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ABSTRACT

This article provides some new construction methods of partially balanced bipartite block (PBBB) designs for comparing test treatments with more than one control. Partially balanced incomplete block (PBIB) designs based on some association schemes such as triangular association, Latin-square association, group divisible association, and cyclic association are used for developing these methods of construction. A catalog of efficient PBBB designs is included for parameter values v_1 (number of test treatments) $\leq 10, v_2$ (number of control treatments) $= 2$, b (number of blocks) ≤ 8, r_1 (replications of test treatments) ≤ 10 and k (block size) ≤ 6 along with computed variances.

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Group divisible association; group divisible partially balanced bipartite design; partially balanced incomplete block design; partially balanced bipartite block design; triangular association

1. Introduction

In many fields of experiments such as agricultural, industrial and biological experiments, the experimenter often desires to compare simultaneously several test treatments with a standard or existing treatment called control treatment. A lot of literature is available for single control situation, see for example, Hedayat, Jacroux, and Majumdar (1988); Majumdar (1996); and Gupta and Parsad (2001). For this situation, an important class of designs called as balanced treatment incomplete block (BTIB) designs was introduced by Bechhofer and Tamhane (1981). Subsequently these designs were studied by Majumdar and Notz (1983); Notz and Tamhane (1983); Hedayat and Majumdar (1984); Stufken (1987, 1988); Cheng et al. (1988); Parsad, Gupta, and Prasad (1995); Das et al. (2005); Mandal, Gupta, and Parsad (2017); and Mandal, Parsad, and Dash (2020). BTIB designs may not be always available for all parametric combinations or even if it is available, it may require a large number of replications. To circumvent this problem, Jacroux (1987) introduced group divisible treatment (GDTD) designs. Further studies on these designs were fostered by Jacroux (1986, 1988); Jacroux and Majumdar (1989); and Stufken (1991).

However, there exists an experimental situation, wherein the experimenter is interested in comparing several test treatments with more than one control. For this situation, initially, Corsten (1962) formulated balanced block designs with two different numbers of replications along with their statistical analysis. Later, Kageyama and Sinha (1988) formally introduced a valuable class of designs called balanced bipartite block (BBPB) designs for comparing several test treatments with more than one control and also gave tables of these designs.

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These designs are subsequently studied by Sinha and Kageyama (1990); Jaggi, Gupta, and Parsad (1996); and Mandal, Parsad, and Dash (2018). BBPB designs may not be always obtainable for all combination of parameters or even if it is obtainable may require a large number of replications. For such situations, Rao (1966) initially developed partially balanced block designs with distinct replications and their statistical analysis. Later, most suitable class of designs, that is, group divisible partially balanced bipartite (GD PBB) designs was first proposed by Kageyama and Sinha (1991). The advantages of these designs were discussed by Kageyama and Sinha (1988). According to Kageyama and Sinha (1991), these designs are also useful for conducting factorial experiments; see for example, Puri and Kageyama (1985); and Puri, Mehta, and Kageyama (1986). This design is defined in the sequel.

Definition 1.1. An incomplete block design with a set of v_1 treatments occurring r_1 times and another set of v_2 treatments occurring r_2 $(r_1 \neq r_2)$ times arranged in b blocks of size k, is called GD PBB design if: (i) the treatments in the j^{th} set can be divided into p_j groups each of size q_j $(j = 1, 2)$ and any two treatments in the same group are called first associates, otherwise second associates; (ii) any two treatments in the j^{th} set which are i^{th} associates occur together in $\lambda_{j(i)}$ blocks for $i = 1, 2$; (iii) any two treatments from different sets occur together in $\lambda_{12} = \lambda_{21} (> 0)$ blocks.

