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# GENERALIZED STAIRCASE DESIGNS

# AND

# THEIR APPLICATIONS

BY

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# C O N T E N T S



NOTE: SUFFIX (') WITH $Z_{mi}$' AND $T_{mi}$' SHOULD BE READ EVERYWHERE AS $Z'_{mi}$ AND $T'_{mi}$ RESPECTIVELY IN THIS THESIS WORK.

# I. INTRODUCTION

In progeny row trials as also in experiments with animals as the experimental units it becomes necessary to adopt designs with unequal number of replications as also unequal blocks.

A good deal of research has been done to get designs with unequal block sizes as also unequal number of replications. The quasi-factorial designs, given by Yates (1936a) when $v = pq$, have blocks of sizes 'p' and 'q'. Again the designs developed by Kishen (1941) have, in general, 'm' different block sizes, $k_1, k_2, \ldots\ldots k_m$. More recently Graybill and Pruitt (1958) introduced a series of designs called staircase designs which accommodates blocks of all sizes less than and equal to the number of treatments. Recently Bose and Shrikhande (1959) have used such incomplete block designs with unequal block sizes and $\lambda = 1$ to get orthogonal latin squares of sides $4t + 2$.

Nair and Rao (1942) introduced the intra and inter-group B.I.B. designs in which the block size is constant but the number of replicates of the different groups of treatments are unequal.

For meeting experimental situation in physical, chemical and other sciences, Youden and Connor (1953) developed the chain-block designs, which have great flexibility and require only smaller number of replicates which may be unequal in number. Das (1957, 1958) has also some designs with unequal replications and blocks.

As mentioned above, Graybill and Pruitt (1958) introduced a design called staircase design which provides for different block sizes and replications. These designs are thus suitable for the above types of experiments. One drawback in this design is that it does not provide for any block size which can be greater than the number of treatments. Hence in experiments on litter-mates, for which alone, they say, these experiments are more useful, all the animals in those litters of which the size is greater than the number of treatments, cannot be utilized and this may lead to some wastage of animals. Cox (1958) has also emphasized on obtaining design to suit such situations. A class of "generalized staircase designs" to suit all such complexities has thus been defined. A method of its analysis which does not follow from their method, has been worked out together with the expressions for finding the variance of treatment differences.

A further problem which is encountered with non-orthogonal data is to have suitable partitioning of the adjusted sum of squares. There seems to be no method of getting such subdivision with more than one degree of freedom. An attempt has been made to evolve a suitable systematic methods for getting such sub-divisions.

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## II. DEFINITION OF STAIRCASE DESIGN AND ITS GENERALIZATION.

Graybill and Pruitt defined the staircase design as an incomplete randomized block design with t- treatments and b - blocks such that when the treatments are arranged in some order the $j^{th}$ block contains the first $k_j$ treatments where $k_j = t$ for at least one block and takes values from 1 to (t-1) not necessaily all of them for the other blocks. This definition implies that each block will have two types of frequencies of occurrence of the treatments, viz., 1 and 0 such that the $j^{th}$ block will have the frequency 1 in the first $k_j$ cells and 0 in the rest and also the treatments will have unequal replications.

All the blocks having the same value of $k_j$ will be said to belong to a step of blocks. Again all the treatments which have the same replications will be said to belong to a step of treatments.

For agricultural experiments using somewhat larger plots, such designs cannot be of much use as the blocks are unequal and the intrablock variance is dependent on the block size. But in experiments with the animals as experimental unit and the blocks being the litters, unequal blocks do not offer any difficulty as it is not so much likely that the intrablock variance is dependent on the litter size. In experiments with smaller plots unequal blocks may not be of much objection if the blocks do not differ much in size.

One more peculiarity with these designs is that the treatments are unequally replicated and hence different treatments are estimated with different precision. As such the authors in the original paper suggested that the treatments should first be arranged in an ascending order of importance so that the treatment considered most important will have the highest number of replications.

One restriction in the design originally defined was that there was no provision of blocks of size greater than the number of treatments. This may lead to some wastage of experimental resources, because of the necessity of discarding animals, say, from certain litters. In order to remove this restriction, as also to get the method of analysis of a particular type of non-orthogonal data the staircase designs have been defined in a general way as given below.

The "generalized staircase design" has been defined to be a design with t- treatments and b- blocks, such that when the treatments are written down in some suitable order the $j^{th}$ block contains each of the first $k_j$ treatments in it $(n_j+x_j)$ times and the rest of the treatments $n_j$ times, where $x_j \geqslant 0$ and $n_j \geqslant 0$ provided both $x_j$ and $n_j$ do not vanish together. By giving suitable values to $n_j$ and $x_j$ various designs can be obtained. Thus when

| | |
|---|---|
| (i) $x_j = 0$ and $n_j = 1$, | We obtain a randomized block design with one observation per cell. |
| (ii) $x_j = 0$ and $n_j = n$, | We get randomized block design with one fixed type of class frequency 'n'. |
| (iii) $x_j = 0$ and $n_j = n_j$, | Randomized block design with proportionate class frequencies is obtained. |

| | |
|---|---|
| (iv) $x_j = 1$ and $n_j = n_j$, | We get a more popular generalized staircase design (To be discussed in detail in article 3). |
| (v) $x_j = 1$ and $n_j = 0$, | Graybill and Pruitt's design is obtained. |
| (vi) $x_j = 1$ and $n_j = 1$, | We get a particular type of generalized design with cell frequencies 2 and 1. |
| (vii) $x_j = x_j$ and $n_j = n_j$, | Most general type of generalized staircase design is obtained. |

It is necessary that in each of these designs, there is at least a block which contains all the treatments of the design.

One more fact of interest with these designs is that they constitute one more type of non-orthogonal data for which the algebraic solution of the normal equations is possible.

For obtaining the layout of the design, the same principles as described in the case of Graybill and Pruitt's design hold.

The frequencies in the generalized design are shown below in a tabular form together with the number of replications, block sizes and the different block and treatment steps.

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# TABLE 1.

$m \longleftarrow$ TREATMENT-STEPS

FREQUENCIES IN THE DIFFERENT BLOCK×TREATMENT CELLS IN THE GENERALIZED DESIGN $[x_{kj} = x_{kj},\ n_{kj} = n_{kj}]$

| BLOCK-STEPS $\longrightarrow k$ | | | step (v). $t_{v1}, t_{v2}, \ldots, t_{vs_v}$ ($s_v$) | step (v-1). $t_{(v-1)1}, t_{(v-1)2}, \ldots, t_{(v-1)s_{(v-1)}}$ ($s_{v-1}$) | ..... | step 3. $t_{31}, t_{32}, \ldots, t_{3s_3}$ ($s_3$) | step 2. $t_{21}, t_{22}, \ldots, t_{2s_2}$ ($s_2$) | step 1. $t_{11}, t_{12}, \ldots, t_{1s_1}$ ($s_1$) | BLOCK-SIZE $N_{kj}$ | BLOCK-TOTAL $B_{kj}$ | BLOCK-AVERAGE $\beta_{kj}=\frac{B_{kj}}{N_{kj}}$ | $P_k=\sum_{j=1}^{r_k}\frac{1}{N_{kj}}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $r_v$. | step (v) | $b_{vr_v}$. | $n_{vr_v}+x_{vr_v}, \ldots, n_{vr_v}+x_{vr_v}$ | $n_{vr_v}, n_{vr_v}$, | ..... | ..... | ..... | $n_{vr_v}$ | $N_{vr_v}$ | $B_{vr_v}$ | $\beta_{vr_v}$ | $P_v=\sum_{j=1}^{r_v}\frac{1}{N_{vj}}$ |
| 2. | | $b_{v2}$. | | | | | | | $N_{v2}$ | $B_{v2}$ | $\beta_{v2}$ | |
| 1. | | $b_{v1}$. | $n_{v1}+x_{v1}, \ldots, n_{v1}+x_{v1}$ | $n_{v1}, n_{v1}$, | ..... | ..... | ..... | $n_{v1}$ | $N_{v1}$ | $B_{v1}$ | $\beta_{v1}$ | |
| $r_{v-1}$. | step (v-1) | $b_{(v-1)r_{(v-1)}}$. | $n_{(v-1)r_{(v-1)}}+x_{(v-1)r_{(v-1)}}, \ldots,$ | $n_{(v-1)r_{(v-1)}}+x_{(v-1)r_{(v-1)}}$ | $n_{(v-1)r_{(v-1)}}, n_{(v-1)r_{(v-1)}}$, | ..... | ..... | $n_{(v-1)r_{(v-1)}}$ | $N_{(v-1)r_{(v-1)}}$ | $B_{(v-1)r_{(v-1)}}$ | $\beta_{(v-1)r_{(v-1)}}$ | $P_{(v-1)}=\sum_{j=1}^{r_{v-1}}\frac{1}{N_{(v-1)j}}$ |
| 2. | | $b_{(v-1)2}$ | | | | | | | $N_{(v-1)2}$ | $B_{(v-1)2}$ | $\beta_{(v-1)2}$ | |
| 1. | | $b_{(v-1)1}$ | $n_{(v-1)1}+x_{(v-1)1}, \ldots,$ | $n_{(v-1)1}+x_{(v-1)1}$ | $n_{(v-1)1}, n_{(v-1)1}$, | ..... | ..... | $n_{(v-1)1}$ | $N_{(v-1)1}$ | $B_{(v-1)1}$ | $\beta_{(v-1)1}$ | |
| ⋮ | | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| $r_3$. | step 3. | $b_{3r_3}$. | $n_{3r_3}+x_{3r_3}, n_{3r_3}+x_{3r_3}, \ldots$ | ..... | ..... | $n_{3r_3}+x_{3r_3}$, | $n_{3r_3}, n_{3r_3}$, | $n_{3r_3}$ | $N_{3r_3}$ | $B_{3r_3}$ | $\beta_{3r_3}$ | $P_3=\sum_{j=1}^{r_3}\frac{1}{N_{3j}}$ |
| 2. | | $b_{32}$ | | | | | | | | | | |
| 1. | | $b_{31}$ | $n_{31}+x_{31}, n_{31}+x_{31}, \ldots$ | ..... | ..... | $n_{31}+x_{31}$ | $n_{31}, n_{31}, \ldots$ | $n_{31}$ | $N_{31}$ | $B_{31}$ | $\beta_{31}$ | |
| $r_2$. | step 2. | $b_{2r_2}$. | $n_{2r_2}+x_{2r_2}, n_{2r_2}+x_{2r_2}$, | ..... | ..... | ..... | $n_{2r_2}+x_{2r_2}$ | $n_{2r_2}, n_{2r_2}, \ldots, n_{2r_2}$ | $N_{2r_2}$ | $B_{2r_2}$ | $\beta_{2r_2}$ | $P_2=\sum_{j=1}^{r_2}\frac{1}{N_{2j}}$ |
| 2. | | $b_{22}$ | $n_{22}+x_{22}, n_{22}+x_{22}$, | ..... | ..... | ..... | $n_{22}+x_{22}$ | $n_{22}, n_{22}, \ldots, n_{22}$ | $N_{22}$ | $B_{22}$ | $\beta_{22}$ | |
| 1. | | $b_{21}$ | $n_{21}+x_{21}, n_{21}+x_{21}$, | ..... | ..... | ..... | $n_{21}+x_{21}$ | $n_{21}, n_{21}, \ldots, n_{21}$ | $N_{21}$ | $B_{21}$ | $\beta_{21}$ | |
| $r_1$. | step 1. | $b_{1r_1}$. | $n_{1r_1}+x_{1r_1}, n_{1r_1}+x_{1r_1}$, | ..... | ..... | ..... | ..... | $n_{1r_1}+x_{1r_1}$ | $N_{1r_1}$ | $B_{1r_1}$ | $\beta_{1r_1}$ | $P_1=\sum_{j=1}^{r_1}\frac{1}{N_{1j}}$ |
| 2. | | $b_{12}$ | $n_{12}+x_{12}, n_{12}+x_{12}$, | ..... | ..... | ..... | ..... | $n_{12}+x_{12}$ | $N_{12}$ | $B_{12}$ | $\beta_{12}$ | |
| 1. | | $b_{11}$ | $n_{11}+x_{11}, n_{11}+x_{11}$, | ..... | ..... | ..... | ..... | $n_{11}+x_{11}$ | $N_{11}$ | $B_{11}$ | $\beta_{11}$ | |
| | | $R_m$ | $R_v, R_v, \ldots, R_v$ | $R_{v-1}, R_{v-1}, \ldots, R_{v-1}$ | ..... | $R_3, R_3, \ldots, R_3$ | $R_2, R_2, \ldots, R_2$ | $R_1, R_1, \ldots, R_1$ | $N$ | | | |
| | | $T_{mi}$ | $T_{v1}, T_{v2}, \ldots, T_{vs_v}$ | $T_{(v-1)1}, \ldots, T_{(v-1)s_{(v-1)}}$ | ..... | $T_{31}, T_{32}, \ldots, T_{3s_3}$ | $T_{21}, T_{22}, \ldots, T_{2s_2}$ | $T_{11}, T_{12}, \ldots, T_{1s_1}$ | $\sum_{m=k=1}^{v}\sum_{i=1}^{s_m}\sum_{j=1}^{r_k} y_{ij} = G$ | | | |

## III. METHOD OF ANALYSIS

On the model

$y_{ij} = \mu + t_{mi} + b_{kj} + e_{ij}$ where

$\mu$ is a constant; $t_{mi}$, the effect of $i^{th}$ treatment in the $m^{th}$ step; $b_{kj}$, the effect of the $j^{th}$ block in the $k^{th}$ step and $e_{ij}$ a random variable with zero mean and constant variance $\sigma^2$, the treatment and block effects can be estimated by solving the following normal equations obtained from the principles of least squares. The case $x_{kj} = 1$, gives the more useful case as mentioned earlier. Solution to this case has been provided first. Solution to the more general case when $n_j = n_j$ has been indicated in a latter section.

General normal equation for treatments is

$$R_m t_{mi} + (b_{11} + b_{12} + \dots + b_{1r_1}) + (b_{21} + \dots + b_{2r_2}) + \dots + (b_{m1} + \dots + b_{mr_m}) + \sum_{k=1}^{q} \sum_{j=1}^{r_k} n_{kj} b_{kj} = T_{mi} \quad (1)$$

Equation (1) can be written as

$$R_m t_{mi} + \sum_{k=1}^{m} \sum_{j=1}^{r_k} b_{kj} = T_{mi} - S = T_{mi}' \quad (2)$$

$$\text{Where } S = \sum_{k=1}^{q} \sum_{j=1}^{r_k} b_{kj} n_{kj} \quad (3)$$

$(m = 1, 2, \dots q)$
$(i = 1, 2, \dots s_m)$

Similarly, general normal equation for blocks is

$$N_{kj} b_{kj} + (n_{kj} + 1) \sum_{m=1}^{q} \sum_{i=1}^{s_m} t_{mi} - (t_{11} + t_{12} + \dots + t_{1s_1}) - (t_{21} + t_{22} + \dots + t_{2s_2}) \dots - (t_{(k-1)1} + t_{(k-2)2} + \dots + t_{(k-1)s_{(k-1)}}) = B_{kj} \quad (4)$$

Equation (4) can be written as:

$$N_{kj}b_{kj} - \sum_{m=1}^{k-1}\sum_{i=1}^{S_m} t_{mi} = B_{kj} \qquad (5)$$

Where $$\sum_{m=1}^{q}\sum_{i=1}^{S_m} t_{mi} = 0 \qquad (6)$$

$(j = 1, 2, \ldots\ldots, r_k)$

$(k = 1, 2, \ldots\ldots, q\ )$

q being the total number of steps in the design.

$S_m$ = number of treatments in the $m^{th}$ step (treatmentwise)

$r_k$ = number of blocks in the $k^{th}$ step (blockwise)

Further, the total number of steps, blockwise and treatmentwise are the same, each being equal to q.

In equations (1) and(4), $T_{mi}$ and $B_{kj}$ denote respectively the corresponding block and treatment totals.

<u>Solution of normal equations:</u>

Putting k = 1 in (5)

$N_{1j}b_{1j} = B_{1j}$, as the second term on right hand side becomes non existent.

