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# Minimal balanced repeated measurements designs

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# Minimal balanced repeated measurements designs

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ABSTRACT *Experimental designs in which treatments are applied to the experimental units, one at a time, in sequences over a number of periods, have been used in several scientific investigations and are known as repeated measurements designs. Besides direct effects, these designs allow estimation of residual effects of treatments along with adjustment for them. Assuming the existence of first-order residual effects of treatments, Hedayat & Afsarinejad (1975) gave a method of constructing minimal balanced repeated measurements* [RM($v,n,p$)] *design for $v$ treatments using $n = 2v$ experimental units for $p[= (v + 1)/2]$ periods when $v$ is a prime or prime power. Here, a general method of construction of these designs for all odd $v$ has been given along with an outline for their analysis. In terms of variances of estimated elementary contrasts between treatment effects (direct and residual), these designs are seen to be partially variance balanced based on the circular association scheme.*

## 1 Introduction

In Repeated Measurements (RM) designs, a set of treatments is applied, one at a time, to each of the experimental units over a number of periods and the observations are recorded during each period to evaluate the performance of the treatments. These designs economize on the use of experimental material and make precise comparisons among treatment effects by eliminating the inter-experimental unit variation and have been advantageously used in many scientific investigations (Patterson & Lucas, 1962; France *et al.*, 1991; Cook, 1995; Yao *et al.*, 1998). Assuming the existence of first-order residual effects of treatments along with direct, period and experimental unit effects, a number of studies have

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been carried out on various aspects of RM designs (see for example Hedayat & Afsarinejad, 1975; Mathews, 1990, 1994; Bellavance *et al.*, 1996; Kunert, 1991, 1998).

Hedayat & Afsarinejad (1975) defined the minimal balanced RM designs as follows.

**Definition 1.1:** A RM$(v, n, p)$ design for $v$ treatments using $n$ experimental units for $p$ periods is said to be balanced with respect to direct and first-order residual effects if (i) each treatment occurs equally frequently, say $\delta_1$ times in each period, and (ii) in the order of application, each treatment is preceded by every other treatment equally often, say $\delta_2$ times. Evidently, the following relations hold for a balanced RM$(v, n, p)$ design:

$$\text{(i)}\ \ n = \delta_1 v; \qquad \text{(ii)}\ \ n(p-1) = \delta_2 v(v-1)$$

**Definition 1.2:** For given $v$ and $p$, a balanced RM$(v, n, p)$ design is said to be minimal if its parameter $\delta_1$ is the smallest integer such that $\delta_1(p-1) \equiv 0[\text{mod}(v-1)]$. Thus, a minimal balanced design contains the minimum possible number of experimental units.

For $p = v$, $\delta_1$ takes the minimum value of one and therefore $n = v$. Methods of constructing minimal balanced RM$(v, v, v)$ designs for all even $v$ have been given by Williams (1949), Bradley (1958), and Sharma (1975). When $\delta_1 = 2$, minimal balanced RM$(v, 2v, v)$ designs can easily be constructed for all odd values of $v$ (see, for example Williams, 1949; Sharma, 1975).

In the above situation, each of the experimental units is to receive all the $v$ treatments. This may not be possible in many experiments such as drug testing or other medical experiments. In such situations, designs with $p < v$ are required. Hedayat & Afsarinejad (1975) have shown that for given $v$ and $p(< v)$, a minimal balanced RM$(v, n, p)$ design requires at least $n = 2v$ experimental units and gave a method of constructing minimal balanced RM $[v, 2v, (v+1)/2]$ design when $v$ is a prime or prime power. The existing methods do not cover the experimental situations involving odd non-prime or the non-prime power number $v$ of the treatments. Here, therefore, a method of constructing minimal balanced RM $[v, 2v, (v+1)/2]$ design for all values of odd $v$ has been proposed along with an outline of the analysis of data generated through these designs. In terms of variances of estimated elementary contrasts between direct or residual effects, these designs are partially variance balanced based on the circular association scheme.

## 2 Construction of minimal balanced RM$[v, 2v, p = (v+1)/2]$ design where $v$ is an odd integer

Let $v = 2t + 1$ treatments be arranged in their natural order $0, 1, 2, \ldots, 2t$.