The symbols $v_1, v_2, b, r_1, r_2, k, \lambda_{j(i)}$ $(j = 1, 2; i = 1, 2, \ldots, t)$, are called parameters of the design. This design is generally known as partially balanced bipartite block (PBBB) design. If PBBB design based on group divisible (GD) association scheme, then it is called as GD PBB design. These designs have the property of partial balance within the two groups of treatments (treatments in each group having constant replication) and of balance structure between the groups. Note that when $\lambda_{12} = 0$, the design is disconnected and hence we give the restriction $\lambda_{12} > 0$. Furthermore, when $\lambda_{2(1)} = \lambda_{2(2)}$, it is denoted by $\lambda_{2()}$. The following parametric relations hold for a PBBB design:

1. $v_1r_1+v_2r_2=bk$
2. $r_1\ (k-1) = n_{11}\lambda_{1(1)}+n_{12}\lambda_{1(2)}+v_2\lambda_{12}$
3. $r_2\ (k-1) = n_{21}\lambda_{2(1)}+n_{22}\lambda_{2(2)}+v_1\lambda_{12}$

If $\lambda_{2(1)} = \lambda_{2(2)}$ then (iii) reduced as $r_2\ (k-1) = (v_2 - 1)\lambda_{2()} + v_1\lambda_{12}$. Here, n_{ji}, is the number of i^{th} $(i = 1, 2)$ associates of a given treatment from the j^{th} $(j = 1, 2)$ set.

The article is organized as follows. Some new construction methods of PBBB designs along with illustrations are given in Section 2. Section 3 depicts the information about efficiency and variance structure of these designs. In Section 4, a comparative study of these designs with existing designs has been discussed. Finally, a list of these designs with two-associate classes for parameter values v_1 (number of test treatments) ≤ 10, v_2 (number of control treatments) $= 2$, b (number of blocks) ≤ 8, r_1 (replications of test treatments) ≤ 10, k (block size) ≤ 6 is also given in Appendix.

2. Construction methods of PBBB designs

This section reveals some new methods for obtaining PBBB designs using PBIB designs based on some usual association schemes.

Method 2.1. Let there exists a t-associate class PBIB design with parameters as $v^{'}, b^{'}, r^{'}, k^{'}, \lambda_i^{'}$, $\forall\, i = 1, 2, \ldots, t$. By reinforcement of v_2 control treatments to each of the $b^{'}$ blocks of PBIB design, implies a PBBB design with parameters $v_1 = v^{'}, v_2, b = b^{'}, r_1 = r^{'}, r_2 = b^{'}, k = k^{'} + v_2$, $\lambda_{1(i)} = \lambda_i, \forall\, i = 1, 2, \ldots, t, \lambda_{12} = r^{'}, \lambda_{2()} = b^{'}$.

Example 2.1. Consider a PBIB design based on two class triangular association scheme with parameters $v^{'} = 10, b^{'} = 5, r' = 2, k^{'} = 4, \lambda_1^{'} = 1, \lambda_2^{'} = 0, n = 5$, which is T28 in Clatworthy (1973) and its block layout as

$$\begin{array}{cccc} (1, & 2. & 3, & 4) \\ (1, & 5, & 6, & 7) \\ (3, & 5, & 8, & 9) \\ (4, & 7, & 9, & 10) \end{array}$$

Then by using the procedure of Method 2.1, on addition of $v_2 = 2$ controls *viz.*, 0_1 and 0_2, one can get a PBBB design based on two-class triangular association scheme with block layout as follows:

$$\begin{array}{cccccc} (1, & 2, & 3. & 4, & 0_1, & 0_2) \\ (1, & 5, & 6, & 7, & 0_1, & 0_2) \\ (2, & 5, & 8, & 9, & 0_1, & 0_2) \\ (3, & 6, & 8, & 10, & 0_1, & 0_2) \\ (4, & 7, & 9, & 10, & 0_1, & 0_2) \end{array}$$

The parameters of the above design are $v_1= 10$, $v_2= 2$, $b= 5$, $r_1= 2$, $r_2= 5$, $k= 6$, $\lambda_{1(1)}= 1$, $\lambda_{1(2)}= 0$, $\lambda_{12}= 2$, $\lambda_{2()} = 5$.