Hence, $$b_{1j} = \frac{B_{1j}}{N_{1j}} = \beta_{1j} \quad \text{(say)} \qquad (7)$$

$$\therefore \sum_{j=1}^{r_1} b_{1j} = \sum_{j=1}^{r_1} \beta_{1j} \qquad (8)$$

Putting m = 1 in (2), we get

$$R_1 t_{1i} + \sum_{j=1}^{r_1} b_{1j} = T_{1i}'$$

but from (8) $$\sum_{j=1}^{r_1} b_{1j} = \sum_{j=1}^{r_1} \beta_{1j}$$

$$\therefore\ t_{1i} = \frac{T_{1i}' - \sum_{j=1}^{r_1}\beta_{1j}}{R_1} = Z_{1i}' \quad \text{Say.}$$

and $$\sum_{i=1}^{S_1} t_{1i} = \sum_{i=1}^{S_1} Z_{1i}' \qquad (9)$$

Putting k = 2 in (5) we get

$$N_{2j}b_{2j} - \sum_{i=1}^{S_1} t_{1i} = B_{2j}$$

Substituting for $\sum_{i=1}^{s_1} t_{1i}$ from (9)

$$b_{2j} = \frac{B_{2j}}{N_{2j}} + \frac{\sum_{i=1}^{s_1} Z_{1i}'}{N_{2j}}, \text{ where } x_{kj} = 1$$

$$b_{2j} = \beta_{2j} + \frac{\sum_{i=1}^{s_1} Z_{1i}'}{N_{2j}}$$

Adding over j from 1 to $r_2$

$$\sum_{j=1}^{r_2} b_{2j} = \sum_{j=1}^{r_2} \beta_{2j} + \left( \sum_{j=1}^{r_2} \frac{1}{N_{2j}} \right) \left( \sum_{i=1}^{s_1} Z_{1i}' \right)$$

$$\sum_{j=1}^{r_2} b_{2j} = \sum_{j=1}^{r_2} \beta_{2j} + P_2 \left( \sum_{i=1}^{s_1} Z_{1i}' \right) \tag{10}$$

Where in general $P_k = \sum_{j=1}^{r_k} \frac{1}{N_{kj}}$, see table 1. (11)

Adding (8) and (10)

$$\sum_{j=1}^{r_1} b_{1j} + \sum_{j=1}^{r_2} b_{2j} = \sum_{j=1}^{r_1} \beta_{1j} + \sum_{j=1}^{r_2} \beta_{2j} + P_2 \sum_{i=1}^{s_1} Z_{1i}' \tag{12}$$

Putting m = 2 in equation (2)

$$R_2 t_{2i} + \sum_{j=1}^{r_1} b_{1j} + \sum_{j=1}^{r_2} b_{2j} = T_{2i}'$$

Substituting from (12)

$$t_{2i} = \frac{T_{2i}' - \sum_{j=1}^{r_1} \beta_{1j} - \sum_{j=1}^{r_2} \beta_{2j}}{R_2} - \frac{P_2}{R_2} \sum_{i=1}^{s_1} Z_{1i}'$$

$$t_{2i} = \frac{T_{2i}' - \sum_{k=1}^{2} \sum_{j=1}^{r_k} \beta_{kj}}{R_2} - \frac{P_2}{R_2} \sum_{i=1}^{s_1} Z_{1i}'$$

Defining in general

$$\frac{T_{mi}' - \sum_{k=1}^{m} \sum_{j=1}^{r_k} \beta_{kj}}{R_m} = Z_{mi}' \tag{13}$$

and taking $\frac{P_2}{R_2} = C^1_2$, $P_2 = \sum_{j=1}^{r_2} \frac{1}{N_{2j}}$ (14)

equation of $t_{2i}$ becomes

$$t_{2i} = Z_{2i}' - C^1_2 \sum_{i=1}^{s_1} Z_{1i}'$$

Adding over i from 1 to $S_2$

$$\sum_{i=1}^{S_2} t_{2i} = \sum_{i=1}^{S_2} Z_{2i}' - S_2C^1{}_2 \sum_{i=1}^{S_1} Z_{1i}' \tag{15}$$

Repeating the above process, we put

K = 3 in equation (5) so that

$$N_{3j}b_{3j} = \sum_{i=1}^{S_1} t_{1i} - \sum_{i=1}^{S_2} t_{2i} = B_{3j}$$

Substituting for $\sum_{i=1}^{S_1} t_{1i}$ from (9) and $\sum_{i=1}^{S_2} t_{2i}$ from (15)

$$N_{3j}b_{3j} = \sum_{i=1}^{S_1} Z_{1i}' + \sum_{i=1}^{S_2} Z_{2i}' - S_2C^1{}_2 \sum_{i=1}^{S_1} Z_{1i}' + B_{3j}$$

$$b_{3j} = \frac{(1-S_2C^1{}_2) \sum_{i=1}^{S_1} Z_{1i}' + \sum_{i=1}^{S_2} Z_{2i}' + B_{3j}}{N_{3j}}, \text{ since } P_3 = \sum_{j=1}^{r_3} \frac{1}{N_{3j}}$$

we have

$$\sum_{j=1}^{r_3} b_{3j} = P_3(1-S_2C^1{}_2) \sum_{i=1}^{S_1} Z_{1i}' + P_3 \sum_{i=1}^{S_2} Z_{2i}' + \sum_{j=1}^{r_3} \beta_{3j} \tag{16}$$

with the help of (8), (10) and (16) we have

$$\sum_{j=1}^{r_1} b_{1j} + \sum_{j=1}^{r_2} b_{2j} + \sum_{j=1}^{r_3} b_{3j} = \sum_{j=1}^{r_1} \beta_{1j} + \sum_{j=1}^{r_2} \beta_{2j} + P_2 \sum_{i=1}^{S_1} Z_{1i}' + P_3(1-S_2C^1{}_2) \sum_{i=1}^{S_1} Z_{1i}' + P_3 \sum_{i=1}^{S_2} Z_{2i}' + \sum_{j=1}^{r_3} \beta_{3j}$$

$$= \sum_{k=1}^{3} \sum_{j=1}^{r_k} \beta_{kj} + \sum_{i=1}^{S_1} Z_{1i}' (P_2 + P_3 - P_3S_2C^1{}_2) + P_3 \sum_{i=1}^{S_2} Z_{2i}' \tag{17}$$

Again putting m = 3 in (2)

$$R_3t_{3i} + \sum_{j=1}^{r_1} b_{1j} + \sum_{j=1}^{r_2} b_{2j} + \sum_{j=1}^{r_3} b_{3j} = T_{3i}'$$

Substituting for (17) in the above equation

$$t_{3i} = \frac{T_{3i}' - \sum_{k=1}^{3} \sum_{j=1}^{r_k} \beta_{kj}}{R_3} + \frac{(P_3S_2C^1{}_2 - P_2 - P_3)}{R_3} \sum_{i=1}^{S_1} Z_{1i}' - \frac{P_3}{R_3} \sum_{i=1}^{S_2} Z_{2i}'$$

Putting

$$\left.\begin{aligned} R_3C^1{}_3 &= P_3S_2C^1{}_2 - P_2 - P_3 \\ &= P_3( S_2C^1{}_2 - 1) - P_2 \\ \text{and} \quad R_3C^2{}_3 &= P_3 = \sum_{j=1}^{r_3} \frac{1}{N_{3j}} \end{aligned}\right\} \tag{18}$$

We have

$$t_{3i} = Z_{3i}' - C^2{}_3 \sum_{i=1}^{S_2} Z_{2i}' + C^1{}_3 \sum_{i=1}^{S_1} Z_{1i}'$$

$$\text{Where } Z_{3i}' = \frac{T_{3i}' - \sum_{k=1}^{3} \sum_{j=1}^{r_k} \beta_{kj}}{R_3}$$

$$\therefore \sum_{i=1}^{S_3} t_{3i} = \sum_{i=1}^{S_3} Z_{3i}' - S_3C^2{}_3 \sum_{i=1}^{S_2} Z_{2i}' + S_3C^1{}_3 \sum_{i=1}^{S_1} Z_{1i}' \quad (19)$$

For $K = 4$ and $m = 4$ we get similarly, expression for $\sum_{i=1}^{S_4} t_{4i}$ as given below:

$$\sum_{i=1}^{S_4} t_{4i} = \sum_{i=1}^{S_4} Z_{4i}' - S_4C^3{}_4 \sum_{i=1}^{S_3} Z_{3i}' + S_4C^2{}_4 \sum_{i=1}^{S_2} Z_{2i}' - S_4C^1{}_4 \sum_{i=1}^{S_1} Z_{1i}' \quad (20)$$

Where

$$\left.\begin{aligned} R_4C^3{}_4 &= P_4 = \sum_{j=1}^{r_4} \frac{1}{N_{4j}} \\ R_4C^2{}_4 &= P_4S_3C^2{}_3 - P_4 - P_3 \\ &= P_4(S_3C^2{}_3 - 1) - P_3 \\ R_4C^1{}_4 &= P_4S_3C^1{}_3 - S_2C^1{}_2(P_3+P_4) + (P_4+P_3+P_2) \\ &= P_4(S_3C^1{}_3 - S_2C^1{}_2 + 1) - R_3C^1{}_3 \end{aligned}\right\} \quad (21)^*$$

and $P_k$ and $Z_{mi}'$ are obtained from (11) and (13) respectively.

Lastly when $m = 5$ and $k = 5$ we get exactly on the same lines,

$$\sum_{i=1}^{S_5} t_{5i} = \sum_{i=1}^{S_5} Z_{5i}' - S_5C^4{}_5 \sum_{i=1}^{S_4} Z_{4i}' + S_5C^3{}_5 \sum_{i=1}^{S_3} Z_{3i}' - S_5C^2{}_5 \sum_{i=1}^{S_2} Z_{2i}' + S_5C^1{}_5 \sum_{i=1}^{S_1} Z_{1i}' \quad (22)$$

where

$$\begin{aligned} R_5C^4{}_5 &= P_5 = \sum_{j=1}^{r_5} \frac{1}{N_{5j}} \\ R_5C^3{}_5 &= P_5S_4C^3{}_4 - (P_4 + P_5) \\ &= P_5(S_4C^3{}_4 - 1) - P_4 \end{aligned}$$

$$\left.\begin{aligned} R_5C^2{}_5 &= P_5S_4C^2{}_4 - S_3C^2{}_3\,(P_5+P_4) + (P_5+P_4+P_3) \\ &= P_5(S_4C^2{}_4 - S_3C^2{}_3 + 1) - R_4C^2{}_4 \\ R_5C^1{}_5 &= P_5S_4C^1{}_4 - S_3C^1{}_3(P_5+P_4) + S_2C^1{}_2(P_5+P_4+P_3) \\ &= P_5\,(S_4C^1{}_4 - S_3C^1{}_3 + S_2C^1{}_2 - 1) - R_4C^1{}_4 \end{aligned}\right\} \quad (23)^*$$

(* For further details about the coefficients $C^1{}_2$, $C^1{}_3$ etc .... see formulae (31), (32), (33) in note 1, further).

Proceeding on the same lines the solution for $t_{mi}$ comes out as:

$$t_{mi} = Z_{mi}' - C_m^{m-1} \sum_{i=1}^{S_{m-1}} Z_{(m-1)i}' + C_m^{m-2} \sum_{i=1}^{S_{m-2}} Z_{(m-2)i}' \cdots\cdots$$
$$+ (-1)^{r-1}\, C_m^{m-r+1} \sum_{i=1}^{S_{m-r+1}} Z_{(m-r+1)i}' + \cdots\cdots$$
$$+ (-1)^{m-1}\, C_m^{1} \sum_{i=1}^{S_1} Z_{1i}' \qquad (24)$$

Where

$$R_mC_m^p = P_m\,(S_{m-1}\,C_{m-1}^p - S_{m-2}C^p{}_{m-2} + \cdots\cdots + (-1)_{\dagger}^{m-p-1}.1\,)$$
$$- R_{m-1}C_{m-1}^p \qquad (25)^*$$

and $p < m$, $m = 1$ to $q$

$$\left.\begin{array}{l} \text{When } p = m,\ S_mC_m^p = 1 \\ \text{When } p < m,\ S_mC_m^p \text{ exists and is non zero} \\ \qquad C_m^o = 0,\quad C_m^m = 0 \end{array}\right\} \quad (26)$$

(* See note 1, further).

## Calculation of S.

The final solution of $t_{mi}$ cannot be obtained from (24) as the solution is a function of $T_{mi}'$ which involves the unknown quantity $S = \sum_{k=1}^{v}\sum_{j=1}^{r_k} b_{kj}n_{kj}$

S has been obtained by using the relation $\sum_{m=1}^{q}\sum_{i=1}^{S_m} t_{mi} = 0$ as shown below.

Equation (24) can be written in summation form as

$$t_{mi} = Z_{mi}' + \sum_{r=2}^{m}\left\{\sum_{i=1}^{S_{m-r+1}} Z'_{(m-r+1)i}\, C_m^{m-r+1}\, (-1)^{r-1}\right\}$$

Separating S we obtain

$$t_{mi} = Z_{mi} + \sum_{r=2}^{m}\left\{(-1)^{r-1} C_m^{m-r+1} \sum_{i=1}^{S_{m-r+1}} Z_{(m-r+1)i}\right\} - S\left\{\sum_{r=2}^{m} (-1)^{r-1} C_m^{m-r+1} \frac{S_{m-r+1}}{R_{m-r+1}} + \frac{1}{R_m}\right\} \qquad (27)$$

$$i \leqslant (m-1)$$

where $Z_{mi} = \dfrac{T_{mi} - \sum_{k=1}^{m}\sum_{j=1}^{r_k}\beta_{kj}}{R_m}$, $Z_{mi}' + \dfrac{S}{R_m} = Z_{mi}$

Summing over all the treatments in $m^{th}$ step and then over all steps,

$$0 = \sum_{m=1}^{v}\sum_{i=1}^{S_m} t_{mi}$$

$$= \sum_{m=1}^{v}\left[\sum_{i=1}^{S_m} Z_{mi} + S_m \sum_{r=2}^{m} (-1)^{r-1} C_m^{m-r+1} \sum_{i=1}^{S_{m-r+1}} Z_{(m-r+1)i}\right] - S\sum_{m=1}^{v} S_m\left\{\frac{1}{R_m} + \sum_{r=2}^{m} (-1)^{r-1} C_m^{m-r+1} \frac{S_{m-r+1}}{R_{m-r+1}}\right\}$$

$$\therefore S = \frac{\sum_{m=1}^{v}\left\{\sum_{i=1}^{S_m} Z_{mi} + S_m \sum_{r=2}^{m} (-1)^{r-1} C_m^{m-r+1} \sum_{i=1}^{S_{m-r+1}} Z_{(m-r+1)i}\right\}}{\sum_{m=1}^{v} S_m\left\{\frac{1}{R_m} + \sum_{r=2}^{m} (-1)^{r-1} C_m^{m-r+1} \frac{S_{m-r+1}}{R_{m-r+1}}\right\}} \qquad (28)$$

Substituting this value of S in all $Z_{mi}'$'s we can get any $t_{mi}$ from (24).

If all the $n_{kj}$'s are zero, S becomes zero from the relation (3) and so we get from the first term of (27) the solution of the design of Graybill and Pruitt. Further here $Z_{mi}' = Z_{mi}$ and $T_{mi}' = T_{mi}$.

Formula (28) can also be expressed by collecting the coefficients of different $Z_{mi}$'s as below:

(28) gives

$$\text{Numerator of } S = \sum_{i=1}^{S_1} Z_{1i}\,(\,1-S_2C^1_2+S_3C^1_3\,\ldots\ldots + S_qC^1_q)$$
$$+ \sum_{i=1}^{S_2} Z_{2i}(\,1-S_3C^2_3+ S_4C^2_4\ldots\ldots - S_qC^2_q)$$
$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$+ \sum_{i=1}^{S_q} Z_{qi}(1-\cancel{S_{q+1}C^q_{q+1}}),\ \text{since from (26) } C^q_{q+1} = 0$$
$$= \sum_{m=1}^{q}\sum_{i=1}^{S_m} Z_{mi}\Big\{1+ \sum_{r=1}^{q-m} (-1)^r S_{m+r}\, C^m_{m+r}\Big\} \qquad (29)$$

Similarly, denominator of S

$$= \sum_{m=1}^{q} \frac{S_m}{R_m}\Big\{1+ \sum_{r=1}^{q-m} (-1)^r S_{m+r}C^m_{m+r}\Big\} = \triangle \quad \text{say} \qquad (30)$$

This $\triangle$, the denominator of S is independent of $Z_{mi}$, therefore will not affect variance of two treatment differences. Further from (29) and (30), coefficient of $\sum_{i=1}^{S_m} Z_{mi}$ is same as coefficient of $(S_m/R_m)$, so these coefficients can be calculated once for all, in any example.