*Case* 1. *t odd.* Retain the first $(t+1)/2$ treatments and the last $(t+1)/2$ treatments. Obtain two sequences of treatments, first by interlacing the last $(t+1)/2$ treatments in their reverse order within the first $(t+1)/2$ treatments, and the second by interlacing the first $(t+1)/2$ treatments within the reverse order of last $(t+1)/2$ treatments. On developing these sequences mod $v$, we get an arrangement of $v$ treatments in $2v$ rows and $(v+1)/2$ columns. This arrangement forms a minimal

TABLE 1. The minimal balanced RM(11,22,6) design with $\delta_1 = 2$, $\delta_2 = 1$

| | Periods | | | | | | | Periods | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Units | 1 | 2 | 3 | 4 | 5 | 6 | Units | 1 | 2 | 3 | 4 | 5 | 6 |
| 1 | 0 | 10 | 1 | 9 | 2 | 8 | 12 | 10 | 0 | 9 | 1 | 8 | 2 |
| 2 | 1 | 0 | 2 | 10 | 3 | 9 | 13 | 0 | 1 | 10 | 2 | 9 | 3 |
| 3 | 2 | 1 | 3 | 0 | 4 | 10 | 14 | 1 | 2 | 0 | 3 | 10 | 4 |
| 4 | 3 | 2 | 4 | 1 | 5 | 0 | 15 | 2 | 3 | 1 | 4 | 0 | 5 |
| 5 | 4 | 3 | 5 | 2 | 6 | 1 | 16 | 3 | 4 | 2 | 5 | 1 | 6 |
| 6 | 5 | 4 | 6 | 3 | 7 | 2 | 17 | 4 | 5 | 3 | 6 | 2 | 7 |
| 7 | 6 | 5 | 7 | 4 | 8 | 3 | 18 | 5 | 6 | 4 | 7 | 3 | 8 |
| 8 | 7 | 6 | 8 | 5 | 9 | 4 | 19 | 6 | 7 | 5 | 8 | 4 | 9 |
| 9 | 8 | 7 | 9 | 6 | 10 | 5 | 20 | 7 | 8 | 6 | 9 | 5 | 10 |
| 10 | 9 | 8 | 10 | 7 | 0 | 6 | 21 | 8 | 9 | 7 | 10 | 6 | 0 |
| 11 | 10 | 9 | 0 | 8 | 1 | 7 | 22 | 9 | 10 | 8 | 0 | 7 | 1 |

balanced RM$[v, 2v, (v+1)/2]$ design with rows representing experimental units and columns the periods.

**Example 2.1:** Suppose $v = 11$ giving $t = 5$. These treatments are arranged as $0, 1, 2, \ldots, 10$. We retain the first $(t+1)/2 = 3$ treatments $(0, 1, 2)$ and the last $(t+1)/2 = 3$ treatments $(8, 9, 10)$.

First sequence: $\{0, 10, 1, 9, 2, 8\}$
Second sequence: $\{10, 0, 9, 1, 8, 2\}$

By developing these sequences mod 11, we get the design given in Table 1.

*Case* 2. *t even*. The first sequence is obtained by interlacing the last $t/2$ treatments in their descending order within the first $(t/2) + 1$ treatments in their natural order. The second sequence is obtained by interlacing the first $t/2$ treatments in their natural order within the last $(t/2) + 1$ treatments in their descending order. On developing these sequences mod $v$, a minimal balanced RM$[v, 2v, (v+1)/2]$ design is obtained with rows representing experimental units and columns the periods.

**Example 2.2:** Let $v = 9$ giving $t = 4$. Arrange these treatments as $0, 1, 2, \ldots, 8$. The first sequence $\{0, 8, 1, 7, 2\}$ is obtained by interlacing $8, 7$ within $0, 1, 2$ and the second sequence $\{8, 0, 7, 1, 6\}$ is the result of interlacing $0, 1$ within $8, 7, 6$. These sequences when developed mod 9 yield the minimal balanced RM$(9, 18, 5)$ design given in Table 2.

TABLE 2. Minimal balanced RM(9,18,5) design with $\delta_1 = 2$, $\delta_2 = 1$

| | Periods | | | | | | Periods | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Units | 1 | 2 | 3 | 4 | 5 | Units | 1 | 2 | 3 | 4 | 5 |
| 1 | 0 | 8 | 1 | 7 | 2 | 10 | 8 | 0 | 7 | 1 | 6 |
| 2 | 1 | 0 | 2 | 8 | 3 | 11 | 0 | 1 | 8 | 2 | 7 |
| 3 | 2 | 1 | 3 | 0 | 4 | 12 | 1 | 2 | 0 | 3 | 8 |
| 4 | 3 | 2 | 4 | 1 | 5 | 13 | 2 | 3 | 1 | 4 | 0 |
| 5 | 4 | 3 | 5 | 2 | 6 | 14 | 3 | 4 | 2 | 5 | 1 |
| 6 | 5 | 4 | 6 | 3 | 7 | 15 | 4 | 5 | 3 | 6 | 2 |
| 7 | 6 | 5 | 7 | 4 | 8 | 16 | 5 | 6 | 4 | 7 | 3 |
| 8 | 7 | 6 | 8 | 5 | 0 | 17 | 6 | 7 | 5 | 8 | 4 |
| 9 | 8 | 7 | 0 | 6 | 1 | 18 | 7 | 8 | 6 | 0 | 5 |

It may be seen that the designs obtained by the proposed method are partially variance balanced repeated measurements designs with $t$-associate classes based on the following circular association scheme.