Method 2.2. Suppose there exists two class PBIB design based on group divisible (GD) association scheme with parameters $v^{'}, b^{'}, r', k^{'}, \lambda_i^{'}, i = 1, 2$. The group divisible association scheme on $v = mn$ treatments, for integers $m \geq 4$ and $n \geq 2$, arranged in $m \times n$ array as

$$\begin{array}{ccccc} 1 & m+1 & 2m+1 & \ldots & m(m-1)+1 \\ 2 & m+2 & 2m+2 & \ldots & m(m-1)+1 \\ 3 & m+3 & 2m+3 & \ldots & m(m-1)+1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ m & 2m & 3m & \ldots & mn \end{array}$$

A PBBB design can be obtained by merging treatments of any $v_2 \in [2, m-2]$ rows of the above $m \times n$ array and the parameters of this design become as $v_1 = v^{'} - v_2 n = n(m - v_2)$, $v_2, b = b^{'}, r_1 = r^{'}, r_2 = nr^{'}, k = k^{'}, \lambda_{1(1)} = \lambda_1^{'}, \lambda_{1(2)} = \lambda_2^{'}, \lambda_{12} = n\lambda_2^{'}, \lambda_{2()} = n^2\lambda_2^{'}$.

Example 2.2. A singular GD design, S19 in Clatworthy (1973) with parameters as $v^{'} = 8$, $b^{'} = 8, r^{'} = 6, k^{'} = 6, \lambda_1^{'} = 6, \lambda_2^{'} = 4, m = 4$ and $n = 2$ based on group divisible association scheme as follows;

$$\begin{array}{cc} 1 & 5 \\ 2 & 6 \\ 3 & 7 \\ 4 & 8 \end{array}$$

The block contents of this design are:

$$\begin{array}{llllllllllll}
(1, & 5, & 2, & 7, & 3, & 6); & (5, & 1, & 7, & 2, & 6, & 3) \\
(2, & 6, & 1, & 5, & 4, & 8); & (6, & 2, & 5, & 1, & 8, & 4) \\
(3, & 7, & 4, & 8, & 1, & 5); & (7, & 3, & 8, & 8, & 5, & 1) \\
(4, & 8, & 3, & 6, & 2, & 7); & (8, & 4, & 6, & 6, & 7, & 2)
\end{array}$$

On merging of any $v_2 = 2$ (say) of the rows, without loss of any generality, one can get a PBBB design. Suppose first two rows [1, 5] and [2, 6] are merged by placing control treatments '0_1' and '0_2' respectively. After merging, the above array becomes as

$$\begin{array}{cc}
0_1 & 0_1 \\
0_2 & 0_2 \\
3 & 7 \\
4 & 8
\end{array}$$

One gets a PBBB design with parameters as $v_1 = 4,\ v_2 = 2,\ b = 8,\ r_1 = 6,\ r_2 = 12,\ k = 6,\ \lambda_{1(1)} = 6,\ \lambda_{1(2)} = 4,\ \lambda_{12} = 8,\ \lambda_{2()} = 16$. The blocks of the design after renumbering of treatments shown below:

$$\begin{array}{llllllllllll}
(0_1, & 0_1 & 0_2 & 3, & 1, & 0_2); & (0_1, & 0_1, & 3, & 0_2, & 0_2, & 1) \\
(0_2, & 0_2, & 0_1, & 0_1, & 2, & 4); & (0_2, & 0_2, & 0_1, & 0_1, & 4 & 2) \\
(1, & 3, & 2, & 4, & 0_1, & 0_1); & (3, & 1 & 4, & 2, & 0_1 & 0_1) \\
(2 & 4, & 1, & 0_2, & 3); & & (4, & 2, & 0_2, & 1, & 3 & 0_2)
\end{array}$$

The designs generated by Methods 2.1 and 2.2 are catalogued in Appendix for parameter values $v_1 \le 10, v_2 = 2, b \le 8, r_1 \le 3$ and $k \le 6$.

3. Efficiency and variance structure

For studying the efficiencies of obtained PBBB designs, we use the results of A-optimality of BBPB designs for comparing test treatments with multiple controls. For this purpose, we consider $D\,(v_1, v_2,\ b,\ k)$ as the class of all connected block designs in which v_1 test treatments and v_2 control treatments are arranged in b blocks of size k each. We make use of the sufficient condition for establishing the A-optimality of BBPB designs discussed by Jaggi, Gupta, and Parsad (1996). The sufficient condition gives the lower bound to the trace of variance-covariance matrix of all the test treatments versus control treatments contrasts. A design that attains the lower bound is termed as A-optimal. The condition is given in Result 3.1.