* NOTE 1. It is evident that we will have to calculate every $C^p_m$ so as to find all $t_{mi}$'s which require the help of formulae (14), (18), (21) and (23), and other similar expressions. This indicates that every time each $C^p_m$ will have to be calculated independently. However it is possible to obtain relation

between any $C^p_m$ and the preceding $C^p_{m-1}$, as shown below:

Let us take the relation (21), as (14) and (18) are not at all difficult to calculate

$$R_4C^1{}_4 = P_4S_3C^1{}_3 - S_2C^1{}_2(P_4+P_3) + (P_4+P_3+P_2)$$

$$= P_4(S_3C^1{}_3 - S_2C^1{}_2 + 1) + (P_2+P_3-P_3S_2C^1{}_2)$$

$$\therefore R_4C^1{}_4 = P_4(S_3C^1{}_3 - S_2C^1{}_2 + 1) - R_3C^1{}_3, \text{ from (18)} \qquad (31)$$

Since $C^1{}_3$, $C^1{}_2$ have already been calculated from (14) and (18), $C^1{}_4$ now can be obtained easily.

Similarly in (23) we have:

$$R_5C^2{}_5 = P_5S_4C^2{}_4 - S_3C^2{}_3(P_5+P_4) + (P_5+P_4+P_3)$$

$$\therefore R_5C^2{}_5 = P_5(S_4C^2{}_4 - S_3C^2{}_3 + 1) - R_4C^2{}_4 \qquad (32)$$

Where $C^2{}_4$ has been obtained in (21) and $(S_3C^2{}_3 - 1)$ has been evaluated while calculating $C^2{}_4$.

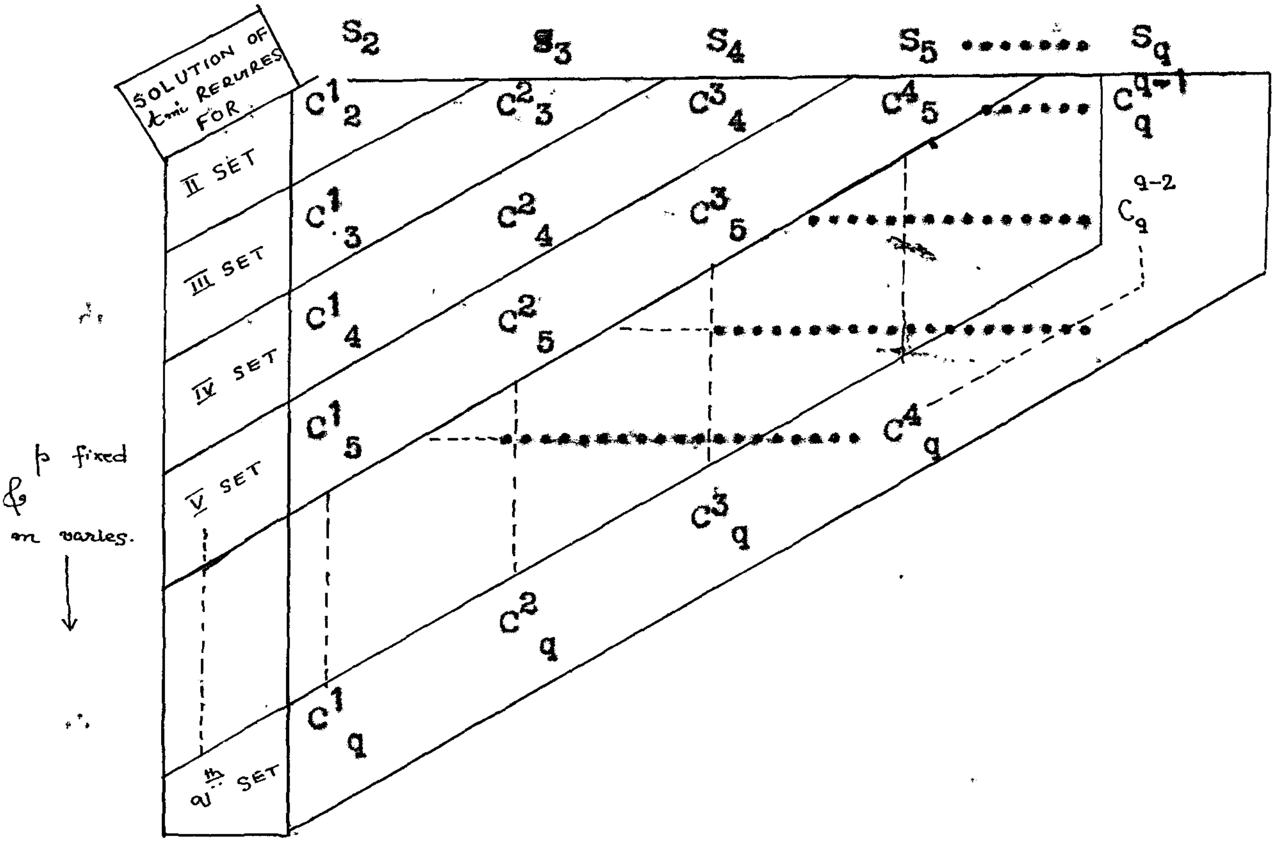
Lastly $R_5C^1{}_5 = P_5(S_4C^1{}_4 - S_3C^1{}_3 + S_2C^1{}_2 - 1) - R_4C^1{}_4$ (33)

where $C^1{}_4$ and $(S_3C^1{}_3 - S_2C^1{}_2 + 1)$ have already been obtained in (31).

Note 2. The following table shows the different $C^p_m$'s required for the solution of the normal equation. The entries are to be obtained columnwise as the coefficients in the same column are connected through recurrence relation. After the table has been completed then the entries within the $(m-1)^{th}$ oblique row are required for the solution of $t_{mi}$.

Table 2

m and p vary $\longrightarrow$




Where any particular $S_m$ occurs with each of $C^p_m$ successively, (shown obliquely), $p = 1, 2, \ldots\ldots (m-1)$, in the expression of $t_{mi}$. This table No. 2 helps us in obtaining any $C^p_m$ if all the preceding calculated $C^p_m$'s are represented in the above indicated tabular form.

<u>Analysis of variance:</u>

After the solution has been obtained thus, the partitioning for the analysis of variance can be obtained as below:

The adjusted sum of squares due to treatments, (A), say $= \sum_{m=1}^{v}\sum_{i=1}^{S_m} t_{mi}Q_{mi}$, where $Q_{mi}$ is the adjusted total for the $(mi)^{th}$ treatment and is given by

$$Q_{mi} = T_{mi} - \sum_{k=1}^{v}\sum_{j=1}^{n_k} n_{mi(kj)} \cdot \beta_{kj}\ , \tag{34}$$

where $n_{mi(kj)}$ denotes the number of observations under the $(mi)^{th}$ treatment in the $j^{th}$ block and $\beta_{kj}$, the average of the $j^{th}$ block, in $k^{th}$ step (blockwise).

Next the unadjusted block sum of squares (B) is to be obtained as usual. The block and treatment interaction sum of squares (I) is to be obtained from

$$I = P_{bt}-(A)-(B) \tag{35}$$

where $P_{bt}$ denotes the total sum of squares due to the cell totals (with their proper divisors) in the treatment x block, table. The error sum of squares is now obtained by subtraction, from the total sum of squares of the table, the quantities (A), (B), (I). Other way is to subtract $P_{bt}$ from total sum of squares and obtain error sum of squares directly, then interaction sum of squares is obtained by usual subtraction from all quantities.

The analysis is shown below in a tabular form.

Table 3

Analysis of variance for generalized staircase design
$(x_{kj} = 1,\ n_{kj} = n_{kj})$

| Source of variation | d.f. | Sum of squares | Mean squares |
|---|---|---|---|
| Blocks unadjusted | $\left(\sum_{k=1}^{v} r_k - 1\right)$ | $\sum_{k=1}^{v}\sum_{j=1}^{r_k} \frac{B_{kj}^2}{N_{kj}} = (B)$ | |
| Treatments adjusted | $\left(\sum_{m=1}^{v} S_m - 1\right)$ | $\sum_{m=1}^{v}\sum_{i=1}^{S_m} t_{mi} Q_{mi} = (A)$ | |
| Interaction | $\left(\sum_{k=1}^{v} r_k - 1\right)\left(\sum_{m=1}^{v} S_m - 1\right)$ | $I = P_{bt}-(A)-(B)$ | |
| Error | $N-\left(\sum_{k=1}^{v} r_k\right)\left(\sum_{m=1}^{v} S_m\right)$ | by difference | Est. of $\sigma^2 = s^2$ |
| Total | $(N-1)$ | $\sum_{m=k=1}^{v}\sum_{i=1}^{S_m}\sum_{j=1}^{r_k} y_{ij}^2$ | |

where N = total number of experimental units in the generalized design

$$= \sum_{k=1}^{v}\sum_{j=1}^{r_k} n_{kj} \cdot \sum_{m=1}^{v} S_m + \sum_{m=1}^{v}\sum_{k=1}^{m} r_k S_m \tag{36}$$

where G = Grand total $= \sum_{k=1}^{q} \sum_{j=1}^{n_k} B_{kj} = \sum_{m=1}^{q} \sum_{i=1}^{S_m} T_{mi}$

Variance of treatment differences:

The variance has been obtained by using the technique given by Das (1953).

The variance of $(t_{mi}-t_{mi'})$ where both the treatments i and i' belong to the same $m^{th}$ step, comes out to be $2\sigma^2/R_m$. (37)

The variance of $(t_{mi}-t_{\overline{m+g}i'})$ has been obtained as below by first collecting the coefficients of $T_{mi}$ and $T_{(m+g)i'}$ and then taking the difference, (m) and (m+g) being two different steps, $m+g \leqslant q$, $g \geqslant 1$.

We have from (24) by replacing (m) by (m+g) and taking difference:

$$\left[t_{mi} - t_{(m+g)i'}\right] = \left[Z_{mi} - Z_{(m+g)i'}\right] + \left\{ \sum_{r=2}^{m} (-1)^{(r-1)} C_m^{m-r+1} \sum_{i=1}^{S_{m-r+1}} Z_{(m-r+1)i} - \sum_{r=2}^{m+g} (-1)^{r-1} C_{m+g}^{m+g-r+1} \sum_{i=1}^{S_{m+g-r+1}} Z_{(m+g-r+1)i'} \right]$$
$$+ S \left[ (1/R_{m+g} - 1/R_m) + \left\{ \sum_{r=2}^{m+g} (-1)^{r-1} \times C_{m+g}^{m+g-r+1} \frac{S_{m+g-r+1}}{R_{m+g-r+1}} - \sum_{r=2}^{m} (-1)^{r-1} C_m^{m-r+1} \times \frac{S_{m-r+1}}{R_{m-r+1}} \right\} \right] \qquad (38)$$

With the help of (29) and (30), after substituting for S, which will contribute later towards variance, we get:

$$\left[t_{mi} - t_{(m+g)i'}\right] = Z_{mi} \left[ 1 - \frac{1}{\Delta} \left\{ 1 + \sum_{r=1}^{q-m} (-1)^r S_{m+r} C_{m+r}^{m} \right\} \times \left\{ \sum_{r=2}^{m} (-1)^{r-1} C_m^{m-r+1} \frac{S_{m-r+1}}{R_{m-r+1}} + \frac{1}{R_m} \right\} \right]$$
$$+ \sum_{r=2}^{m} \left[ (-1)^{r-1} C_m^{m-r+1} \sum_{i=1}^{S_{m-r+1}} Z_{(m-r+1)i} \right]$$

(Continued)

$$-Z_{(m+g)i'}\left[1-\frac{1}{\Delta}\left\{1+\sum_{r=1}^{q-m-g}(-1)^r S_{m+g+r}C_{m+g+r}^{m+g}\right\}\times\right.$$

$$\left.\left\{\sum_{r=2}^{m}(-1)^{r+1}C_{m+g}^{m-r+g+1}\ \frac{S_{m+g-r+1}}{R_{m+g-r+1}}+\frac{1}{R_{m+g}}\right\}\right]$$

$$-\sum_{r=2}^{m+g}\left\{(-1)^{r-1}C_{m+g}^{m+g-r+1}\sum_{i=1}^{S_{m+g-r+1}}Z_{(m+g-r+1)i'}\right\}$$

Replacing $Z_{mi}$ by $+\ 1/R_m$ and $Z_{(m+g)i'}$ by $-\ 1/R_{m+g}$ and putting rest of the $Z_{mi}$'s equal to zero, we obtain the variance of $\{t_{mi}-t_{(m+g)i'}\}$. However $Z_{(m-r+1)i}$ will be zero but $Z_{(m+g-r+1)i'}$ will be zero except when $r = g + 1$, thus $Z_{mi}'$ will contribute towards variance. Lastly it has been observed that when 'g' is odd than coefficent of $Z_{mi}$ has positive sign, when 'g' is even then coefficient of $Z_{mi}$ bears negative sign.

Putting all these points in notation form we obtain finally the algebraic form of the variance formula together with the contribution of S towards variance, as

$$V\{t_{mi}-t_{(m+g)i'}\} = \left\{\frac{1-(-1)^g C_{m+g}^{m}}{R_m}+\frac{1}{R_{(m+g)}}\right\}\sigma^2$$

$$+\ \frac{\sigma^2}{\Delta}\left[\left(\frac{1}{R_{m+g}}-\frac{1}{R_m}\right)+\sum_{r=2}^{m+g}\frac{(-1)^{r-1}C_{m+g}^{m+g-r+1}}{R_{m+g-r+1}}\right.$$

$$\left.-\sum_{r=2}^{m}\frac{(-1)^{r-1}C_{m}^{m-r+1}}{R_{m-r+1}}\right]\times$$

$$\left[\frac{1+\sum_{r=1}^{q-m}(-1)^r S_{m+r}\,C_{m+r}^{m}}{R_m}-\right.$$

$$\left.\frac{1+\sum_{r=1}^{q-m-g}(-1)^r S_{m+g+r}C_{m+g+r}^{m+g}}{R_{m+g}}\right] \qquad (39)$$

$$m+g \leqslant q,\ g \geqslant 1$$

If we put $g = 0$, $C_m^m = 0$ in (39), we obtain

$$V\left\{t_{mi} - t_{(m+g)i'}\right\} = \frac{2\sigma^2}{R_m} = V(t_{mi} - t_{mi'}), \text{ same as (37).}$$

Note: From formula (39) we can get ${}^qC_2$ possible number of variance expressions, where $q$ = total number of steps in the design.

## IV. A PARTICULAR CASE : when $n_{kj} = 0$

Case I:

When $n_{kj}$ is zero the general design reduces to the design considered by Graybill and Pruitt. The different results in this case come out as indicated below:

(i) Since $n_{kj} = 0$

$$S = \sum_{k=1}^{q}\sum_{j=1}^{n_k} b_{kj} n_{kj} = 0$$

(ii) $T_{mi}' = T_{mi}$

and $Z_{mi}' = Z_{mi}$

where $Z_{mi} = \dfrac{T_{mi} - \sum_{k=1}^{m}\sum_{j=1}^{n_k}\beta_{kj}}{R_m}$, $\beta_{kj} = \dfrac{B_{kj}}{N_{kj}}$

(iii) Thus formula (24) reduces to:

$$t_{mi} = Z_{mi} - C_m^{m-1}\sum_{i=1}^{s_{m-1}} Z_{(m-1)i} + C_m^{m-2}\sum_{i=1}^{s_{m-2}} Z_{(m-2)i} + \dots + (-1)^{m-1} C_m^{1}\sum_{i=1}^{s_1} Z_{1i} \tag{40a}$$

(iv) $C_m^p$ can now be obtained easily through the recurrence relation:

$$\frac{C_m^p}{R_p} + \frac{C_m^{p+1}}{R_{p+1}} = \frac{(-1)^{m-p-1}}{\sum_{k=p+1}^{v} S_k}\left(1/R_p - 1/R_{p+1}\right), \quad p < m \leqslant q \tag{40 b}$$

conveniently, $C_m^{m-1}$ is to be obtained from $pR_m/R_m$, as the other $C_m^p$'s being obtained from the relation (40 b). This relation could be established through a method which has been discussed in section 10, page ( 54 ).

(v) $Q_{mi} = R_m Z_{mi}$ (41)

where $Q_{mi} = T_{mi} - \sum_{k=1}^{m}\sum_{j=1}^{r_k} \frac{B_{kj}}{N_{kj}}$ , because for $k = (m+1)$ to $q$ , $n_{kj} = 0$ here.

Hence treatment sum of squares (adjusted)

$$= \sum_{m=1}^{q}\sum_{i=1}^{s_m} t_{mi}\, Q_{mi}$$

$$= \sum_{m=1}^{q}\sum_{i=1}^{s_m} R_m Z_{mi} t_{mi} \qquad (42)$$

and block sum of squares (unadjusted)

$$= \sum_{k=1}^{q}\sum_{j=1}^{r_k} \frac{B_{kj}^2}{N_{kj}}$$

Now analysis of variance table can be completed.

