimation procedure and sum of squares and F statistics have been computed. Similarly two data sets have been generated for Gomperz error distribution with the parameter values η =1 and 2 The SAS code have been developed and generated data sets have been analyzed and presented. Finally, SAS code has been developed for the computation of size of the test.

SUMMARY

Factorial experiments are widely used in agriculture and allied sciences. In the present study, 2×3^2 factorial experiments have been considered under the assumption that the error follows non-normal distribution. Under the assumption of normality and independence of observations, the normal equations obtained from maximum likelihood function are linear and solvable. On the other hand when the data do not follow the normal distribution, the equations obtained of MLE are not linear and so these equations are difficult to handle.

The present study focuses on the development of analytical procedure for the factorial experiments in order to tackle the situations where error term violates normality assumptions. Here, factorial experiments have been considered where error follows non-normal distribution. Two non-normal distributions have been considered from which one is generalized logistic distribution and another is Gompertz distribution. The theory of modified maximum likelihood estimation has been applied and efficient estimators have been developed. New modified maximum likelihood estimates have been developed and the estimates of parameters are obtained for both the situations of non-normality.

The developed procedure is applied in the analysis of 2×3^2 factorial experiments in which error follows the generalized logistic and Gompertz error distributions. Data have been generated for the simulation studies for which error follows generalized logistic distribution. Three data sets have been generated for parameter values (θ =0.5, 1 and 2) in 2×3^2 factorial set up where the data are positively skewed, symmetric and negatively skewed respectively. In the same way, two data sets have been generated with the parameter values (η =1 and 2) where error follows Gompertz distribution. These data sets are analyzed through developed procedure.

SAS codes have been developed for analysis of the data sets generated through 2×3^2 factorial experiments where error follows logistic and Gompertz distributions. The output for the data sets of all mentioned five parameter values *i.e.* θ =0.5, 1, 2 and η =1, 2 are given in table 4.4, 4.5, 4.6, 4.10 and 4.11 of Chapter 4 where sum of squares and F^* statistics have been given. The probability P ($F^*>F_{0.05}$) (v_1, v_{18}) is calculated empirically for the developed F^* statistics. Further, size of the test is computed with 5000 Monte Carlo runs using re-sampling technique.

This present investigation would help scientist, research scholars and students under NARES dealing with factorial experiments where error follows generalized logistic distribution and Gompertz distribution. For easy accessibility by the users, the SAS codes have been developed which provide a readymade solution.

सारांश

बहुउपादानी परीक्षण कृषि और सम्बद्ध विज्ञान में व्यापक रूप से उपयोग किए जाते हैं। वर्तमान अध्ययन में, 2×3^2 बहुउपादानी परीक्षणों के अन्तर्गत माना गया है कि त्रुटि नॉन-नॉर्मल वितरण का अनुसरण करती है। नॉर्मलटी और अवलोकनों की स्वतंत्रता की के अन्तर्गत, अधिकतम संभावना आंकलन से प्राप्त नॉर्मल समीकरण रैखिक और हल करने योग्य होते हैं। दूसरी ओर जब अवलोकन नॉर्मल वितरण का पालन नहीं करते हैं, तो अधिकतम संभावना आंकलन के प्राप्त समीकरण रैखिक नहीं होते हैं और इसलिए इन समीकरणों को हल करना मुश्किल हो जाता है।

वर्तमान अध्ययन उन परिस्थितियों से निपटने के लिए बहुउपादानी परीक्षणों के लिए विश्लेषणात्मक प्रक्रिया के विकास पर केंद्रित है, जहां त्रुटि नॉर्मलटी का उल्लंघन करता है। यहाँ, बहुउपादानी परीक्षणों पर विचार किया गया है जहां त्रुटि नॉन-नॉर्मल वितरण का अनुसरण करती है। दो नॉन-नॉर्मल वितरणों पर विचार किया गया उनमें से एक सामान्यीकृत लॉजिस्टिक वितरण और दूसरा गोम्पर्ट्ज वितरण है। संशोधित अधिकतम संभावना आंकलन के सिद्धांत को लागू किया गया है और कुशल अनुमानक विकसित किए गए हैं। नये संशोधित अधिकतम संभावना आंकलक विकसित किये गए है और त्रुटि वितरण के दोनों नॉन-नॉर्मल परिस्थितियों के लिए मापदंडों का आंकलन प्राप्त किया जाता है।

