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Generation of 2^n series fractional factorial plans robust against linear-trend

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Factorial experiments have been widely used in agricultural, biological and industrial experiments. However, they occur in experimental situations where the response is dependent on the spatial or temporal position of the experimental units within a block and thus trend in the experimental units becomes an important nuisance factor. In such situations, a common polynomial trend of a specified degree over units within experimental units may be appropriately assumed. Trend-free plans for single factor and at two level factorial experiments (complete and fractional both) are available in literature but trend-free multifactor plans could not be traced in the literature except some discussion by John (1990) where factors are at three levels.

Factorial/fractional factorial experiments are often used in various experiments. If there is trend in the experimental units then it is essential to sequence these factorial experiments such that factor effects are orthogonal to the trend. The resulting plans are termed as minimum cost linear trend-free factorial/fractional factorial experiments.

Trends may occur in the experimental units when the land is irrigated and the fertilizers supply the nutrients but because of the slope, the distribution of nutrients is not uniform. In the presence of trends, it is desired to allocate the treatment combinations to experimental units in such a manner that the main effects and interactions of interest are estimated free from the linear trend effects. Such fractional factorial plans are called robust against linear trend and the ordered application of treatments to experimental units is called run order (Yeh and Bradley 1983, Cheng 1988, Coster and Cheng 1988, Majumdar and Martin 2002, Lal *et al.* 2005; 2007, Sarkar *et al.* 2009, Nguyen 2013).

Consider, a complete factorial plan D with four factors each at two levels, i.e. D is a 2^4 plan. The four factors are denoted as A, B, C and D and the two levels of the factors are presence or absence of the small alphabet of that factor. In this factorial plan, treatment combination ab is complementary to the treatment combination cd because it is easy to obtain one from the other by simply changing the levels of all the factors in a combination. First, we give two Lemmas.

Lemma 1.1: {John 1990} Take a factorial plan in n runs (say plan D). Its fold over obtain a plan D′. Juxtapose the two plan to obtain plan $D^* = D \cup D'$ in 2n runs, which become linear trend-free for main effects.

Lemma 1.2: {John 1990} Let D, D′ and D* be as in Lemma 1.1. Then the main effects which are linear trend-free in D are quadratic trend-free in D*.

Consider a 2^{k-p} fractional factorial plan, which has k factors, denoted by 1,2,...,k and 2^{k-p} 2^{k-p} runs uniquely determined by p independent defining words (contrasts). A word consists of letters that are names of factors denoted by 1,2,...,k or a, b, … The group formed by p defining words can be represented by:

$$I = w_1 = w_2 = ... = w_{p-1}$$

or the treatment defining contrasts sub group $G = \{I, w_1, ..., w_{p-1}\}$.

For obtaining the plan that is linear trend-free for main effects by the technique of fold over, we have two rules other than the Lemma 1.1 and Lemma 1.2. If we start with a defining word and obtain a half replicate plan say D. Its fold over is D′. Combining the two, we have plan D*. Thus the two rules are:

Rule 1.1: If the letters in the defining word for plan D are even, then plan D′ is mirror image of D and we have only as much information in D* as contained in D (half replicate).

Rule 1.2: If the letters in the defining word for plan D are odd, then the plan D′ will contain the different treatment combination as contained in D. Then we have the plan D* that will have more information than D in terms of estimation of factorial effects and will be a complete factorial plan.

Generation of fractional factorial plan: We shall denote the algorithm that generates a plan for a fractional

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factorial plan of desired fraction in a single replication. Let p independent treatment combinations are taken in defining contrasts, and then this algorithm generates $\frac{1}{2^p}$ replicate of size 2^{k-p}.

Step 1: Generate a $n \times k$ matrix, representing n treatment combinations of a 2^k factorial plan in lexicographic order. Lower order is presented by 0 and higher order by 1.

Step 2: Solve the following equations for the chosen defining contrast:

$$\sum x_j = 0 \qquad (\text{mod } 2)$$
$$\sum x_j = 1; (j = 1, \cdots, k) \qquad (\text{mod } 2)$$

Where j takes the values of chosen contrast for (j = 1,2,...k) the treatment combination to be taken in defining contrast.

Step 3: Out of the two equations, retain the treatment combinations of any one, say of $\sum x_j = 0$, for having the half fraction.

Step 4: Similarly, p equations are solved for p independent treatment combinations and these are to be taken in defining contrasts by following the principle of defining contrast.