Table 4

Graybill and Pruitt's design, when $n_{kj} = 0$

| Source of variation | d.f. | Sum of squares | Mean squares | F-ratio |
|---|---|---|---|---|
| Blocks | $\left(\sum_{k=1}^{q} r_k - 1\right) = R-1$ | $\sum_{k=1}^{q}\sum_{j=1}^{r_k} \frac{B_{kj}^2}{N_{kj}}$ | | |
| Treatments | $\left(\sum_{m=1}^{q} s_m - 1\right) = N-1$ | $\sum_{m=1}^{q}\sum_{i=1}^{s_m} R_m Z_{mi} t_{mi}$ | T | T/E |
| Error | $(R-1)(N-1)$ | By difference | E | |
| Total | $(RN-1)$ | $\sum_{m=k=1}^{q}\sum_{i=1}^{s_m}\sum_{j=1}^{r_k} y_{ij}^2$ | | |

where $R_m = \sum_{k=1}^{m} r_k$ , $R = \sum_{k=1}^{q} r_k$

$N_{kj} = \sum_{m=k}^{q} S_m$ , $N = \sum_{m=1}^{q} S_m$

Further, since $S = 0$, its contibution towards variance is nil. Hence from formula (39)

$$V\left[t_{mi} - t_{(m+g)i'}\right] = \left\{ \frac{1-(-1)^g\, C_{m+g}^{m}}{R_m} + \frac{1}{R_{m+g}} \right\} \sigma^2 \qquad (43)$$

Whenever $g = 0$, $C_m^m = 0$, i.e. two treatments belong to the same step,

VK

$$V(t_{mi}-t_{mi}') = \frac{2\sigma^2}{R_m}$$, same as (37), which also follows from the above relation if we take $C_m^m = 0$.

## V. A CLASS OF MORE GENERALIZED DESIGNS ($x_{kj} = x_{kj}$, $n_{kj} = n_{kj}$)

We have got the analysis of the design for which the frequency of observation in any cell in the $(kj)^{th}$ block is either $(n_{kj}+1)$ or $(n_{kj})$. This however is a particular case of more generalized design obtainable by taking the frequencies in the cells as $(n_{kj}+x_{kj})$ and $(n_{kj})$ as defined earlier.

The solution of the normal equation for this design does not involve any fresh difficulty. As a matter of fact the expression giving the solution of $t_{mi}$ remains the same but for the new meaning of some of the notations.

Thus in this case, we have:

(i) $$t_{mi} = Z_{mi}' - C_m^{m-1} \sum_{i=1}^{S_{m-1}} Z_{(m-1)i}' + \cdots\cdots + (-1)^{m-1} C_m^1 \sum_{i=1}^{S_1} Z_{1i}' \tag{44}$$

which has the same form as (24).

(ii) $$Z_{mi} = \frac{T_{mi} - \sum_{k=1}^{m} \sum_{j=1}^{n_k} x_{kj}\beta_{kj}}{R_m}, \quad Z_{mi}' = Z_{mi} - \frac{S}{R_m} \tag{45}$$

(iii) $$P_k = \sum_{j=1}^{n_k} \frac{x_{kj}^2}{N_{kj}} \tag{46}$$

while $$S = \sum_{k=1}^{v} \sum_{j=1}^{n_k} b_{kj} n_{kj}$$

Finding S on the same lines as indicated in article 3, we note that its expression is also unaltered and is given by

$$S = \frac{\sum_{m=1}^{v} \left\{ \sum_{i=1}^{S_m} Z_{mi} + S_m \sum_{r=2}^{m} (-1)^{r+1} C_m^{m-r+1} \sum_{i=1}^{S_{m-r+1}} Z_{(m-r+1)i} \right\}}{\sum_{m=1}^{v} \left[ S_m \left\{ \frac{1}{R_m} + \sum_{r=2}^{m} (-1)^{r-1} C_m^{m-r+1} \frac{S_{m-r+1}}{R_{m-r+1}} \right\} \right]} \tag{47}$$

Variance formula:

The form will remain same as in (39). Hence the variance of two treatments difference, lying in different steps is given by:

$$V\left[t_{mi}-t_{(m+g)i'}\right] = \left\{\frac{1-(-1)^{g}C_{m+g}^{m}}{R_m} + \frac{1}{R_{m+g}}\right\}\sigma^2$$
$$+ \frac{\sigma^2}{\Delta}\left\{\left(\frac{1}{R_{m+g}}-\frac{1}{R_m}\right) + \sum_{r=2}^{m+g}\frac{(-1)^{r-1}C_{m+g}^{m+g-r+1}}{R_{m+g-r}}\right.$$
$$\left. - \sum_{r=2}^{m}\frac{(-1)^{r-1}C_{m}^{m-r+1}}{R_{m-r}}\right\} \times$$
$$\left\{\frac{1+\sum_{r=1}^{v-m}(-1)^{r}S_{m+r}C_{m+r}^{m}}{R_m} - \frac{1+\sum_{r=1}^{v-m-g}(-1)^{r}S_{m+g+r}C_{m+g+r}^{m+g}}{R_{m+g}}\right\} \tag{48}$$

$$m+g \leqslant q, \quad g \geqslant 1$$

where $x_{kj}$ is involved directly in $R_m$ and $R_{m+g}$, here.

Analysis of variance remains unaltered as indicated in table number 3 of the design ($n_{kj} = n_{kj}$, $x_{kj} = 1$). Method of obtaining various sums of squares is also unaffected, and has been described fully in article 3.

Table 5

Analysis of variance of more general staircase design
($x_{kj} = x_{kj}$, $n_{kj} = n_{kj}$)

| Source of variation | d.f. | Sum of squares | Mean squares |
|---|---|---|---|
| Blocks unadjusted | $\left(\sum_{k=1}^{v} r_k - 1\right)$ | $\sum_{k=1}^{v}\sum_{j=1}^{r_k}\frac{B_{kj}^2}{N_{kj}} = (B)$ | |
| Treatments adjusted | $\left(\sum_{m=1}^{v} S_m - 1\right)$ | $\sum_{m=1}^{v}\sum_{i=1}^{S_m} t_{mi}Q_{mi} = (A)$ | |
| Interaction | $\left(\sum_{k=1}^{v} r_k - 1\right)\left(\sum_{m=1}^{v} S_m - 1\right)$ | $I = P_{bt} - (A) - (B)$ | |
| Error | $N - \left(\sum_{k=1}^{v} r_k\right)\left(\sum_{m=1}^{v} S_m\right)$ | By difference | Est. of $\sigma^2 = s^2$ |
| Total | $(N-1)$ | $\sum_{m=k=1}^{v}\sum_{i=1}^{S_m}\sum_{j=1}^{r_k} y_{ij}^2$ | |

where N = total number of experimental units in this design.

VI. <u>PARTICULAR CASE:</u> obtained from the more general designs ($x_{kj} = x_{kj}$, $n_{kj} = n_{kj}$)

<u>Case II.</u>

When $x_{kj} = 0$ and $n_{kj} = 1$ we get the randomized block design. The solution of $t_{mi}$ comes out from the formula (44) as given below:

In $n_{kj} = 1$ and $x_{kj} = 0$ designs we have

$$R_m = \sum_{k=1}^{v} r_k = R \text{ say}, \quad N_{kj} = \sum_{m=1}^{v} S_m = N \text{ say},$$

From formulae (7), (10) and (46) and other similar expressions it is clear that when $x_{kj} = 0$, we have

$$\left.\begin{array}{l} \text{(i) } P_k = \displaystyle\sum_{j=1}^{n_k} \frac{x_{kj}^2}{N_{kj}} = 0 \\ \text{(ii) } C_m^p = 0 \text{ for all p and m, from (14), (18), (21), (23) and (25)} \\ \text{and (iii) } \displaystyle\sum_{j=1}^{n_k} b_{kj} = \sum_{j=1}^{n_k} \frac{B_{kj}}{N}, \text{ from (8),(10), and (16) since every } P_k \text{ is zero here.} \\ \text{i.e. } b_{kj} = \dfrac{B_{kj}}{N} \text{ only when } x_{kj} = \text{ and } P_k = 0, \text{ hence} \\ b_{kj} = \beta_{kj} \text{ in randomized block design.} \end{array}\right\} \quad (49)$$

Hence $S = \sum_{k=1}^{v} \sum_{j=1}^{n_k} b_{kj} n_{kj}$ reduces to, since $n_{kj} = 1$

$= G/N$ in randomized block design, with the help of (49)

Further in ( $x_{kj} = x_{kj}$, $n_{kj} = n_{kj}$) design from (45) and (46)

$$Z_{mi}' = \frac{T_{mi}' - \sum_{k=1}^{m} \sum_{j=1}^{n_k} \beta_{kj}\, x_{kj}}{R_m}$$, which in randomized block design reduces to, as $x_{kj} = 0$

$$Z_{mi}' = \frac{T_{mi}'}{R} = \frac{T_{mi} - \frac{G}{N}}{R}, \quad Z_{mi} = \frac{T_{mi}}{R} \qquad (50)$$

Thus ultimately formula (24) for $t_{mi}$ with the help of (50), becomes:

$$\left.\begin{array}{l} t_{mi} = Z_{mi}' = \left( \dfrac{T_{mi}}{R} - \dfrac{G}{NR} \right) \text{ in randomized block design as every } C_m^p = 0 \\ \text{and } b_{kj} = B_{kj}/N, \text{ since } x_{kj} = 0 \end{array}\right\} \quad (51)$$

In analysis of variance ($x_{kj} = x_{kj}$, $n_{kj} = n_{kj}$), from table number 5, original interaction component will now become error component. Thus interaction is absent in the randomized block design.

Further treatment sum of squares $= \sum_{m=1}^{v}\sum_{i=1}^{S_m} t_{mi}Q_{mi}$, from (42) &(51) becomes $= \sum_{m=1}^{v}\sum_{i=1}^{S_m}\left(\frac{T_{mi}^2}{R} - \frac{G^2}{NR}\right)$ in randomized block design.

where $Q_{mi} = T_{mi} - \sum_{k=1}^{v}\sum_{j=1}^{n_k} \frac{B_{kj}}{N} = \left(T_{mi} - \frac{G}{N}\right)$ (52)

and block sum of squares $= \sum_{k=1}^{v}\sum_{j=1}^{n_k} b_{kj}B_{kj}$

$$= \sum_{k=1}^{v}\sum_{j=1}^{n_k} \frac{B_{kj}^2}{N}\text{, the usual one.}$$

Below given table summarizes these results.

Table 6

Randomized block design with one observation per cell
($n_{kj} = 1$, $x_{kj} = 0$)

| Source of variation | d.f. | Sum of squares | Mean squares | F-ratio |
|---|---|---|---|---|
| Blocks | $\left(\sum_{k=1}^{v} n_k - 1\right) = (R-1)$ | $\sum_{k=1}^{v}\sum_{j=1}^{n_k}\frac{B_{kj}^2}{N}$ | | |
| Treatments | $\left(\sum_{m=1}^{v} S_m - 1\right) = (N-1)$, | $\sum_{m=1}^{v}\sum_{i=1}^{S_m}\left(\frac{T_{mi}^2}{R} - \frac{G^2}{NR}\right)$ | T | T/E |
| Error | $(R-1)(N-1)$ | By difference | E | |
| Total | $(RN-1)$ | $\sum_{m=k=1}^{v}\sum_{i=1}^{S_m}\sum_{j=1}^{n_k} y_{ij}^2$ | | |

Since $S = G/N$ and every $C_m^p = 0$ in randomized block design, the variance formula (39), for two treatments' difference lying in different steps, becomes:

$$V\left[t_{mi} - t_{(m+g)i'}\right] = \frac{2\sigma^2}{R} = V(t_{mi} - t_{mi'}) \qquad (53)$$

because $R_m = R_{m+g}$, $g = 0$ and contribution of S towards variance is nil.

Note: Whenever $x_{kj} = 0$ we always get a randomized block design with one or more observations per cell, as indicated in article 2.

## VII. PARTITIONING OF ADJUSTED SUM OF SQUARES

In many experiments particularly in progeny row trials it becomes necessary to partition the adjusted treatment sum of squares into components. The method of partitioning in the case of orthogonal data is known. In the case of non-orthogonal data methods are available for obtaining components with only 1 degree of freedom each. But for such data there seems to be no suitable method for obtaining components based on more than 1 degree of freedom. It is known that in the case of non-orthogonal data, through the total of two or more components each of 1 degree of freedom, the sum of squares due to the total degree of freedom can not be obtained.

An attempt has thus been made to evolve a suitable method of obtaining components of adjusted sum of squares having more than one degree of freedom.

General method:

The method for finding a component with 1 degree of freedom consists in first obtaining a solution of the normal equations and then substituting these values in $\sum_i l_i t_i$ where $t_i$'s are the treatment effects and $\sum_i l_i t_i$ a contrast among the $t_i$'s following from the hypothesis corresponding to the degree of freedom. The sum of squares is then obtained from

$$\frac{(\sum_i l_i t_i)^2}{\sum_i l_i a_i}, \text{ where } \sigma^2 \sum_i l_i a_i \text{ is the variance of the}$$

contrast $\sum_i l_i t_i$.

When a component having, say, 'p' degrees of freedom is to be obtained, first 'p' mutually orthogonal linear contrasts among the $t_i$'s are to be defined such that the total of the sum of squares from these contrasts lead to the required component in the orthogonal case. Let us denote these contrasts by

$$L_1, L_2 \ldots\ldots\ldots, L_p.$$

Next define (v-p-1) more contrasts among the $t_i$'s which are orthogonal among themselves as also to any contrasts $L_j$, ($j = 1, 2, \ldots, p$) where v denotes the total number of treatments. Let these latter contrasts be denoted by

$$L_{p+1}, L_{p+2}, \ldots\ldots, L_{v-1}.$$

When obtained as usual, the normal equations suitable for two-way non-orthogonal data, come as

$$\sum_k C_{ik} t_k = Q_i$$

where $Q_i$ is the adjusted total of the $i^{th}$ treatment and $C_{ik}$'s are functions of the cell frequencies of observations in the two-way table. These equations are obtainable easily by following Kempthorne (1952) or Das (1953).

Next let us form contrasts among the $Q_i$'s to be obtained by replacing $t_i$ by $Q_i$ in the contrasts $L_j$, ( $j = 1, 2, \ldots, (v-1)$). Let these contrasts of $Q_i$'s be denoted by $P_j$, ($j = 1, 2 \ldots, (v-1)$), such that $P_j$ corresponds to $L_j$. Taking the expected value of $Q_i$, to be $\sum_k C_{ik} t_k$, the expected values of $P_j$'s are written and these will give another set of (v-1) equations, one corresponding to each $P_j$. Let these equations be denoted by

$$\sum_i K_{ij} t_i = P_j, \; (j = 1, 2, \ldots\ldots (v-1)). \qquad \text{(A)}$$

The next step consists in expressing each $t_i$ as functions of $L_j$'s. This can be done with the help of the table given below.

Table 7

| | $t_1$ | $t_2$ | $t_3$ | ...... | $t_i$ | .......... | $t_v$ |
|---|---|---|---|---|---|---|---|
| $L_1$ | $l_{11}$ | $l_{12}$ | $l_{13}$ | | $l_{1i}$ | | $l_{1v}$ |
| $L_2$ | $l_{21}$ | $l_{22}$ | $l_{23}$ | | $l_{2i}$ | | $l_{2v}$ |
| $L_3$ | $l_{31}$ | $l_{32}$ | $l_{33}$ | | $l_{3i}$ | | $l_{3v}$ |
| $L_j$ | $l_{j1}$ | $l_{j2}$ | $l_{j3}$ | | $l_{ji}$ | | $l_{jv}$ |
| $L_{(v-1)}$ | $l_{(v-1)1}$ | $l_{(v-1)2}$ | $l_{(v-1)3}$ | | $l_{(v-1)i}$ | | $l_{(v-1)v}$ |

The $l_{ij}$'s are the constants defining the L contrasts, such that $L_j = \dfrac{\sum_i l_{ji} t_i}{\sum_i l_{ji}^2}$ .

We can now get $t_i$ as functions of $L_j$'s from the table through the relation

$$t_i = l_{1i}L_1 + l_{2i}L_2 + \ldots\ldots\ldots + l_{ji}L_j + \ldots\ldots + l_{(v-1)i}L_{v-1}$$

$$= \sum_{j=1}^{v-1} l_{ji}L_j \qquad \text{(B)}$$

Next substituting $t_i$ as obtained above in equations (A) we shall get (v-1) equations in $L_j$'s.

The expression $\sum_j L_j P_j$ will give the adjusted sum of squares due to treatments having (v-1) degrees of freedom. In order to get a component with p. degrees of freedom due to the first p contrasts $L_j$'s, we make the hypothesis that $L_1 = L_2 = \ldots = L_p = 0$ and solve (v-p-1) equations corresponding to $P_j$, ($j = p+1, \ldots\ldots, (v-1)$) under the assumption that $L_1 = L_2 = L_3 = \ldots\ldots = L_p$ .