विकसित प्रक्रिया, 2×3^2 बहुउपादानी परीक्षणों के विश्लेषण में लागू की गयी है जिसमें त्रुटि लॉजिस्टिक और गोम्पर्ट्ज वितरण का अनुसरण करती है, सिमुलेशन अध्ययन के लिए अवलोकनों को उत्पन्न किया गया है। पैरामीटर मानों जो $\theta = 0.5, 1$ और 2 के लिए 2×3^2 बहुउपादानी परीक्षणों में अवलोकनों के तीन सेट उत्पन्न किए गए हैं जिनके लिए अवलोकन क्रमशः वैषम्य, सममित और ऋणात्मक वैषम्य है। उसी तरह, पैरामीटर मान $\eta = 1$ और 2 के साथ अवलोकनों के दो सेट उत्पन्न किए गए हैं जहां त्रुटि गोम्पर्ट्ज वितरण का अनुसरण करती है। इन अवलोकनों के सेटों का विश्लेषण विकसित प्रक्रिया के माध्यम से किया गया है।

2×3^2 बहुउपादानी परीक्षणों के माध्यम से उत्पन्न अवलोकनों के विश्लेषण के लिए SAS कोड विकसित किए गए हैं जहां त्रुटि लॉजिस्टिक और गोम्पर्ट्ज वितरण का अनुसरण करती है। सभी उल्लिखित पाँच पैरामीटर मानों जोकि $\theta = 0.5, 1, 2$ और $\eta = 1, 2$ के अवलोकनों के सेट के लिए आउटपुट, अध्याय 4 के तालिका-4.4, 4.5, 4.6, 4.10 और 4.11 में दिए गए हैं जिनमें वर्गों का योग और F स्टेटिस्टिक्स हैं। संभाव्यता $P(F^* > F_{0.05})(v_1, v_{18})$ की गणना आनुभविक रूप से की गयी है। इसके अलावा, परीक्षण के आकार की गणना 5000 मोंटे कार्लो के साथ की गयी है, जो पुनर्नमूने की तकनीक का उपयोग करता है।

यह वर्तमान अध्ययन, वैज्ञानिकों, एनएआरईएस के अन्तर्गत अनुसंधान विद्वानों और छात्रों को बहुउपादानी परीक्षणों के विश्लेषण में मदद करेगी जहां त्रुटि सामान्यीकृत लॉजिस्टिक वितरण और गोम्पर्ट्ज वितरण का अनुसरण करती हैं। उपयोगकर्ताओं की सुगमता के लिए , SAS कोड विकसित किए गए हैं जो एक रेडीमेड समाधान प्रदान करते हैं।