*Association scheme*. Arrange the $v(=2t+1)$ treatments denoted by $0, 1, \ldots, 2t$ on the circumference of a circle. Now, a treatment $\phi$ is said to be the $i$th associate $(i = 1, 2, \ldots, t)$ of treatment $\theta$, if $\phi$ occupies the $i$th position from $\theta$ on the circumference. Thus, every treatment has two treatments, one each either side at the $i$th position, as its $i$th associates. The parameters of the association scheme are $v = 2t + 1$, $n_i = 2$, $\lambda_i = 2(t - i + 1)$, and $P_i$ $(i = 1, 2, \ldots, t)$ matrices of order $t$ as

$$P_1 = (p^1_{\alpha\beta})$$

where

$$p^1_{\alpha\beta} = 1, \text{ if } \alpha = i \text{ and } \beta = i+1, \text{ or } \alpha = i+1 \text{ and } \beta = i \ (i = 1, 2, \ldots, t-1), \text{ or}$$
$$\alpha = t \text{ and } \beta = t$$
$$= 0, \text{ otherwise}$$

$$P_2 = (p^2_{\alpha\beta})$$

where

$$p^2_{\alpha\beta} = 1, \text{ if } \alpha = i \text{ and } \beta = i+2, \text{ or } \alpha = i+2 \text{ and } \beta = i \ (i = 1, 2, \ldots, t-2), \text{ or}$$
$$\alpha = t-1 \text{ and } \beta = t, \text{ or } \alpha = t \text{ and } \beta = t-1, \text{ or } \alpha = \beta = 1$$
$$= 0, \text{ otherwise}$$

etc. and $P_t = (p^t_{\alpha\beta})$

where

$$p^t_{\alpha\beta} = 1, \text{ if } \alpha = i \text{ and } \beta = t-i, \text{ or } \alpha = i \text{ and } \beta = t-i+1 \ (i = 1, 2, \ldots, t-1), \text{ or}$$
$$\alpha = t \text{ and } \beta = 1$$
$$= 0, \text{ otherwise}$$

$$(\alpha, \beta = 1, 2, \ldots, t)$$

The parameters of the association scheme for the design given in Example 2.2 are $v = 9$ (giving $t = 4$), $n_1 = n_2 = n_3 = n_4 = 2$, $\lambda_1 = 8$, $\lambda_2 = 6$, $\lambda_3 = 4$, $\lambda_4 = 2$, and

$$P_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \quad P_3 = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix},$$

$$P_4 = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

**Remark:** When residual effects and period effects are ignored and experimental units are taken as blocks, the design obtained by the proposed method for $t > 1$ forms a partially balanced incomplete block design with $t$-associate classes based on the circular associaton scheme and it has the parameters $v = 2t + 1$, $b = 2(2t + 1)$, $r = 2(t + 1)$, $k = t + 1$, $\lambda_i = 2(t - i + 1)$, $i = 1, 2, \ldots, t$.

## 3 Outline of analysis

We assume an additive fixed effects linear model in which the observations are represented by constants for general mean, direct and residual effects of treatments, period and experimental unit effects and contain errors that are identically, independently distributed normally with mean zero and constant variance. As the observations collected during the first period do not contain the residual effects of the treatments, the corresponding residual effect constants will not appear in the model for these observations. Minimization of the residual sum of squares yields, after some simplification, reduced normal equations for adjusted treatment effects (direct + residual). Adjusted treatment sum of squares (direct + residual) with $2(v - 1)$ degrees of freedom can now be obtained by using a solution of these equations and the adjusted treatment totals. To test the significance of direct or residual effects, the adjusted treatment sum of squares is partitioned in the following two ways:

$$\text{(i)} \begin{cases} \text{Direct effects (eliminating residual effects) with } (v-1) \text{ d.f.} \\ \text{Residual effects (ignoring direct effects) with } (v-1) \text{ d.f.} \end{cases}$$

$$\text{(ii)} \begin{cases} \text{Direct effects (ignoring residual effects) with } (v-1) \text{ d.f.} \\ \text{Residual effects (eliminating direct effects) with } (v-1) \text{ d.f.} \end{cases}$$

The sum of squares due to periods, and experimental units (ignoring direct and residual effects) are obtained in the usual manner. The details of the analysis can be easily worked out on the lines of the analysis of a partially balanced changeover design given in Patterson & Lucas (1962). However, standard statistical software packages may be utilized for carrying out the analysis of the data under the model used.

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