The sum of squares $\sum_{j} L_j'P_j$, $(j = p+1, \ldots\ldots, (v-1))$ where $L_j'$ is the solution from the set of equations, will have $(v-p-1)$ degrees of freedom.

The component of sum of squares with the $p$ degrees of freedom can now be obtained from

$$\sum_{j=1}^{v-1} L_jP_j - \sum_{j=p+1}^{v-1} L_j'P_j \qquad \text{(C)}$$

The sum of squares $\sum_{j=1}^{v-1} L_jP_j$ can also be obtained by first obtaining a solution of the $t_i$'s and then by getting $L_j$'s through the relation

$$L_j = \frac{\sum_i l_{ji}t_i}{\sum_i l_{ji}^2} .$$

But the sum of squares $\sum_{j} L_j'P_j$ $(j = p+1, \ldots\ldots, (v-1))$ cannot always be obtained from a solution of the original normal equations. If, however, none of the $(v-p-1)$ equations corresponding to $L_j$ $(j = 1, 2, \ldots\ldots, (v-p-1))$ contains the rest of the $L_j$'s $(j = v-p, v-p+1, \ldots\ldots, (v-1))$, the sum of squares $\sum_{j} L_j'P_j$ can be obtained directly from the solution of the original normal equations in $t_i$'s.

## VIII. AN EXAMPLE

The example given below, illustrates the method of analysis of the generalized staircase design with $x_{kj}=1$.

The generalized staircase design given below has twelve treatments and six blockwise as well as six treatmentwise, steps. The different block-steps in this design have unequal number of blocks. Similarly the different treatment-steps are also unequal in size.

In the following table the generalized staircase design has been presented together with the observations and their frequencies in the different cells. The figure written first in a cell gives the frequency of the observations in the cell and the figures below are the corresponding observations. The observations have actually been taken from a uniformity trial data on Malvi cotton as presented by Panse and Sukhatme (1954).

# TABLE 8.

i ← TREATMENTS

YIELD OF SEED-COTTON PER PLOT ($\frac{1}{2000}$ acre) IN GRAMS, TOGETHER WITH THE FREQUENCIES OF OBSERVATIONS IN THE DIFFERENT CELLS.

| j ↑ BLOCKS | | | step 6: $t_{61}$ | step 6: $t_{62}$ | step 6: $t_{63}$ | step 5: $t_{51}$ | step 5: $t_{52}$ | step 4: $t_{41}$ | step 4: $t_{42}$ | step 3: $t_{31}$ | step 3: $t_{32}$ | step 2: $t_{21}$ | step 1: $t_{11}$ | step 1: $t_{12}$ | BLOCK-SIZE. $N_{kj}$ | BLOCK-TOTAL $B_{kj}$ | BLOCK-AVERAGE $\beta_{kj}=\frac{B_{kj}}{N_{kj}}$ | $P_k=\sum_{j=1}^{r_k}\frac{1}{N_{kj}}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 10. | step 6. | $b_{62}$ | 3 (51,51,83) | 3 (63,22,55) | 3 (39,65,36) | 2 (63,70) | 2 (77,45) | 2 (85,50) | 2 (68,97) | 2 (87,64) | 2 (81,27) | 2 (83,73) | 2 (71,98) | 2 (16,34) | 27 | 1654 | 1654/27 | $P_6 =$ |
| 9. | step 6. | $b_{61}$ | 2 (29,18) | 2 (94,51) | 2 (30,79) | 1 (20) | 1 (69) | 1 (22) | 1 (40) | 1 (98) | 1 (72) | 1 (20) | 1 (56) | 1 (20) | 15 | 718 | 718/15 | $(1/15+1/27$ |
| 8. | step 5. | $b_{53}$ | 1 (11) | 1 (72) | 1 (65) | 1 (71) | 1 (18) | | | | | | | | 5 | 237 | 237/5 | |
| 7. | step 5. | $b_{52}$ | 2 (81,69) | 2 (40,23) | 2 (72,51) | 2 (39,75) | 2 (17,26) | 1 (99) | 1 (76) | 1 (89) | 1 (37) | 1 (20) | 1 (70) | 1 (90) | 17 | 974 | 974/17 | $P_5 = (1/5+1/17+$ |
| 6. | step 5. | $b_{51}$ | 2 (60,73) | 2 (96,53) | 2 (97,86) | 2 (37,48) | 2 (60,82) | 1 (29) | 1 (81) | 1 (30) | 1 (15) | 1 (39) | 1 (46) | 1 (15) | 17 | 947 | 947/17 | |
| 5. | step 4. | $b_{41}$ | 1 (38) | 1 (26) | 1 (61) | 1 (70) | 1 (14) | 1 (68) | 1 (38) | | | | | | 7 | 315 | 315/7 | $P_4 = (1/7)$ |
| 4. | step 3. | $b_{31}$ | 3 (46,30,99) | 3 (15,48,67) | 3 (26,43,18) | 3 (14,23,98) | 3 (61,67,70) | 3 (52,85,11) | 3 (98,35,55) ~~(23,36,67)~~ | 3 (23,36,67) ~~(68,49,38)~~ | 3 (68,49,38) | 2 (96,21) | 2 (44,25) | 2 (27,99) | 33 | 1654 | 1654/33 | $P_3 = (1/33)$ |
| 3. | step 2. | $b_{22}$ | 4 (41,11,53,44) | 4 (10,13,85,57) | 4 (78,37,16,28) | 4 (43,63,61,62) | 4 (42,26,71,95) | 4 (36,79,88,54) | 4 (37,21,34,17) | 4 (68,86,96,83) | 4 (23,83,45,19) | 4 (90,70,99,14) | 3 (29,19,34) | 3 (34,87,83) | 46 | 2364 | 2364/46 | $P_2 = (1/22+1/4$ |
| 2. | step 2. | $b_{21}$ | 2 (27,55) | 2 (49,98) | 2 (65,97) | 2 (38,20) | 2 (46,68) | 2 (43,28) | 2 (16,36) | 2 (49,52) | 2 (83,51) | 2 (39,84) | 1 (51) | 1 (67) | 22 | 1154 | 1154/22 | |
| 1. | step 1. | $b_{11}$ | 3 (11,52,49) | 3 (10,43,67) | 3 (29,70,80) | 3 (62,89,23) | 3 (16,17,17) | 3 (95,70,45) | 3 (80,44,38) | 3 (88,39,54) | 3 (86,97,37) | 3 (44,13,74) | 3 (63,52,52) | 3 (51,41,90) | 36 | 1879 | 1879/36 | |
| | | $R_m$ | 23 | 23 | 23 | 21 | 21 | 18 | 18 | 17 | 17 | 16 | 14 | 14 | 225. | | | |
| | | $T_{mi}$ | 1082 | 1149 | 1268 | 1080 | 1004 | 1039 | 911 | 1109 | 911 | 879 | 710 | 754 | | 11896. | | |

From the above given table, we obtain the following parameters of the design.

| Replications $R_m$ | No. of treatments in the different steps | Sum of reciprocals of block sizes in the different block steps |
|---|---|---|
| $R_1 = 14$ | $S_1 = 2$ | |
| $R_2 = 16$ | $S_2 = 1$ | $P_2 = (1/46+1/22)$ |
| $R_3 = 17$ | $S_3 = 2$ | $P_3 = (1/33)$ |
| $R_4 = 18$ | $S_4 = 2$ | $P_4 = (1/7)$ |
| $R_5 = 21$ | $S_5 = 2\cancel{2}$ | $P_5 = (1/5+1/17+1/17)$ |
| $R_6 = 23$ | $S_6 = 3$ | $P_6 = (1/27+1/15)$ |

For writ~~t~~ing the normal equations the following are to be obtained first:

(i) $Z_{mi} = \dfrac{T_{mi} - \sum_{k=1}^{m}\sum_{j=1}^{n_k}\beta_{kj}}{R_m}$, Giving m and i different values.

$Z_{11} = \dfrac{T_{11} - \beta_{11}}{R_1}$, as there is only one block in the first block step.

$$= \frac{710 - 1879/36}{14} = 46.986111$$

$$Z_{12} = \frac{T_{12} - \beta_{11}}{R_1} = \frac{754 - 1879/36}{14} = 50.128968, \quad \sum_{i=1}^{S_1} Z_{11} = \underline{97.115079}$$

$$Z_{21} = \frac{T_{21} - \beta_{11} - \beta_{21} - \beta_{22}}{R_2}$$

$$= \frac{879 - 1879/36 - 1154/22 - 2364/46}{16}$$

$$= 45.184981, \quad \sum_{i=1}^{S_2} Z_{21} = \underline{45.184981}$$

$$Z_{31} = \frac{T_{31} - \beta_{11} - \beta_{21} - \beta_{22} - \beta_{31}}{R_3}$$

$$= \frac{1109 - 1879/36 - 1154/22 - 2364/46 - 1654/33}{17}$$

$$= 53.108146$$

$$\left.\begin{aligned} Z_{32} &= \frac{T_{32}-\beta_{11}-\beta_{21}-\beta_{22}-\beta_{31}}{R_3} \\ &= \frac{911-1879/36-1154/22-2364/46-1654/33}{17} \\ &= 41.461087 \end{aligned}\right\}, \quad \sum_{i=1}^{S_3} Z_{3i} = \underline{94.569233}$$

Similarly the other $Z_{mi}$'s have been obtained as given below:

$Z_{41} = 43.768805$, $\quad Z_{42} = 36.657690$, $\quad \sum_{i=1}^{S_4} Z_{4i} = \underline{80.426495}$

$Z_{51} = 31.830404$, $\quad Z_{52} = 28.211356$, $\quad \sum_{i=1}^{S_5} Z_{5i} = \underline{60.041760}$

$Z_{61} = 24.404894$, $\quad Z_{62} = 27.317937$, $Z_{63} = 32.491850$, $\quad \sum_{i=1}^{S_6} Z_{6i} = \underline{84.214681}$

(ii) Next we have to calculate various $C^p_m$'s from the following recurrence relation.

$$R_m C^p_m = P_m \underbrace{(S_{m-1}C^p_{m-1} - S_{m-2}C^p_{m-2} + \cdots\cdots + (-1)^{m-p-1}\quad)}_{(m-p)\text{ terms}} - R_{m-1}C^p_{m-1}$$

$p < m.$

As indicated in note 2, article 3 we calculate $C^p_m$'s for various values of 'm', keeping 'p' fixed till $m = q$. Thus

(a) $R_2C^1_2 = P_2$

$C^1_2 = P_2/R_2 = \dfrac{(1/46+1/22)}{16} = \underline{.004200}$

(b) $R_3C^1_3 = P_3(S_2C^1_2-1)-R_2C^1_2$

$= 1/33(1 \times .004200-1) - (1/46+1/22)$

$\therefore C^1_3 = -\underline{.005728}$, where $R_3 = 17$.

(c) $R_4C^1_4 = P_4(S_3C^1_3-S_2C^1_2+1) - R_3C^1_3$

$= 1/7\ (-2 \times .005728 - 1 \times .004200+1) + 17 \times .005728$

$\therefore C^1_4 = \underline{.013221}$

(d) $R_5C^1_5 = P_5(S_4C^1_4-S_3C^1_3+S_2C^1_2 -1) -R_4C^1_4$

$= (1/5 + 2/17)(2\times.013221+2 \times.005728+1\times.004200-1)$

$-18\times.013221$

$\therefore C^1_5 = \underline{-.025822}$

(e) $R_6C^1_6 = P_6(S_5C^1_5 - S_4C^1_4 + S_3C^1_3 - S_2C^1_2 + 1) - R_5C^1_5$

$= (1/27+1/15)(-2x.025822-2x.013221-2x.005728-1x.004200+1)$

$+ .025822 \times 21$

$\therefore C^1_6 = \underline{.027662}$, where $R_6 = 23$

Similarly for other values of p, we obtain all the remaining $C^p_m$'s and represent them in the tabular form:

Table 9

| | $C^1_m$ | $C^2_m$ | $C^3_m$ | $C^4_m$ | $C^5_m$ |
|---|---|---|---|---|---|
| | $C^1_2=.004200$ | $C^2_3=.001783$ | $C^3_4=.007937$ | $C^4_5=.015126$ | $C^5_6=.004509$ |
| $C^p_2$ | $C^1_3=-.005728$ | $C^2_4=-.009592$ | $C^3_5=-.021689$ | $C^4_6=-.018183$ | |
| $C^p_3$ | $C^1_4=.013221$ | $C^2_5=.023003$ | $C^3_6=.024044$ | | |
| $C^p_4$ | $C^1_5=-.025822$ | $C^2_6=-.025202$ | | | |
| $C^p_5$ | $C^1_6=.027662$ | | | | |
| $C^p_6$ | | | | | |

(3) Calculation of S: It is convenient to use formulae (29) and (30), getting the numerator and denominator of S, separately.

Numerator of S $= \sum_{i=1}^{S_1} Z_{1i}(1-S_2C^1_2+S_3C^1_3-S_4C^1_4+S_5C^1_5-S_6C^1_6)$

$+ \sum_{i=1}^{S_2} Z_{2i}(1-S_3C^2_3+S_4C^2_4-S_5C^2_5+S_6C^2_6)$

$+ \sum_{i=1}^{S_3} Z_{3i}(1-S_4C^3_4+S_5C^3_5-S_6C^3_6)$

$+ \sum_{i=1}^{S_4} Z_{4i}(1-S_5C^4_5+S_6C^4_6)$

$+ \sum_{i=1}^{S_5} Z_{5i}(1-S_6C^5_6)$

$+ \sum_{i=1}^{S_6} Z_{6i}(1)$

Substituting the values:

The numerator $= \sum_{i=1}^{S_1} Z_{1i}(1-1 \times .004200 - 2 \times .005728 - 2 \times .013221 - 2 \times .025822 - 3 \times .027662)$.

$+ \sum_{i=1}^{S_2} Z_{2i}(1-2x.001783-2x.009592-2x.023003-3 \times .025202) + \cdots\cdots$

$+ \sum_{i=1}^{S_6} Z_{6i}\ (1)$

$= 97.115079 \times .823272 + 45.184981 \times .855638 + \ldots\ldots$

$+ 84.214681 \times 1.000000$

$= \underline{417.810224}$

Similarly, denominator of S is given by:

$$\text{Denominator of } S = \frac{S_1}{R_1}(1-S_2C^1{}_2+S_3C^1{}_3-S_4C^1{}_4+S_5C^1{}_5-S_6C^1{}_6)$$
$$+ \frac{S_2}{R_2}(1-S_3C^2{}_3+S_4C^2{}_4-S_5C^2{}_5+S_6C^2{}_6)$$
$$+ \frac{S_3}{R_3}(1-S_4C^3{}_4+S_5C^3{}_5-S_6C^3{}_6)$$
$$+ \frac{S_4}{R_4}(1-S_5C^4{}_5+S_6C^4{}_6)$$
$$+ \frac{S_5}{R_5}(1-S_6C^5{}_6)$$
$$+ \frac{S_6}{R_6}(1)$$

Denominator of $S = 2/14 \times .823272 + 1/16 \times .855638 + \ldots\ldots$

$+ 3/23 \times 1.000000$

$= \underline{.599351}$

$$\text{Hence, } S = \frac{417.810224}{.599351} = \underline{\underline{697.104408}}$$

(iv) Calculation of $Z_{mi}$'s and their totals over the different steps can now be affected easily with the help of formula:

$$Z_{mi}' = Z_{mi} - \frac{S}{R_m}, \quad \left.\begin{array}{l} i = 1 \text{ to } S_m \\ m = 1 \text{ to } 6 \end{array}\right\}$$

$$\left.\begin{array}{l} Z_{11}' = 46.986111 - \dfrac{697.104408}{14} = -2.807061 \\ Z_{12}' = 50.128968 - \dfrac{697.104408}{14} = 0.335796 \end{array}\right\}, \quad \sum_{i=1}^{S_1} Z_{1i}' = \underline{-2.471265}$$

Similarly

$Z_{21}' = 1.615956 \}$, $\qquad \sum_{i=1}^{S_2} Z_{2i}' = \underline{1.615956}$

$Z_{31}' = 12.102005, \quad Z_{32}' = .454946 \}$, $\qquad \sum_{i=1}^{S_3} Z_{3i}' = \underline{12.556951}$

$Z_{41}' = 5.040783$, $Z_{42}' = -2.070332\}$, $\sum_{i=1}^{S_4} Z_{4i}' = \underline{2.970451}$

$Z_{51}' = -1.365044$, $Z_{52}' = -4.984092\}$, $\sum_{i=1}^{S_5} Z_{5i}' = \underline{-6.349136}$