Result 3.1. An PBBB design is A-optimal in the class of all designs with the same values of v_1, v_2, b, k if

$$g\left(w, q\right) = \min \left\{g\left(x, z\right),\ \left(x, z\right) \in \Delta\right\} \tag{3.1}$$

where $\Delta = \{(x, z)\,;\ x{=}\ 0,\ 1,\ \ldots,\ \text{int}\left[k/v_2\right] - 1; z{=}\ 0,\ 1,\ \ldots, b \text{ with } z{>}\ 0 \text{ when } x{=}\ 0\}$,

$$g\left(x, z\right) = \frac{1}{A\left(x, z\right)} + \frac{a}{B\left(x, z\right)} + \frac{d}{C\left(x, z\right)}$$

$$a{=}v_2(v_1{-}1)^2,\ d{=}v_1(v_2{-}1)$$

$$A\,(x,z) = \{k(bx+z)\; - \;v_2\left(bx^2+2xz+z\right)\}\,/v_1k,$$

$$B\,(x,z) = [bkv_1\,(k-1)\,-v_2\,\{v_1\,(k-1)\,+k\}\,(bx+z)$$

$$+\;v_2^2\left(bx^2+2xz+z\right)]\,/v_1k,$$

$C\,(x,z) = bx + z$. Here int [.] denotes the greatest integer function.

We adopt the approach of Stufken (1988) to obtain A-efficiency (E) of PBBB designs. The A-efficiency is the ratio of A-value of a hypothetical A-optimal design whose criterion value given in (3.1) is minimum for comparing test treatments with controls in a given class of designs, to the A-value of the design whose A-efficiency to be obtained in the same class of designs. Here, A-value is the trace of variance-covariance matrix of the estimated treatment contrasts of interest. A-efficiencies of these designs are obtained by using Result 3.1. Further, a design is A-optimal if the A-efficiency is 1.000. Some of the A-optimal designs within the scope of parameters $v_1 \leq 10, v_2 = 2, b \leq 8, r_1 \leq 3, k \leq 6$ are observed in Appendix. For cataloguing purpose, we have restricted for v_2 (number of control treatments) = 2. Furthermore, computed variances of these designs for the estimated differences in effect between two treatments, $V_{1(1)}\sigma^2$, $V_{1(2)}\sigma^2$, $V_{12}\sigma^2$ and $V_{2()}\,\sigma^2$ have also been given. Here, $V_{1(1)}\sigma^2$ denotes variance for the estimated differences in effect between two test treatments (first associates) both from the first set; $V_{1(2)}\sigma^2$ denotes variance for the estimated differences in effect between two test treatments (second associates) both from the first set; $V_{2()}\,\sigma^2$ denotes variance for two control treatments both from the second set; and $V_{12}(= V_{21})\sigma^2$ denotes variance for two treatments (test with control) from different sets. These quantities $V_{1(1)}$, $V_{1(2)}$, V_{12} and $V_{2()}$ are also helpful to judge whether the designs are practical or not. Experimenters and practicing statisticians may choose design, suitable to their practical purpose, from the tables by reading values $V_{1(1)}$, $V_{1(2)}$, V_{12} and $V_{2()}$ for given v_1, v_2 and b. The letters S, SR, R, LS, T, C in Appendix indicate the source designs from Clatworthy (1973).