ANNEXURE

SAS code for the analysis when error follows logistic error distribution

```
data mrin;
input x;
datalines;
<data set>
;
proc iml;
use mrin;
read all into data;
obs=data(|,1|);
b=1;/* value of parameter b*/
trt=18;/* number of treatment*/
n=3;/* number of replication*/
dfc={-1     1     -1     1     -1     1     -1     1     -1     1     -1     1
      -1     1     -1     1     -1     1,
-1    -1     0      0     1      1     -1    -1     0      0     1      1     -1
      -1     0      0     1      1,
1     -1     0      0     -1     1     1     -1     0      0     -1     1     1
      -1     0      0     -1     1,
1     1      -2     -2    1      1     1     1      -2     -2    1      1     1
      1      -2     -2    1      1,
-1    1      2      -2    -1     1     -1    1      2      -2    -1     1     -1
      1      2      -2    -1     1,
-1    -1     -1     -1    -1     -1    0     0      0      0     0      0     1
      1      1      1     1      1,
1     -1     1      -1    1      -1    0     0      0      0     0      0     -1
      1      -1     1     -1     1,
1     1      0      0     -1     -1    0     0      0      0     0      0     -1
      -1     0      0     1      1,
-1    1      0      0     1      -1    0     0      0      0     0      0     1
      -1     0      0     -1     1,
-1    -1     2      2     -1     -1    0     0      0      0     0      0     1
      1      -2     -2    1      1,
1     -1     -2     2     1      -1    0     0      0      0     0      0     -1
      1      2      -2    -1     1,
1     1      1      1     1      1     -2    -2     -2     -2    -2     -2    1
      1      1      1     1      1,
-1    1      -1     1     -1     1     2     -2     2      -2    2      -2    -1
      1      -1     1     -1     1,
-1    -1     0      0     1      1     2     2      0      0     -2     -2    -1
      -1     0      0     1      1,
1     -1     0      0     -1     1     -2    2      0      0     2      -2    1
      -1     0      0     -1   1,
1     1      -2     -2    1      1     -2    -2     4      4     -2     -2    1
      1      -2     -2    1      1,
-1    1      2      -2    -1     1     2     -2     -4     4     2      -2    -1
      1      2      -2    -1     1};/* define contrast*/
k=nrow(obs);
c=sum(obs);
mean=b/k;
p=j(k,1,mean);
sq=obs-p;
f1=sq#sq;
f2=sum(f1);
var=f2/k;
sd=sqrt(var);
```

```
Z=sq/sd;
%let h=n;
p2=j(&h,1,0);
do t=1 to &h;
 p2[t,]=t;
 a=exp(t);
 end;
 qr=p2/(n+1);
 ql=repeat(qr,trt);
 g1=-(1/b);
 mq1=ql##g1;
 p1=j(k,1,1);
 dmq1=mq1-p1;
 tl=-log(dmq1);
 %let h=k;
dell=j(&h,1,0);
gal=j(&h,1,0);
do t=1 to &h;
a=exp(tl[t]);
b1=tl[t];
 dell[t,]=(a/((1+a)*(1+a)));
 gal[t,]=((1+b1*a+a)/((1+a)*(1+a)));
 end;
 cont=dell#obs;
%let h1=trt;
pc1=j(&h1,1,0);
do t=1 to &h1;
 pc1[t,]=t;
 end;
 pc2=repeat(pc1,n);
 call sort(pc2,1);
 pc3=design(pc2);
 cont1=cont`;
 contf=cont1*pc3;
 contfn=contf`;
 m=sum(dell)/trt;
 muij=contfn/m;
 muijj=pc1||muij;
 crast=dfc*contfn;
 muijjj=repeat(muijj,n);
 call sort (muijjj,1);
 start delcol(x,i);
return(x[,setdif(1:ncol(x),i)]);
finish;
muijl=delcol(muijjj,1);
 new=obs-muijl;
 f=1/(b+1);
 new1=repeat(f,k);
 new2=new1-gal;
 new3=new2#new;
 f1=1+b;
 new4=repeat(f1,k);
 bij=new4#new3;
 new5=new#new;
 new6=dell#new5;
 cij=f1#new6;
 ray=sum(bij);
 ray1=sum(cij);
 nume=ray+sqrt(ray*ray+4*k*ray1);
 denom=2*sqrt(k*(k-18));
 sigma=nume/denom;
```

```
 jani=crast#crast;
 trtdiv={18 12    12    36    36    12    12    8     8     24    24    36
      36    24    24    72    72}` *m;
 sss=jani/trtdiv;
 sig=sigma*sigma;
 fstat=sss/sig;
print sigma sss fstat;
 %mend;
```