$Z_{61}' = -5.903993$, $Z_{62}' = -2.990950$, $Z_{63}' = 2.182963\}$,

$\sum_{i=1}^{S_6} Z_{6i}' = \underline{-6.711980}$

(v) Calculation of the treatment effect $t_{mi}$, where

$$t_{mi} = Z_{mi}' - C_m^{m-1} \sum_{i=1}^{S_{m-1}} Z_{(m-1)i}' + \ldots\ldots\ldots + (-1)^{m-1} C^1{}_m \sum_{i=1}^{S_1} Z_{1i}'$$

Substituting the required values the estimates are:

$t_{11} = Z_{11}' = \underline{-2.807061}$

$t_{12} = Z_{12}' = \underline{0.335796}$

$t_{21} = Z_{21}' - C^1{}_2 \sum_{i=1}^{S_1} Z_{1i}'$

$= 1.615956 + .004200 \times 2.471265$

$= \underline{1.626335}$

$t_{31} = Z_{31}' - C^2{}_3 \sum_{i=1}^{S_2} Z_{2i}' + C^1{}_3 \sum_{i=1}^{S_1} Z_{1i}'$

$= 12.102005 - .001783 \times 1.615956 + .005728 \times 2.471265$

$= \underline{12.113279}$

$t_{32} = Z_{32}' - C^2{}_3 \sum_{i=1}^{S_2} Z_{2i}' + C^1{}_3 \sum_{i=1}^{S_1} Z_{1i}'$

$= .454946 - .001783 \times 1.615956 + .005728 \times 2.471265$

$= \underline{.466220}$

$t_{41} = Z_{41}' - C^3{}_4 \sum_{i=1}^{S_3} Z_{3i}' + C^2{}_4 \sum_{i=1}^{S_2} Z_{2i}' - C^1{}_4 \sum_{i=1}^{S_1} Z_{1i}'$

$= 5.040783 - .007937 \times 12.556951 - .009592 \times 1.615956$
$+ .013221 \times 2.471265$

$= \underline{4.958291}$

$t_{42} = Z_{42}' - C^3{}_4 \sum_{i=1}^{S_3} Z_{3i}' + C^2{}_4 \sum_{i=1}^{S_2} Z_{2i}' - C^1{}_4 \sum_{i=1}^{S_1} Z_{1i}'$

$= -2.070332 - .007937 \times 12.556951 - .009592 \times 1.615956$
$+ .013221 \times 2.471265$

$= \underline{-2.152824}$

$$t_{51} = Z_{51}' - C^4_5 \sum_{i=1}^{S_4} Z_{4i}' + C^3_5 \sum_{i=1}^{S_3} Z_{3i}' - C^2_5 \sum_{i=1}^{S_2} Z_{2i}' + C^1_5 \sum_{i=1}^{S_1} Z_{1i}'$$

$= -1.365044 - .015126 \times 2.970451 - .021689 \times 12.556951$

$- .023003 \times 1.615956 + .025822 \times 2.471265$

$= \underline{-1.655682}$

$$t_{52} = Z_{52}' - C^4_5 \sum_{i=1}^{S_4} Z_{4i}' + C^3_5 \sum_{i=1}^{S_3} Z_{3i}' - C^2_5 \sum_{i=1}^{S_2} Z_{2i}' + C^1_5 \sum_{i=1}^{S_1} Z_{1i}'$$

$= -4.984092 - .015126 \times 2.970451 - .021689 \times 12.556951$

$- .023003 \times 1.615956 + .025822 \times 2.471265$

$= \underline{-5.274730}$

$$t_{61} = Z_{61}' - C^5_6 \sum_{i=1}^{S_5} Z_{5i}' + C^4_6 \sum_{i=1}^{S_4} Z_{4i}' - C^3_6 \sum_{i=1}^{S_3} Z_{3i}' + C^2_6 \sum_{i=1}^{S_2} Z_{2i}'$$

$$- C^1_6 \sum_{i=1}^{S_1} Z_{1i}'$$

$= -5.903993 + .004509 \times 6.349136 - .018183 \times 2.970451$

$- .024044 \times 12.556951 - .025202 \times 1.615956$

$+ .027661 \times 2.471265$

$= \underline{-6.203661}$

$$t_{62} = Z_{62}' - C^5_6 \sum_{i=1}^{S_5} Z_{5i}' + C^4_6 \sum_{i=1}^{S_4} Z_{4i}' - C^3_6 \sum_{i=1}^{S_3} Z_{3i}' + C^2_6 \sum_{i=1}^{S_2} Z_{2i}'$$

$$- C^1_6 \sum_{i=1}^{S_1} Z_{1i}'$$

$= -2.990950 + .004509 \times 6.349136 - .018183 \times 2.970451$

$- .024044 \times 12.556951 - .025202 \times 1.615956$

$+ .027662 \times 2.471265$

$= \underline{-3.290618}$

$$t_{63} = Z_{63}' - C^5_6 \sum_{i=1}^{S_5} Z_{5i}' + C^4_6 \sum_{i=1}^{S_4} Z_{4i}' - C^3_6 \sum_{i=1}^{S_3} Z_{3i}' + C^2_6 \sum_{i=1}^{S_2} Z_{2i}'$$

$$- C^1_6 \sum_{i=1}^{S_1} Z_{1i}'$$

$= 2.182963 + .004509 \times 6.349136 - .018183 \times 2.970451$

$- .024044 \times 12.556951 - .025202 \times 1.615956$

$+ .027662 \times 2.471265$

$= \underline{1.883295}$

As $\sum_{m=1}^{6} \sum_{i=1}^{S_m} t_{mi} = .001360$, the check is satisfied.

(vi) Calculation of $Q_{mi}$'s, the usual adjusted totals from

$$Q_{mi} = T_{mi} - \sum_{k=1}^{v}\sum_{j=1}^{r_k} n_{mi(kj)} \cdot \beta_{kj}$$

Substituting the different values

$$Q_{11} = 710 - \frac{2 \times 1654}{27} - \frac{718 \times 1}{15} - \frac{237 \times 0}{5} - \frac{974 \times 1}{17}$$
$$- \frac{947 \times 1}{17} - \frac{315 \times 0}{7} - \frac{1654 \times 2}{33} - \frac{3 \times 2364}{46} - \frac{1154 \times 1}{22}$$
$$- \frac{1879 \times 3}{36}$$
$$= -36.83940$$

Similarly,

$Q_{12} = 7.16060$

$Q_{21} = 28.31476$

$Q_{31} = 208.19355$, $Q_{32} = 10.19355$

$Q_{41} = 93.19355$ , $Q_{42} = -34.80645$

$Q_{51} = -26.20645$, $Q_{52} = -102.20645$

$Q_{61} = -133.33237$, $Q_{62} = -66.33237$, $Q_{63} = 52.66763$

As $\sum_{m=1}^{6}\sum_{i=1}^{s_m} Q_{mi} = .00015$, the check is satisfied.

(vii) Hence the adjusted sum of squares for 11 degrees of freedom is given by:

$$\sum_{m=1}^{6}\sum_{i=1}^{s_m} t_{mi}Q_{mi}$$

$= -2.807061 \times -36.83940 + .335796 \times 7.16060 + 1.626335 \times 28.31476 + \ldots\ldots\ldots + 1.883295 \times 52.66763$

$= 4942.657044$ on 11 degrees of freedom.

(viii) Analysis of variance: from table 8, we obtain:

(a) Unadjusted block sum of squares (B) $= \sum_{k=1}^{v}\sum_{j=1}^{g_{k}} \frac{B_{kj}^2}{N_{kj}}$

$= 3698.5195$

(b) Adjusted treatment sum of squares(A) $= \sum_{m=1}^{6}\sum_{i=1}^{s_m} t_{mi}Q_{mi}$

$= 4942.657044$

(c) Total sum of squares $= \sum_{m=1,\,k=1}^{6} \sum_{i=1}^{s_m} \sum_{j=1}^{n_k} y_{ij}^2$

$$= 51^2 + 51^2 + \ldots\ldots + 90^2 - C.F.$$

$$= \underline{150175.2622}$$

(d) $P_{bt}$ = Sum of squares due to cell - totals

$$= \frac{185^2}{3} + \frac{140^2}{3} + \frac{140^2}{3} + \frac{122^2}{2} + \ldots\ldots + \frac{183^2}{3} - C.F.$$

$$= \underline{78915.4289}$$

(e) Hence interaction sum of squares(I) $= P_{bt} - (A) - (B)$

$$= 78915.4289 - 4942.6571 - 3698.5195$$

$$= \underline{70274.2523}$$

(f) and the error sum of squares = 150175.2622–70274.2523–4942.6571 – 3698.5195

$$= \underline{71259.8333}$$

$$= \underline{\underline{\text{Total S.S.} - P_{bt}}}$$

Representing these in a tabular form:

Table 10

Analysis of variance table

| Source of variation | d.f. | Sum of squares | Mean squares | F-ratio |
|---|---|---|---|---|
| Blocks unadjusted | 9 | 3698.5195 | | |
| Treatments adjusted | 11 | 4942.6571 | 449.3325 | 0.6621 |
| Interaction | 99 | 70274.2523 | 709.8409 | 1.0459 |
| Error | 105 | 71259.8333 | 678.6651 $= \sigma^2$ | |
| Total | 224 | 150175.2622 | | |

(ix) Variance of treatment differences:

(a) When two treatments lie within the same step the variance of their difference is $\frac{2\,\sigma^2}{R_m}$, calculation of which presents no difficulty.

(b) But when two treatments lie in different steps, we can get it from (39). This has been applied to obtain $V(t_{41}-t_{61})$ and $V(t_{21}-t_{51})$, as shown below:

(I) Putting $m = 4$, $m+g = 6$, $g = 2$, $q = 6$ in (39), and expanding we have

$$V(t_{41}-t_{61}) = \left( \frac{1-C^4{}_6}{R_4} + \frac{1}{R_6} \right) \sigma^2$$
$$+ \frac{\sigma^2}{\Delta} \left\{ \left( \frac{1}{R_6} - \frac{1-C^4{}_6}{R_4} \right) - \frac{C^5{}_6}{R_5} + \frac{C^3{}_4-C^3{}_6}{R_3} + \frac{C^2{}_6-C^2{}_4}{R_2} \right.$$
$$\left. + \frac{C^1{}_4-C^1{}_6}{R_1} \right\} \times \left\{ \frac{1-S_5C^4{}_5 + S_6C^4{}_6}{R_4} - \frac{1}{R_6} \right\}$$, where $\Delta$ is the denominator of S.

Substituting the values we have:

$$= \sigma^2 \left( \frac{1+.018183}{18} + \frac{1}{23} \right) + \frac{\sigma^2}{.599351} \left\{ \left( \frac{1}{23} - \frac{1+.018183}{18} \right) \right.$$
$$- \frac{.004509}{21} + \frac{.007937-.024044}{17} + \frac{-.025202 + .009592}{16}$$
$$\left. + \frac{.013221-.027662}{14} \right\} \times \left\{ \frac{1-2 \times .015126-3 \times .018183}{18} - \frac{1}{23} \right\}$$

$$= \sigma^2 \left( \frac{41.418209}{18 \times 23} \right) - \sigma^2 (1.668471)\left( \frac{5.418209}{23x18} + \frac{.004509}{21} \right.$$
$$\left. + \frac{.016107}{17} + \frac{.015610}{16} + \frac{.014441}{14} \right) \times \left( \frac{3.049577}{18 \times 23} \right)$$

$$= \frac{41.418209}{18 \times 23} \sigma^2 - (1.668471)\sigma^2 \times \left( \frac{6407.248730}{23x18 \times 7 \times 17x8} \right) \times \left( \frac{3.049577}{18 \times 23} \right)$$

$$= \frac{41.418209}{18 \times 23} \sigma^2 - \frac{5.088131 \times 6407.248730}{18x23x18x23x7x17x8} \sigma^2$$

$$= \frac{16291471.41512}{163168992} \sigma^2$$

$$V(t_{41}-t_{61}) = \underline{\underline{.099844}} \; \sigma^2$$

(II) Similarly when m = 2, m+g = 5, g = 3, q = 6

$$V(t_{21}-t_{51}) = \left(\frac{1+C^2{}_5}{R_2} + \frac{1}{R_5}\right)\sigma^2 + \frac{\sigma^2}{\Delta}\left\{\left(\frac{1}{R_5} - \frac{1+C^2{}_5}{R_2}\right) - \frac{C^4{}_5}{R_4} + \frac{C^3{}_5}{R_3} + \frac{C^1{}_5+C^1{}_2}{R_1}\right\} \times \left\{\frac{1-S_3C^2{}_3+S_4C^2{}_4-S_5C^2{}_5+S_6C^2{}_6}{R_2} - \frac{1-S_6C^5{}_6}{R_5}\right\}$$

$$= \left(\frac{1+.023003}{16} + \frac{1}{21}\right)\sigma^2 + \frac{\sigma^2}{.599351}\left\{\left(\frac{1}{21} - \frac{1.023003}{16}\right) - \frac{.015126}{18} + \frac{.021689}{17} + \frac{.004200-.025822}{14}\right\} \times \left\{\frac{1-2x.001783-2x.009592-2x.023003-3x.025202}{16} - \frac{1+3\ x\ .004509}{21}\right\}$$

$$V(t_{21}-t_{51}) \simeq \frac{641061.741195\sigma^2}{5757696} = \underline{\underline{.111340\ \ \sigma^2}}$$

## IX. PARTITIONING THE ADJUSTED SUM OF SQUARES - an example

(The same data as indicated in table 8 were used)

(1) For partitioning the sum of squares we have first to define the orthogonal contrasts. These contrasts can be defined in an infinite number of ways. Taking the contrasts to represent between and within steps comparisons, we have:

$$L_1 = \frac{t_{61}-t_{62}}{2} \qquad L_7 = \frac{(t_{61}+t_{62}+t_{63}) - 3/2(t_{51}+t_{52})}{15/2}$$

$$L_2 = \frac{t_{61}+t_{62}-2t_{63}}{6} \qquad L_8 = \frac{(t_{61}+t_{62}+t_{63}+t_{51}+t_{52})-5/2(t_{41}+t_{42})}{35/2}$$

$$L_3 = \frac{t_{51}-t_{52}}{2} \qquad L_9 = \frac{(t_{61}+\dots\dots+t_{42})-7/2(t_{31}+t_{32})}{63/2}$$

$$L_4 = \frac{t_{41}-t_{42}}{2} \qquad L_{10} = \frac{(t_{61}+\dots\dots+t_{32})-9(t_{21})}{90}$$

$$L_5 = \frac{t_{31}-t_{32}}{2} \qquad L_{11} = \frac{(t_{61}+\dots\dots+t_{21})+10/2(t_{11}+t_{12})}{60}$$

$$L_6 = \frac{t_{11}-t_{12}}{2}$$

(2) Next step is to form twelve normal equations as required for the analysis of variance of non-orthogonal data in two-way classification (Kempthorne, 1952, article 6.3, page 80). Using table 8, the normal equation corresponding to the first treatment is:

$$\begin{aligned}
&[23-(9/27+4/15+1/5+4/17+4/17+1/7+9/33+16/46+4/22+9/36)]\,(t_{61})\\
&-(9/27+4/15+1/5+4/17+4/17+1/7+9/33+16/46+4/22+9/36)(t_{62}+t_{63})\\
&-(6/27+2/15+1/5+4/17+4/17+1/7+9/33+16/46+4/22+9/36)(t_{51}+t_{52})\\
&-(6/27+2/15+0/5+2/17+2/17+1/7+9/33+16/46+4/22+9/36)(t_{41}+t_{42})\\
&-(6/27+2/15+0/5+2/17+2/17+0/7+9/33+16/46+4/22+9/36)(t_{31}+t_{32})\\
&-(6/27+2/15+0/5+2/17+2/17+0/7+6/33+16/46+4/22+9/36)(t_{21})\\
&-(6/27+2/15+0/5+2/17+2/17+0/7+6/33+12/46+2/22+9/36)(t_{11}+t_{12}) = Q_{61}
\end{aligned}$$

The other equations can be obtained similarly. These equations have been shown in a tabular but reduced form below:

(3) With the help of the following table we can get the

TAB LE 11.

LE CONTAINS COEFFICIENTS OF VARIOUS $t_{mi}$'s,
FOR EVERY FIGURE BEING [16257780].