Step 5: The choice of the order of the fraction is such that main effects are not aliased with two factor interactions; they may be aliased with higher order interactions, though. This ensures the estimation of main effects and two factor interactions without correlations.

Generation of linear-trend free-fractional factorial plan: In this section, an algorithm is given to obtain a fractional factorial plan that is linear-trend-free for main effects and a search will be made from the obtained plan for two factor effects that are linear trend-free/nearly linear trend-free.

The steps of the algorithm to generate the half fraction that is linear trend free for main effects are given below:

Step 1: Generate fractional factorial plan by taking two independent defining contrasts containing an odd number of factors.

Step 2: The two independent defining contrasts must be such that their generalized interaction is the one of the desired half fraction.

Step 3: Replace the symbols 0 and 1 in each column by -1 and +1, respectively and call this plan as D.

Step 4: Generate the plan D′ by taking the fold over of D by replacing all +1 by -1 and -1 by +1.

Step 5: Juxtapose the plan D and D′. This is desired plan D*.

Step 6: The plan D* is now half replicate for which the defining contrast is the generalized interaction of two defining contrasts.

Step 7: We further describe steps to identify linear trend-free two-factor interactions in the plan generated in L6. From the linear main effects trend-free plan generated in L6 generate a new plan matrix $n \times \left[k + \binom{k}{2}\right]$ given by $Z = [X \;\; X^{(2)}]$. Here $X^{(2)}$ contains columns corresponding to the coefficients of the contrasts of all the $\binom{k}{2}$ 2-factors interactions obtained from X.

Step 8: For these 2-factor interactions, identify the columns that are linear trend-free. Then the corresponding 2-factor interactions are linear trend-free. Further, identify the columns from the remaining columns that satisfy the condition $0 < A'T \leq n$. Then the corresponding 2-factor interactions are nearly linear trend-free. The remaining columns are not trend-free.

Illustration: For obtaining the desired plan of ¼th fraction of 2^5 factorial using the algorithm is as follows:

Step 1: For half fraction, we take the defining contrasts as ABC, i.e. I = ABC. From step I4 and I5 of AL1, we retain the treatment combinations corresponding to the equation $x_1 + x_2 + x_3 = 0$.

	A	B	C	D	E		A	B	C	D	E
(1)	0	0	0	0	0	e	0	0	0	0	1
a	1	0	0	0	0	ae	1	0	0	0	1
b	0	1	0	0	0	be	0	1	0	0	1
ab	1	1	0	0	0	abe	1	1	0	0	1
c	0	0	1	0	0	ce	0	0	1	0	1
ac	1	0	1	0	0	ace	1	0	1	0	1
bc	0	1	1	0	0	bce	0	1	1	0	1
abc	1	1	1	0	0	abce	1	1	1	0	1
d	0	0	0	1	0	de	0	0	0	1	1
ad	1	0	0	1	0	ade	1	0	0	1	1
bd	0	1	0	1	0	bde	0	1	0	1	1
abd	1	1	0	1	0	abde	1	1	0	1	1
cd	0	0	1	1	0	cde	0	0	1	1	1
acd	1	0	1	1	0	acde	1	0	1	1	1
bcd	0	1	1	1	0	bcde	0	1	1	1	1
abcd	1	1	1	1	0	abcde	1	1	1	1	1

Step 2: Defining contrasts as ABC, i.e. I = ABC where $x_1 + x_2 + x_3 = 0$.

	A	B	C	D	E
(1)	0	0	0	0	0
Ab	1	1	0	0	0
Ac	1	0	1	0	0
Bc	0	1	1	0	0
D	0	0	0	1	0
Abd	1	1	0	1	0
Acd	1	0	1	1	0
Bcd	0	1	1	1	0
E	0	0	0	0	1
Abe	1	1	0	0	1
Ace	1	0	1	0	1

Bce	0	1	1	0	1
De	0	0	0	1	1
abde	1	1	0	1	1
acde	1	0	1	1	1
bcde	0	1	1	1	1

Step 3: I = ABC = CDE = ABDE.

	A	B	C	D	E
(1)	0	0	0	0	0
Ab	1	1	0	0	0
acd	1	0	1	1	0
bcd	0	1	1	1	0
Ace	1	0	1	0	1
bce	0	1	1	0	1
De	0	0	0	1	1
abde	1	1	0	1	1

Step 4: Replace the symbols 0 and 1 in each column by -1 and 1, respectively denote as Plan D.