SAS Code for computation of Size of the Test

```
dm output 'clear' output;
dm log 'clear' output;
%let b=1;/* value of parameter b*/
%let trt=18;/* number of treatment*/
%let n=3;/* number of replication*/
%let obs=54;/*total number of observations */
%let rep=10;/*total number of iterations */
data mrin;
input id x;
datalines;
<data set>;
run;
data mrin2;
input g a b ab b2 ab2   c ac bc     abc   b2c   ab2c c2     ac2   bc2
      abc2 b2c2 ab2c2 ;
cards;
-1    1     -1    1     -1    1     -1    1     -1    1     -1    1     -1
      1     -1    1     -1    1
-1    -1    0     0     1     1     -1    -1    0     0     1     1     -1
      -1    0     0     1     1
1     -1    0     0     -1    1     1     -1    0     0     -1    1     1
      -1    0     0     -1    1
1     1     -2    -2    1     1     1     1     -2    -2    1     1     1
      1     -2    -2    1     1
-1    1     2     -2    -1    1     -1    1     2     -2    -1    1     -1
      1     2     -2    -1    1
-1    -1    -1    -1    -1    -1    0     0     0     0     0     0     1
      1     1     1     1     1
1     -1    1     -1    1     -1    0     0     0     0     0     0     -1
      1     -1    1     -1    1
1     1     0     0     -1    -1    0     0     0     0     0     0     -1
      -1    0     0     1     1
-1    1     0     0     1     -1    0     0     0     0     0     0     1
      -1    0     0     -1    1
-1    -1    2     2     -1    -1    0     0     0     0     0     0     1
      1     -2    -2    1     1
1     -1    -2    2     1     -1    0     0     0     0     0     0     -1
      1     2     -2    -1    1
1     1     1     1     1     1     -2    -2    -2    -2    -2    -2    1
      1     1     1     1     1
```

```
-1     1     -1     1     -1     1     2     -2     2     -2     2     -2     -1
       1     -1     1     -1     1
-1     -1    0      0     1      1     2     2      0     0      -2    -2     -1
       -1    0      0     1      1
1      -1    0      0     -1     1     -2    2      0     0      2     -2     1
       -1    0      0     -1     1
1      1     -2     -2    1      1     -2    -2     4     4      -2    -2     1
       1     -2     -2    1      1
-1     1     2      -2    -1     1     2     -2     -4    4      2     -2     -1
       1     2      -2    -1     1
;
run;
%macro sunil;
proc iml;
use sunil1;
read all into data;
obs=data(|,1|);
b=&b;
trt=&trt;
n=&n;
use mrin2;
read all into dfc;
k=nrow(obs);
c=sum(obs);
mean=b/k;
p=j(k,1,mean);
sq=obs-p;
f1=sq#sq;
f2=sum(f1);
var=f2/k;
sd=sqrt(var);
Z=sq/sd;
%let h=n;
p2=j(&h,1,0);
do t=1 to &h;
 p2[t,]=t;
 a=exp(t);
 end;
 qr=p2/(n+1);
 ql=repeat(qr,trt);
 g1=-(1/b);
 mq1=ql##g1;
 p1=j(k,1,1);
 dmq1=mq1-p1;
 tl=-log(dmq1);
 %let h=k;
dell=j(&h,1,0);
gal=j(&h,1,0);
do t=1 to &h;
a=exp(tl[t]);
b1=tl[t];
 dell[t,]=(a/((1+a)*(1+a)));
 gal[t,]=((1+b1*a+a)/((1+a)*(1+a)));
 end;
 cont=dell#obs;
%let h1=trt;
pc1=j(&h1,1,0);
do t=1 to &h1;
 pc1[t,]=t;
 end;
 pc2=repeat(pc1,n);
```

```
 call sort(pc2,1);
 pc3=design(pc2);
 cont1=cont`;
 contf=cont1*pc3;
 contfn=contf`;
 m=sum(dell)/trt;
 muij=contfn/m;
 muijj=pc1||muij;
 crast=dfc*contfn;
 muijjj=repeat(muijj,n);
 call sort (muijjj,1);
 start delcol(x,i);
return(x[,setdif(1:ncol(x),i)]);
finish;
muijl=delcol(muijjj,1);
 new=obs-muijl;
 f=1/(b+1);
 new1=repeat(f,k);
 new2=new1-gal;
 new3=new2#new;
 f1=1+b;
 new4=repeat(f1,k);
 bij=new4#new3;
 new5=new#new;
 new6=dell#new5;
 cij=f1#new6;
 ray=sum(bij);
 ray1=sum(cij);
 nume=ray+sqrt(ray*ray+4*k*ray1);
 denom=2*sqrt(k*(k-18));
 sigma=nume/denom;
 jani=crast#crast;
 trtdiv={18 12    12    36    36    12    12    8     8     24    24    36
      36    24    24    72    72}` *m;
 sss=jani/trtdiv;
 sig=sigma*sigma;
 fstat=sss/sig;
 print fstat;
 %mend;


%macro resampling1;
*dm log 'clear' output;
proc surveyselect data=mrin out=sunil method=srs sampsize=&obs seed=-20141
noprint;
run;

proc sql;
create table sunil1 as select x from sunil;
quit;

%sunil;

create Temp_org from fstat[colname="fstat_org"];
append from fstat;
quit;

data final_fstat_org;
set Temp_org ;
run;
%mend;
```

```
%macro resampling2;
%do i=1 %to &rep;
*dm log 'clear' output;
%put ..............;
%put iteration no.&i;
%put ..............;

proc surveyselect data=mrin out=sunil method=urs sampsize=&n outhits seed=-
20141 noprint;
strata id;
run;

proc sql;
create table sunil1 as select x from sunil;
quit;

%sunil;

create Temp_&i. from fstat[colname="fstat_&i."];
append from fstat;
quit;
%end;

data final_fstat;
%do i=1 %to &rep;
set Temp_&i. ;
%end;
run;

proc iml;
use final_fstat_org;
read all into data2;
fmat1=data2;
use final_fstat;
read all into data1;
fmat=data1;
finmat=fmat1||fmat;
power1=1/fmat1;
power2=finmat#power1;
power3=j(nrow(power2), ncol(power2),0);
do i=1 to nrow(power2);
do j=1 to ncol(power2);
if power2[i,j]< 1 then power3[i,j]=1;
end;
end;
power4=power3[,+];
rep=&rep;
power=power4/rep;
print power;
quit;
%mend;
%resampling1;
%resampling2;
```