Page no. 44

| $t_{41}$ | | $t_{42}$ | $t_{31}$ | $t_{32}$ | $t_{21}$ | $t_{11}$ | $t_{12}$ | $\Omega_{mi}$'s calculated in preceding example. |
|---|---|---|---|---|---|---|---|---|
| -29037669 | | -29037669 | -26715129 | -26715129 | -25237149 | -22345449 | -22345449 | -133.3324 |
| -29037669 | | -29037669 | -26715129 | -26715129 | -25237149 | -22345449 | -22345449 | -66.3324 |
| -29037669 | | -29037669 | -26715129 | -26715129 | -25237149 | -22345449 | -22345449 | 52.6676 |
| -87113007 | | -87113007 | -80145387 | -80145387 | -75711447 | -67036347 | -67036347 | |
| -26749537 | | -26749537 | -24426997 | -24426997 | -22949017 | -20057317 | -20057317 | -26.2065 |
| -26749537 | | -26749537 | -24426997 | -24426997 | -22949017 | -20057317 | -20057317 | -102.2065 |
| -53499074 | | -53499074 | -48853994 | -48853994 | -45898034 | -40114634 | -40114634 | |
| +267803183 | | -24836857 | -22514317 | -22514317 | -21036337 | -18144637 | -18144637 | 93.1936 |
| 24836857 | | +267803183 | -22514317 | -22514317 | -21036337 | -18144637 | -18144637 | -34.8065 |
| 242966326 | | +242966326 | -45028634 | -45028634 | -42072674 | -36289274 | -36289274 | |
| 22514317 | | -22514317 | +253867943 | -22514317 | -21036337 | -18144637 | -18144637 | 208.1936 |
| 22514317 | | -22514317 | -22514317 | +253867943 | -21036337 | -18144637 | -18144637 | 10.1936 |
| 45028634 | | -45028634 | +231353626 | +231353626 | -42072674 | -36289274 | -36289274 | |
| 21036337 | | -21036337 | -21036337 | -21036337 | +240073463 | -17159317 | -17159317 | 28.3148 |
| 21036337 | | -21036337 | -21036337 | -21036337 | +240073463 | -17159317 | -17159317 | |
| 144637 | | -18144637 | -18144637 | -18144637 | -17159317 | +212248883 | -15360037 | -36.8394 |
| 8144637 | | -18144637 | -18144637 | -18144637 | -17159317 | -15360037 | +212248883 | 7.1606 |
| 6289274 | | -36289274 | -36289274 | -36289274 | -34318634 | +196888846 | +196888846 | |

16257780

$\sum \Omega_{mi} = 0$

$$t_{21} = -9L_{10}+L_{11}$$
$$t_{11} = L_6-5L_{11}$$
$$t_{12} = -L_6-5L_{11}$$

(3) With the help of

Page no. 44

# TABLE 11.

BODY OF THE TABLE CONTAINS → COEFFICIENTS OF VARIOUS $t_{mi}$'s,
OVERALL DIVISOR FOR EVERY → FIGURE BEING [16257780].

| $t_{mi}$'s → / $Q_{mi}$'s ↓ | $t_{61}$ | $t_{62}$ | $t_{63}$ | $t_{51}$ | $t_{52}$ | $t_{41}$ | $t_{42}$ | $t_{31}$ | $t_{32}$ |
|---|---|---|---|---|---|---|---|---|---|
| $Q_{61}$ | +333840231 | −40088709 | −40088709 | −36114585 | −36114585 | −29037669 | −29037669 | −26715129 | −2671512 |
| $Q_{62}$ | −40088709 | +333840231 | −40088709 | −36114585 | −36114585 | −29037669 | −29037669 | −26715129 | −2671517 |
| $Q_{63}$ | −40088709 | −40088709 | +333840231 | −36114585 | −36114585 | −29037669 | −29037669 | −26715129 | −267151 |
| $\sum_{i=1}^{S_6} Q_{6i}$ | +253662813 | +253662813 | +253662813 | −108343755 | −108343755 | −87113007 | −87113007 | −80145387 | −8014538 |
| $Q_{51}$ | −36114585 | −36114585 | −36114585 | +307586927 | −33826453 | −26749537 | −26749537 | −24426997 | −244269 |
| $Q_{52}$ | −36114585 | −36114585 | −36114585 | −33826453 | +307586927 | −26749537 | −26749537 | −24426997 | −24426 |
| $\sum_{i=1}^{S_5} Q_{5i}$ | −72229170 | −72229170 | −72229170 | +273760474 | +273760474 | −53499074 | −53499074 | −48853994 | −488539 |
| $Q_{41}$ | −29037669 | −29037669 | −29037669 | −26749537 | −26749537 | +267803183 | −24836857 | −22514317 | −2251431 |
| $Q_{42}$ | −29037669 | −29037669 | −29037669 | −26749537 | −26749537 | −24036857 | +267803183 | −22514317 | −2251430 |
| $\sum_{i=1}^{S_4} Q_{4i}$ | −58075338 | −58075338 | −58075338 | −53499074 | −53499074 | +242966326 | +242966326 | −45028634 | −4502863 |
| $Q_{31}$ | −26715129 | −26715129 | −26715129 | −24426997 | −24426997 | −22514317 | −22514317 | +253867943 | −225143 |
| $Q_{32}$ | −26715129 | −26715129 | −26715129 | −24426997 | −24426997 | −22514317 | −22514317 | +22514317 | +253867 |
| $\sum_{i=1}^{S_3} Q_{3i}$ | −53430258 | −53430258 | −53430258 | −48853994 | −48853994 | −45028634 | −45028634 | +231353626 | +231353 |
| $Q_{21}$ | −25237149 | −25237149 | −25237149 | −22949017 | −22949017 | −21036337 | −21036337 | −21036337 | −2103633 |
| $\sum_{i=1}^{S_2} Q_{2i}$ | −25237149 | −25237149 | −25237149 | −22949017 | −22949017 | −21036337 | −21036337 | −21036337 | −2103633 |
| $Q_{11}$ | −22345449 | −22345449 | −22345449 | −20057317 | −20057317 | −18144637 | −18144637 | −18144637 | −1814463 |
| $Q_{12}$ | −22345449 | −22345449 | −22345449 | −20057317 | −20057317 | −18144637 | −18144637 | −18144637 | −181446 |
| $\sum_{i=1}^{S_1} Q_{1i}$ | −44690898 | −44690898 | −44690898 | −40114634 | −40114634 | −36289274 | −36289274 | −36289274 | −3628927 |

$$t_{21} = -9L_{10}+L_{11}$$
$$t_{11} = L_6-5L_{11}$$
$$t_{12} = -L_6-5L_{11}$$

(3) With the help of the following table we can get the solution of $t_{mi}$'s as functions of $L_j$'s.

Table 12

| | $t_{61}$ | $t_{62}$ | $t_{63}$ | $t_{51}$ | $t_{52}$ | $t_{41}$ | $t_{42}$ | $t_{31}$ | $t_{32}$ | $t_{21}$ | $t_{11}$ | $t_{12}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $L_1$ | 1 | -1 | | | | | | | | | | |
| $L_2$ | 1 | 1 | -2 | | | | | | | | | |
| $L_3$ | | | | 1 | -1 | | | | | | | |
| $L_4$ | | | | | | 1 | -1 | | | | | |
| $L_5$ | | | | | | | | 1 | -1 | | | |
| $L_6$ | | | | | | | | | | | 1 | -1 |
| $L_7$ | 1 | 1 | 1 | -3/2 | -3/2 | | | | | | | |
| $L_8$ | 1 | 1 | 1 | 1 | 1 | -5/2 | -5/2 | | | | | |
| $L_9$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | -7/2 | -7/2 | | | |
| $L_{10}$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | -9 | | |
| $L_{11}$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | -5 | -5 |

Thus

$$t_{61} = L_1+L_2+(L_7+L_8+L_9+L_{10}+L_{11})$$

$$t_{62} = -L_1+L_2+(L_7+L_8+L_9+L_{10}+L_{11})$$

$$t_{63} = -2L_2+(L_7+L_8+L_9+L_{10}+L_{11})$$

$$t_{51} = L_3-3/2L_7+(L_8+L_9+L_{10}+L_{11})$$

$$t_{52} = -L_3-3/2L_7+(L_8+L_9+L_{10}+L_{11})$$

$$t_{41} = L_4-5/2L_8+(L_9+L_{10}+L_{11})$$

$$t_{42} = -L_4-5/2L_8+(L_9+L_{10}+L_{11})$$

$$t_{31} = L_5-7/2L_9+(L_{10}+L_{11})$$

$$t_{32} = -L_5-7/2L_9+(L_{10}+L_{11})$$

$$t_{21} = -9L_{10}+L_{11}$$

$$t_{11} = L_6-5L_{11}$$

$$t_{12} = -L_6-5L_{11}$$

(4) The equations in terms of $L_j$'s can now be obtained as below:

The linear contrasts of $Q_{mi}$'s corresponding to those of $\bar{t}_{mi}$'s in $L_j$'s are

(i) $(Q_{61}-Q_{62}) = P_1$

(ii) $(Q_{61}+Q_{62}-2Q_{63}) = P_2$

(iii) $(Q_{51}-Q_{52}) = P_3$

(iv) $(Q_{41}-Q_{42}) = P_4$

(v) $(Q_{31}-Q_{32}) = P_5$

(vi) $(Q_{11}-Q_{12}) = P_6$

(vii) $((Q_{61}+Q_{62}+Q_{63})-3/2(Q_{51}+Q_{52})) = P_7$

(viii) $((Q_{61}+Q_{62}+Q_{63}+Q_{51}+Q_{52}) - 5/2(Q_{41}+Q_{42}) = P_8$

(ix) $((Q_{61}+\cdots\cdots\cdots+Q_{42})- 7/2(Q_{31}+Q_{32}) = P_9$

(x) $((Q_{61}+ \cdots\cdots\cdots+Q_{32}) - 9(Q_{21}) = P_{10}$

(xi) $((Q_{61}+ \cdots\cdots\cdots+Q_{21}) - 5(Q_{11}+Q_{12}) = P_{11}$

The equations corresponding to the above 11 contrasts have been written below, after referring to table 11 of the normal equations.

(i) $\dfrac{333840231+40088709}{16257780}(t_{61}-t_{62}) = (Q_{61}-Q_{62})$

or $23(t_{61}-t_{62}) = (Q_{61}-Q_{62})$

or $46L_1 = (Q_{61}-Q_{62})$

as $(t_{61}-t_{62}) = 2L_1$, from table 12.

(ii) $\dfrac{333840231-40088709+2 \times 40088709}{16257780}(t_{61}+t_{62}-2t_{63})$

$= (Q_{61}+Q_{62}-2Q_{63})$

or $23\ (t_{61}+t_{62})-2t_{63}) = (Q_{61}+Q_{62}-2Q_{63})$

or $138L_2 = (Q_{61}+Q_{62}-2Q_{63})$

as $(t_{61}+t_{62}-2t_{63}) = 6L_2$, from table 12.

Proceeding in this way we obtain the following other normal equations:

(iii) $42L_3 = (Q_{51}-Q_{52})$

(iv) $36L_4 = (Q_{41}-Q_{42})$

(v) $34L_5 = (Q_{31}-Q_{32})$

(vi) $28L_6 = (Q_{11}-Q_{12})$

*Similarly* the remaining normal equations corresponding to between step comparisons have been obtained as shown below:

(vii) We have from table 11

$$16257780 \times (Q_{61}+Q_{62}+Q_{63}) = 253662813(t_{61}+t_{62}+t_{63})-108343755(t_{51}+t_{52}) -87113007(t_{41}+t_{42})- 80145387(t_{31}+t_{32}) -75711447(t_{21})-67036347(t_{11}+t_{12})$$

and

$$16257780 \times 3/2\ (Q_{51}+Q_{52}) = 3/2(-72229170(t_{61}+t_{62}+t_{63}) + 273760474(t_{51}+t_{52})-53499074\ (t_{41}+t_{42}) -48853994(t_{31}+t_{32})-45898034(t_{21}) - 40114634(t_{11}+t_{12}))$$

Hence

$$(Q_{61}+Q_{62}+Q_{63}) - 3/2\ (Q_{51}+Q_{52}) = 362006568(t_{61}+t_{62}+t_{63}) - 518984466(t_{51}+t_{52})-6864396(t_{41}+t_{42}+t_{31}+t_{32}+t_{21}+t_{11}+t_{12}))\ /\ 16257780$$

Since $\sum_{m=1}^{6}\sum_{i=1}^{s_m} t_{mi} = 0$

$$= \left[362006568(t_{61}+t_{62}+t_{63})-518984466(t_{51}+t_{52})+6864396(t_{61}+t_{62}+t_{63}+t_{51}+t_{52})\right]\ /\ 16257780$$

$$= \left[368870964(t_{61}+t_{62}+t_{63})-512120070(t_{51}+t_{52})\right]\ /16257780$$

Now substituting for $t_{mi}$'s from table 12 we get

$$\frac{368870964}{16257780}\left[3(L_7+\ldots\ldots+L_{11})\right]-\frac{512120070}{16257780}\left[-3L_7+2(L_8+\ldots\ldots+L_{11})\right]$$

$$=(Q_{61}+Q_{62}+Q_{63})-3/2(Q_{51}+Q_{52})$$

(vii) i.e., $162.5667L_7+5.0667(L_8+L_9+L_{10}+L_{11})$

$$=(Q_{61}+Q_{62}+Q_{63})-3/2(Q_{51}+Q_{52})$$

Similarly the other equations have been obtained as

(viii) $5.0667L_7+327.1255L_8+12.1255(L_9+L_{10}+L_{11})$

$$=(Q_{61}+Q_{62}+Q_{63}+Q_{51}+Q_{52})-5/2\ (Q_{41}+Q_{42})$$

(ix) $5.0667L_7+12.1255L_8+547.6255L_9+12.1255(L_{10}+L_{11})$

$$=(Q_{61}+\ldots\ldots\ldots\ldots+Q_{42})-7/2\ (Q_{31}+Q_{32})$$

(x) $5.0667L_7+12.1255(L_8+L_9)+1458.6710L_{10}+18.6710L_{11}$

$$=(Q_{61}+\ldots\ldots\ldots\ldots\ldots+Q_{32})-9\ (Q_{21})$$

(xi) $5.0667L_7+12.1255(L_8+L_9)+18.6710L_{10}+871.9516L_{11}$

$$=(Q_{61}+\ldots\ldots\ldots\ldots\ldots+Q_{21})-5\ (Q_{11}+Q_{12})$$

Equations (i) to (vi) correspond to within step comparisons. Here each $L_j$, $j=1$ to 6 is obtainable directly, since each equation involves only one $L_j$.

Thus solving these equations we have

| | |
|---|---|
| $L_1=\frac{-67}{46}=-1.456522$ | $L_4=128/36=3.555556$ |
| $L_2=-305/138=-2.210145$ | $L_5=198/34=5.823529$ |
| $L_3=75/42=1.785714$ | $L_6=-44/28=-1.571428$ |

Equations (vii) to (xi) comprise the second set of equations and correspond to between-step comparisons. Solution of these normal equations is not straight forward. These equations have been solved through the iterative method. The solutions are as given below:

$L_7 = 0.371280$ | $L_{10} = -.153256$
$L_8 = -1.231724$ | $L_{11} = 0.247105$
$L_9 = -1.770291$

Now the adjusted treatment sum of squares for 11 degrees of freedom is given by:

$$\sum_{j=1}^{11} L_j P_j = \sum_{j=1}^{6} L_j P_j + \sum_{j=7}^{11} L_j P_j$$

$$= \left( \frac{67^2}{46} + \frac{305^2}{138} + \ldots\ldots + \frac{44^2}{28} \right) +$$

$$\left\{ (.371280) \times 45.6231 + \ldots + .247105 \times 178.0830 \right\}$$

$$= 2586.5272 + 2356.1302$$

$$= \underline{\underline{4942.6574}} \text{ on 11 degrees of freedom}$$

It will be seen that the sum of squares agrees exactly with that obtained earlier while illustrating the generalized staircase design.

In order to get a partition component of the sum squares to correspond to between step sum of squares, we make the hypothesis that all the step means are equal which is the same as putting $L_7 = L_8 = L_9 = L_{10} = L_{11} = 0$. On this hypothesis the normal equations become the first six equations corresponding to $L_1$ to $L_6$ as shown earlier. As these equations do not involve any $L_j$'s ($j$ = 7 to 11), the solution of these remains the same as before. Hence the sum of squares due to these $L_j$'s ($j$ = 1 to 6) on the hypothesis $L_7 = L_8 = \ldots\ldots = L_{11} = 0$ is

$$\sum_{j=1}^{6} L_j P_j = \underline{2586.5272}$$

So the adjusted sum of squares due to the between step contrasts with 5 degrees of freedom is

$$\sum_{j=1}^{11} L_j P_j - \sum_{j=1}^{6} L_j P_j = \sum_{j=7}^{11} L_j P_j$$

$4942.6571 - 2586.5272 = 2356.1299$ on 5 degrees of freedom.

Similarly the component of within step sum of squares with 6 degrees of freedom comes out as

$$4942.6571 - 2356.1299 = 2586.5272$$

It will be seen that this method of finding the adjusted treatment sum of squares provides an alternative method of analysing non-orthogonal data. We have seen that if the data come from a generalized staircase design, the analysis becomes very much simplified if the contrasts are taken so as to represent the between and within step comparisons. If the contrasts are formed in any other way, the analysis may be complicated, as in that case the equations in $L_j$'s will be more involved. This has been illustrated below.