Plan D

	A	B	C	D	E
(1)	-1	-1	-1	-1	-1
Ab	1	1	-1	-1	-1
acd	1	-1	1	1	-1
bcd	-1	1	1	1	-1
Ace	1	-1	1	-1	1
bce	-1	1	1	-1	1
De	-1	-1	-1	1	1
abde	1	1	-1	1	1

Plan D

	A	B	C	D	E
abcde	1	1	1	1	1
cde	-1	-1	1	1	1
be	-1	1	-1	-1	1
ae	1	-1	-1	-1	1
bd	-1	1	-1	1	-1
ad	1	-1	-1	1	-1
abc	1	1	1	-1	-1
c	-1	-1	1	-1	-1

Combining the plan D and D′. The desired Plan D* is:

	A	B	C	D	E
1	-1	-1	-1	-1	-1
ab	1	1	-1	-1	-1
acd	1	-1	1	1	-1
bcd	-1	1	1	1	-1
ace	1	-1	1	-1	1
bce	-1	1	1	-1	1
de	-1	-1	-1	1	1
abde	1	1	-1	1	1
abcde	1	1	1	1	1
cde	-1	-1	1	1	1
be	-1	1	-1	-1	1
ae	1	-1	-1	-1	1
bd	-1	1	-1	1	-1
ad	1	-1	-1	1	-1
abc	1	1	1	-1	-1
c	-1	-1	1	-1	-1

Now to check all main effects of the plan D^* are orthogonal to linear-trend $X'T = 0$.

	A	B	C	D	E	T
	-1	-1	-1	-1	-1	-15
	1	1	-1	-1	-1	-13
	1	-1	1	-1	-1	-11
	-1	1	1	-1	-1	-9
	-1	-1	-1	1	1	-7
	1	1	-1	1	1	-5
	1	-1	1	1	1	-3
X=	-1	1	1	1	1	-1
	1	1	1	1	1	1
	-1	-1	1	1	1	3
	-1	1	-1	1	1	5
	1	-1	-1	1	1	7
	1	1	1	-1	-1	9
	-1	-1	1	-1	-1	11
	-1	1	-1	-1	-1	13
	1	-1	-1	-1	-1	15

	A	B	C	D	E
	15	15	15	15	15
	-13	-13	13	13	13
	-11	11	-11	11	11
	9	-9	-9	9	9
	7	7	7	-7	-7
	-5	-5	5	-5	-5
	-3	3	-3	-3	-3
X′T =	1	-1	-1	-1	-1
	1	1	1	1	1
	-3	-3	3	3	3
	-5	5	-5	5	5
	7	-7	-7	7	7
	9	9	9	-9	-9
	-11	-11	11	-11	-11
	-13	13	-13	-13	-13
	15	-15	-15	-15	-15
Sum	0	0	0	0	0

Considering the problem of constructing a linear-trend-free fractional factorial plan $\frac{1}{2^r}2^n$ $\frac{1}{2^r}2^n$ where, we have to take $\frac{1}{2^{r+1}}$ fraction of 2^n factorial, which further changes to fold over plan. Here, the defining contrast should be taken in such a way that all the main effects must be estimable and are linear trend-free. Again, different independent interactions are to be selected which further form the previously chosen defining contrast by taking generalized interaction. Hence, one can easily get those independent interactions as some of the other defining contrasts. In a 2^k factorial experiment, the total number of treatment combinations is always even. Therefore, vector T will be:

$$T = \{-(N-1), -(N-3), ..., -3, -1, 1, 3, (N-3), (N-1)\}$$

Based on general definition given by Yeh and Bradley (1980), all the s-factor factorial effects are linear trend-free if $X'_s T = 0$. Here 0_t denotes a t-component vector of all zeros. In particular, a plan for factorial experiment would be linear trend free for all main effects if $X'_1 T = 0$. The condition given by Chai (1995) for a factorial plan run with $n < N$ treatment combinations, the condition for the $n \times 1$ column vector of coefficients of contrast of interest, say A, to be nearly linear trend-free is:

$$0 < A2'T \leq n$$

This condition is used to obtain a factorial plan in which contrasts for the main effects are estimated free from linear trend effects and to identify/search some two and three factor interactions that are estimable free from linear trend effects.

SUMMARY

Trend-free design for single factor at two level factorial experiments (complete and fractional both) are available in literature but trend-free multi-factor design could not be traced in the literature. There are a number of experiments in agronomy, agricultural engineering and virology where factors are multi-level. Multi-level factorial/fractional factorial designs are not readily available, and there is a need to address this problem. Hence an attempt has been made to construct linear trend-free multi-level factorial/fractional factorial main effect design with their cost consideration.

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