4. Comparative study with existing designs

Previously, balanced bipartite block (BBPB) designs for comparing test treatments with more than one control have been reported [Kageyama and Sinha (1988); Sinha and Kageyama (1990); and Jaggi, Gupta, and Parsad (1996)]. However, these designs are available in large number of blocks ($b \leq 52$) for minimum number of treatments ($2 \leq v_1 \leq 10$) [Kageyama and Sinha (1988)]. Additionally, Sinha and Kageyama (1990) presented two designs such as SK4 ($v_1 = 8$, $b = 16$, $r_1 = 6$, $r_2 = 8$, $k = 4$, $\lambda_{11} = 2$, $\lambda_{12} = 8$, $\lambda_{22} = 2$) and SK13 ($v_1 = 10$, $b = 15$, $r_1 = 7$, $r_2 = 10$, $k = 6$, $\lambda_{11} = 3$, $\lambda_{12} = 10$, $\lambda_{22} = 4$) in 16 and 15 blocks, respectively. Jaggi, Gupta, and Parsad (1996) also contain the designs with large number of blocks ($b \geq 15$) *viz.*, JGP1 ($b = 15$, $r_1 = 7$, $r_2 = 9$, $k = 4$, $E = 0.955$) and JGP2 ($b = 15$, $r_1 = 9$, $r_2 = 3$, $k = 4$, $E = 0.483$) for $v_1 = 8$; and JGP3-7 (*i.e.*, $b = 18$, $r_1 = 5$, $r_2 = 16$, $k = 4$, $E = 0.965$; $b = 36$, $r_1 = 10$, $r_2 = 32$, $k = 4$, $E = 0.965$; $b = 16$, $r_1 = 6$, $r_2 = 8$, $k = 4$, $E = 0.937$; $b = 24$, $r_1 = 9$, $r_2 = 12$, $k = 4$, $E = 0.936$; and $b = 22$, $r_1 = 10$, $r_2 = 4$, $k = 4$, $E = 0.517$) for $v_1 = 10$. Here, SK# and JGP# denotes the BBPB design at serial number # in Sinha and Kageyama (1990); and Jaggi, Gupta, and Parsad (1996) respectively. In place of these designs, the proposed PBBB designs (given in Appendix) *i.e.*, either design with serial number 7 or 12; and either design with serial number 13 or 14, respectively, may be practical with good A-efficiencies (*i.e.*, $0.901 \leq E \leq 1.000$) in a smaller number of blocks. Interestingly, the proposed construction methods in the present investigation are simple and straightforward, which yields quite highly efficient PBBB designs (*i.e.*, $0.868 \leq E \leq 1.000$) with a minimum number of blocks ($3 \leq b \leq 8$) compared to existing ones. As a result, these designs can make it easier for experimenters (even if the experimenter is constrained by a lack of resources) to infer elementary contrasts between each test and each control with as much precision as possible in plant and animal experiments.

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There is no potential conflict of interest.

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Appendix

Sl. No.	v_1	v_2	b	r_1	r_2	k	$\lambda_{1(1)}$	$\lambda_{1(2)}$	λ_{12}	$\lambda_{2()}$	$V_{1(1)}$	$V_{1(2)}$	V_{12}	$V_{2()}$	E	Method (source)
1	4	2	8	3	6	3	0	1	2	4	1.000	0.875	0.625	0.375	0.964	2.2 (R54)
2	4	2	4	2	4	4	0	1	2	4	1.333	1.167	0.833	0.500	1.000	2.1 (SR1)
3	5	2	5	2	5	4	1	0	2	5	1.171	1.366	0.807	0.400	1.000	2.1 (C1)
4	5	2	5	3	5	5	2	1	3	5	0.718	0.773	0.565	0.400	1.000	2.1 (C12)
5	6	2	4	2	4	5	0	1	2	4	1.250	1.125	0.813	0.500	1.000	2.1 (SR18)
6	6	2	6	3	6	5	2	1	3	6	0.714	0.774	0.540	0.333	1.000	2.1 (R42)
7	8	2	8	3	8	5	0	1	3	8	0.833	0.774	0.509	0.250	1.000	2.1 (R54)
8	9	2	6	2	6	5	1	0	2	6	1.143	1.286	0.762	0.333	1.000	2.1 (LS7)
9	4	2	4	3	6	6	3	2	4	8	0.667	0.708	0.542	0.375	0.952	2.2 (S18)
10	6	2	3	2	3	6	2	1	2	3	1.000	1.100	0.867	0.667	1.000	2.1 (S1)
11	6	2	6	3	9	6	3	1	3	9	0.667	0.778	0.556	0.333	0.868	2.2 (S27)
12	8	2	6	3	6	6	3	1	3	6	0.667	0.762	0.536	0.333	1.000	2.1 (S6)
13	10	2	5	2	5	6	1	0	2	5	1.111	1.222	0.767	0.400	1.000	2.1 (T28)
14	10	2	7	3	6	6	3	1	2	4	0.667	0.762	0.595	0.429	0.908	2.2 (S32)