SAS code for the analysis when error follows Gompertz error distribution

```
data mrin;
input x;
datalines;
<Data set>
;
proc iml;
use mrin;
read all into data;
obs=data(|,1|);
b=1;/* value of parameter b*/
trt=18;/* number of treatment*/
n=3;/* number of replication*/
dfc={ -1     1      -1     1      -1     1      -1     1      -1     1      -1     1
      -1     1      -1     1      -1     1,
-1    -1     0      0      1      1      -1     -1     0      0      1      1      -1
      -1     0      0      1      1,
1     -1     0      0      -1     1      1      -1     0      0      -1     1      1
      -1     0      0      -1     1,
1     1      -2     -2     1      1      1      1      -2     -2     1      1      1
      1      -2     -2     1      1,
-1    1      2      -2     -1     1      -1     1      2      -2     -1     1      -1
      1      2      -2     -1     1,
-1    -1     -1     -1     -1     -1     0      0      0      0      0      0      1
      1      1      1      1      1,
1     -1     1      -1     1      -1     0      0      0      0      0      0      -1
      1      -1     1      -1     1,
1     1      0      0      -1     -1     0      0      0      0      0      0      -1
      -1     0      0      1      1,
-1    1      0      0      1      -1     0      0      0      0      0      0      1
      -1     0      0      -1     1,
-1    -1     2      2      -1     -1     0      0      0      0      0      0      1
      1      -2     -2     1      1,
1     -1     -2     2      1      -1     0      0      0      0      0      0      -1
      1      2      -2     -1     1,
1     1      1      1      1      1      -2     -2     -2     -2     -2     -2     1
      1      1      1      1      1,
-1    1      -1     1      -1     1      2      -2     2      -2     2      -2     -1
      1      -1     1      -1     1,
-1    -1     0      0      1      1      2      2      0      0      -2     -2     -1
      -1     0      0      1      1,
1     -1     0      0      -1     1      -2     2      0      0      2      -2     1
      -1     0      0      -1   1,
1     1      -2     -2     1      1      -2     -2     4      4      -2     -2     1
      1      -2     -2     1      1,
-1    1      2      -2     -1     1      2      -2     -4     4      2      -2     -1
      1      2      -2     -1     1};/* define contrast*/
k=nrow(obs);
c=sum(obs);
mean=b/k;
p=j(k,1,mean);
sq=obs-p;
f1=sq#sq;
f2=sum(f1);
var=f2/k;
sd=sqrt(var);
Z=sq/sd;
%let h=n;
```

```
p2=j(&h,1,0);
do t=1 to &h;
 p2[t,]=t;
 a=exp(t);
 end;
 qr=p2/(n+1);
 q1=repeat(qr,trt);
 p1=j(k,1,1);
 q2=p1+q1;
 q3=log(q2);
 q4=b*q3;
 dmq1=p1-q4;
 tl=log(dmq1);
 %let h=k;
dell=j(&h,1,0);
gal=j(&h,1,0);
do t=1 to &h;
a=exp(tl[t]);
b1=tl[t];
 dell[t,]=(a);
 gal[t,]=((1-b1)*a);
 end;
 cont=dell#obs;
%let h1=trt;
pc1=j(&h1,1,0);
do t=1 to &h1;
 pc1[t,]=t;
 end;
 pc2=repeat(pc1,n);
 call sort(pc2,1);
 pc3=design(pc2);
 cont1=cont`;
 contf=cont1*pc3;
 contfn=contf`;
 m=sum(dell)/trt;
 muij=contfn/m;
 muijj=pc1||muij;
 crast=dfc*contfn;
 muijjj=repeat(muijj,n);
 call sort (muijjj,1);
 start delcol(x,i);
 return(x[,setdif(1:ncol(x),i)]);
 finish;
 muijl=delcol(muijjj,1);
 new=obs-muijl;
 new1=new#new;
 new2=sum(new1);
 A=b*new2;
 new3=gal-dell;
 new4=new#new3;
 new5=sum(new4);
 B=b*new5;
 new6=sum(gal);
 new7=b*new6;
 new8=b*k;
 C=new7-new8;
 nume=-B+sqrt(B*B-4*A*C);
 denum=2*A;
 sigma=nume/denum;
 jani=crast#crast;
```

```
trtdiv={18 12    12    36    36    12    12    8     8     24    24    36
     36    24    24    72    72}` *m;
sss=jani/trtdiv;
sig=sigma*sigma;
fstat=sss/sig;
print sigma sss fstat;
%mend;
```

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