There are nine treatments giving rise to five steps in the case of generalized design with $x_{kj} = 1$, as shown below:

| | $F_4$ | | | $F_3$ | | $F_2$ | | $F_1$ |
|---|---|---|---|---|---|---|---|---|
| $t_{51}$ | $t_{52}$ | $t_{41}$ | $t_{31}$ | $t_{32}$ | $t_{33}$ | $t_{21}$ | $t_{11}$ | $t_{12}$ |
| 3 | 3 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| 2 | 2 | 2 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 |
| 4 | 4 | 4 | 4 | 4 | 4 | 4 | 3 | 3 |
| 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |

If now the contrasts are formed as below:

$$L_1 = \frac{t_{51}-t_{52}}{2}, \qquad L_2 = \frac{t_{51}+t_{52}-2t_{41}}{6}, \qquad L_3 = \frac{t_{31}-t_{32}}{2}$$

$$L_4 = \frac{t_{31}+t_{32}-2t_{33}}{6}, \qquad L_5 = \frac{t_{21}-t_{11}}{2}$$

$$L_6 = \frac{\frac{(t_{51}+t_{52}+t_{41})}{3} + \frac{(t_{31}+t_{32}+t_{33})}{3} - \frac{(t_{21}+t_{11})}{2} - \frac{(t_{12})}{1}}{13/6}$$

$$L_7 = \frac{\frac{(t_{51}+t_{52}+t_{41})}{3} - \frac{(t_{31}+t_{32}+t_{33})}{3} + \frac{(t_{21}+t_{11})}{2} - \frac{(t_{12})}{1}}{13/6}$$

$$L_8 = \frac{\frac{(t_{51}+t_{52}+t_{41})}{3} - \frac{(t_{31}+t_{32}+t_{33})}{3} - \frac{(t_{21}+t_{11})}{2} + \frac{(t_{12})}{1}}{13/6}$$

The normal equations in $L_i$'s come out as:

$$24L_1 = (Q_{51}-Q_{52})$$

$$67.800L_2 + .600(L_6+L_7+L_8) = (Q_{51}+Q_{52}-2Q_{41})$$

$$20L_3 = (Q_{31}-Q_{32})$$

$$60L_4 = (Q_{31}+Q_{32}-2Q_{33})$$

$$16.971L_5 - .545L_6 + .485(L_7-L_8) = (Q_{21}-Q_{11})$$

and

$$.600L_2 - .545L_5 + 18.633L_6 + 4.178L_7 - 3.278L_8 = \frac{(Q_{51}+Q_{52}+Q_{41})}{3} + \frac{(Q_{31}+Q_{32}+Q_{33})}{3} - \frac{(Q_{21}+Q_{11})}{2} - \frac{(Q_{12})}{1}$$

$$.600L_2 + .485L_5 + 4.178L_6 + 19.359L_7 - 5.126L_8 = \frac{(Q_{51}+Q_{52}+Q_{41})}{3} - \frac{(Q_{31}+Q_{32}+Q_{33})}{3} + \frac{(Q_{21}+Q_{11})}{2} - \frac{(Q_{12})}{1}$$

$$.600L_2 - .485L_5 + 3.279L_6 - 5.126L_7 + 19.359L_8 = \frac{(Q_{51}+Q_{52}+Q_{41})}{3} - \frac{(Q_{31}+Q_{32}+Q_{33})}{3} - \frac{(Q_{21}+Q_{11})}{2} + \frac{(Q_{12})}{1}$$

Evidently these equations are too involved to have any easier solution, thus proving the statement given in preceding paragraph.

It may be mentioned in this connection that in progeny row trials with plants coming from different families, such staircase designs become more suitable. The analysis also turns out very simple if the families are identified with the treatment-steps and the treatments within a step with the plants in a family.

## X. GRAYBILL AND PRUITT'S DESIGN - a particular case

We have seen that in the case of generalized designs solutions for the $L_j$'s corresponding to the between step contrasts is not straight forward as in the case of within step contrasts. In the case of the staircase designs given by Graybill and Pruitt this difficulty also vanishes. For such designs the equations in $L_j$'s come out as obtained below for the present design (table 8) with $n_{kj} = 0$ following the method illustrated earlier.

(i) $2 \times 10L_1 = (Q_{61}-Q_{62}) = P_1$ where $10 = R_6$, $2 = \sum_{i=1}^{2} l_{ij}^2$

(ii) $6 \times 10L_2 = (Q_{61}+Q_{62}-2Q_{63}) = P_2$ where $10 = R_6$, $6 = \sum_{i=1}^{3} l_{ij}^2$

(iii) $2 \times 8L_3 = (Q_{51}-Q_{52}) = P_3$ where $8 = R_5$, $2 = \sum_{i=1}^{2} l_{ij}^2$

(iv) $2 \times 5L_4 = (Q_{41}-Q_{42}) = P_4$ where $5 = R_4$,

(v) $2 \times 4L_5 = (Q_{31}-Q_{32}) = P_5$ where $4 = R_3$

(vi) $2 \times L_6 = (Q_{11}-Q_{12}) = P_6$ where $1 = R_1$

(vii) $8\left[(t_{61}+t_{62}+t_{63})-3/2(t_{51}+t_{52})\right] = (Q_{61}+Q_{62}+Q_{63})-3/2(Q_{51}+Q_{52}) = P_7$

or $8 \times 15/2 L_7 = P_7$, from definition of $L_7$,

$$\text{viz.,}\; L_7 = \frac{(t_{61}+t_{62}+t_{63})-3/2(t_{51}+t_{52})}{15/2}$$

(vii) or $60L_7 = P_7$ where $R_5 = 8$, i.e. the number of blocks where $L_7$ contrast occurs completely within steps. Hence it is possible in Graybill and Pruitt's case to write down normal equations in $L_j$'s directly.

(viii) $87.5L_8 = (Q_{61}+ \ldots\ldots +Q_{52}) - 5/2(Q_{41}+Q_{42}) = P_8$

(ix) $126L_9 = (Q_{61} + \ldots\ldots +Q_{41}) - 7/2(Q_{31}+Q_{32}) = P_9$

(x) $270L_{10} = (Q_{61}+\ldots\ldots\ldots +Q_{32}) - 9Q_{21} = P_{10}$

& (xi) $60L_{11} = (Q_{61}+\ldots\ldots\ldots +Q_{21}) - 5(Q_{11}+Q_{12}) = P_{11}$

Thus the sum of squares due to the contrasts viz.,
$\sum_{j=1}^{11} L_j P_j$ comes out as

$$P_1^2/20 + P_2^2/60 + P_3^2/16 + \ldots\ldots + P_{11}^2/60 \text{ on 11 degrees of freedom.}$$

<u>In general</u> if $L_j$ denotes any contrast $\sum_i l_{ij} t_i$ where all the $t_i$'s belong to the same $m^{th}$ step with say $R_m$ replications the equation corresponding to this within step contrast comes out as

$$R_m \left( \sum_{i=1}^{S_m} l_{ij}^2 \right) L_j = \sum_{i=1}^{S_m} l_{ij} Q_i = P_j, \; j = 1 \text{ to } (v-p-1)$$

where there are $(p+1)$ steps.

Again let $L_i$ denote another contrast among the step-totals as shown below:

$$L_i = t_{q.} + t_{(q-1).} + \ldots\ldots + t_{(i+1).} - (n_i)\, t_{i.},$$

where $(t_{i.})$ denotes the sum of the treatment effects in the $i^{th}$ step and '$n_i$' is so chosen that $L_i$ becomes a contrast among the t's. Actually

$$n_i = \frac{S_{i+1} + S_{i+2} + \ldots\ldots + S_{q-1} + S_q}{S_i}$$

where $S_i$ denotes the number of treatments in the $i^{th}$ step.

The equation for $L_i$ is then:

$$R_i \, L_i \, n_i \, (n_i + 1) \, S_i = P_i, \; i = (v-p) \text{ to } (v-1)$$

where $R_i$ is the replication of a treatment in the $i^{th}$ step having $S_i$ treatments and $P_i$ is the contrast among the Q's corresponding to $L_i$.

Thus the solutions of the equations can be directly obtained for such designs. Moreover the treatment sum of squares can be partitioned into mutually independent contrasts each of 1 degree of freedom. Actually components with <u>any number</u>

of degrees of freedom can be obtained by adding together the sum of squares due to two or more appropriate contrasts.

We have now two methods of solution of the normal equations in $t_{mi}$'s viz. one following from the general method and this solution has been presented in(40a) and the other through the solution of the equations in $L_j$'s and then getting $t_{mi}$'s from the $L_j$'s as indicated in the present section. As both the solutions are linear functions of $Q_{mi}$'s, some identity relations are available by equating the coefficients of $Q_{mi}$'s in the two solutions for $t_{mi}$'s. With the help of these identities relations it has been possible to get a very much simplified procedure of obtaining $C_m^p$'s which are required for the solution of $t_{mi}$'s through the former method.

In general the procedure has come out to obtaining the different $C_m^p$'s through the recurrence relation

$$\frac{C_m^p}{R_p} - \frac{C_m^{p+1}}{R_{p+1}} = \frac{(-1)^{m-p-1}}{\sum_{k=p+1}^{v} S_k}\left(\frac{1}{R_p} - \frac{1}{R_{p+1}}\right), \qquad p < m \leq v.$$

Its derivation has been illustrated below with reference to the example considered for this design.

<u>Recurrence formula for obtaining $C_m^p$'s in Graybill and Pruitt's design.</u>

The general recurrence formula (25) for $C_m^p$ is suitable for all types of staircase designs. As mentioned in article 4, page ( 20 ), it is possible to obtain a very simplified form of this formula in the case of Graybill and Pruitt's design. Derivation of this simplified formula with reference to the same example, as given in table 8, (but with $n_{kj} = 0$) is explained below. The eleven contrasts, $L_j$'s, therefore remain unaffected (page 42, article 9).

Let us derive the recurrence formula for the 6th set, i.e. $m = q = 6$ and $i = 1$.

From table 12, page 45, (which remains unaffected due to contrasts)

$t_{61} = L_1 + L_2 + (L_7 + L_8 + L_9 + L_{10} + L_{11})$

Converting each of these $L_j$'s into $Q_{mi}$'s with the help of normal equations given on page 52, article 10 (Graybill and Pruitt's case), we obtain

$$t_{61} = \frac{(Q_{61}-Q_{62})}{2R_6} + \frac{(Q_{61}+Q_{62}-2Q_{63})}{6R_6} + \frac{(Q_{61}+Q_{62}+Q_{63})-3/2(Q_{51}+Q_{52})}{15/2R_5}$$

$$+ \frac{(Q_{61}+\ldots+Q_{52})-5/2(Q_{41}+Q_{42})}{35/2R_4} + \frac{(Q_{61}+\ldots+Q_{42})-7/2(Q_{31}+Q_{32})}{63/2R_3}$$

$$+ \frac{(Q_{61}+\ldots+Q_{32})-9Q_{21}}{90R_2} + \frac{(Q_{61}+\ldots+Q_{21})-5(Q_{11}+Q_{12})}{60R_1}$$

where $R_1 = 1$, $R_2 = 3$, $R_3 = 4$, $R_4 = 5$, $R_5 = 8$ and $R_6 = 10$ in this design.

$$\text{Now:}\quad \frac{Q_{61}-Q_{62}}{2R_6} + \frac{Q_{61}+Q_{62}-2Q_{63}}{6R_6}$$

$$= \frac{Q_{61}}{R_6} - \frac{Q_{61}+Q_{62}+Q_{63}}{3R_6}$$

Collecting coefficients of $\sum_{i=1}^{S_6} Q_{61}$, $\sum_{i=1}^{S_5} Q_{51}$, $\sum_{i=1}^{S_4} Q_{41}$, and $\sum_{i=1}^{S_3} Q_{31}$, $\sum_{i=1}^{S_2} Q_{21}$ and $\sum_{i=1}^{S_1} Q_{11}$ and eliminating $\sum_{i=1}^{S_6} Q_{61}$ with the help of $\sum_{m=1}^{6}\sum_{i=1}^{S_m} Q_{mi} = 0$ and collecting terms:

$$t_{61} = \frac{Q_{61}}{R_6} + (Q_{51}+Q_{52}) \left(\frac{-3/2}{15/2R_5} - \frac{1}{15/2R_5} + \frac{1}{3R_6}\right) + (Q_{41}+Q_{42}) \times$$

$$\left(-\frac{1}{15/2R_5} - \frac{5/2}{35/2R_4} - \frac{1}{35/2R_4} + \frac{1}{3R_6}\right) + (Q_{31}+Q_{32}) \times$$

$$\left(\frac{-7/2}{63/2R_3} - \frac{1}{63/2R_3} - \frac{1}{35/2R_4} - \frac{1}{15/2R_5} + \frac{1}{3R_6}\right) + (Q_{21}) \times$$

$$\times\left(\frac{-9}{90R_2}-\frac{1}{90R_2}-\frac{1}{63/2R_3}-\frac{1}{35/2R_4}-\frac{1}{15/2R_5}+\frac{1}{3R_6}\right)+$$

$$(Q_{11}+Q_{12})\left(\frac{-5}{60R_1}-\frac{1}{60R_1}-\frac{1}{90R_2}-\frac{1}{63/2R_3}-\frac{1}{35/2R_4}-\frac{1}{15/2R_5}+\frac{1}{3R_6}\right),$$

where coefficients of each $\sum_{i=1}^{s_m} Q_{mi}$, $m = 1$ to 5, are nothing but sum of reciprocals of different $\sum_{i=1}^{s_m} l_{ij}^2 R_m$.

Converting each $Q_{mi}$ into $Z_{mi}$ from the relation $Q_{mi} = R_m Z_{mi}$, we obtain

$$t_{61} = Z_{61}+R_5(Z_{51}+Z_{52})\left(-\frac{1}{3R_5}+\frac{1}{3R_6}\right)+R_4(Z_{41}+Z_{42})\left(-\frac{1}{5R_4}-\frac{1}{15/2R_5}+\frac{1}{3R_6}\right)+R_3(Z_{31}+Z_{32})\left(-\frac{1}{7R_3}-\frac{1}{35/2R_4}-\frac{1}{15/2R_5}+\frac{1}{3R_6}\right)$$

$$+R_2Z_{21}\times\left(-\frac{1}{9R_2}-\frac{1}{63/2R_3}-\frac{1}{35/2R_4}-\frac{1}{15/2R_5}+\frac{1}{3R_6}\right)+$$

$$R_1(Z_{11}+Z_{12})\left(-\frac{1}{10R_1}-\frac{1}{90R_2}-\frac{1}{63/2R_3}-\frac{1}{35/2R_4}-\frac{1}{15/2R_5}+\frac{1}{3R_6}\right) \quad \text{(A)}$$

Comparing the various coefficients of $Z_{mi}$'s, with (40a) i.e.

$$t_{61} = Z_{61}-C^5{}_6\sum_{i=1}^{s_5} Z_{5i}+C^4{}_6\sum_{i=1}^{s_4} Z_{4i}-C^3{}_6\sum_{i=1}^{s_3} Z_{3i}+C^2{}_6\sum_{i=1}^{s_2} Z_{2i}-C^1{}_6\sum_{i=1}^{s_1} Z_{1i} \quad \text{(B)}$$

we obtain

$$\left.\begin{aligned}
&1 = 1\\
&R_5\times\left(-\frac{1}{3R_5}+\frac{1}{3R_6}\right) = -C^5{}_6\\
&R_4\times\left(-\frac{1}{5R_4}-\frac{1}{15/2R_5}+\frac{1}{3R_6}\right) = C^4{}_6\\
&R_3\times\left(-\frac{1}{7R_3}-\frac{1}{35/2R_4}-\frac{1}{15/2R_5}+\frac{1}{3R_6}\right) = -C^3{}_6\\
&R_2\times\left(-\frac{1}{9R_2}-\frac{1}{63/2R_3}-\frac{1}{35/2R_4}-\frac{1}{15/2R_5}+\frac{1}{3R_6}\right) = C^2{}_6\\
&R_1\times\left(-\frac{1}{10R_1}-\frac{1}{90R_2}-\frac{1}{63/2R_3}-\frac{1}{35/2R_4}-\frac{1}{15/2R_5}+\frac{1}{3R_6}\right) = -C^1{}_6
\end{aligned}\right\} \quad \text{(C)}$$

Same result is obtained with $m = 6$, but $i = 2$ and 3 ( since $S_6 = 3$ in this design). Other sets of $C_m^p$ 's can be obtained similarly for $m = 5,4,3,2$ and 1 successively.

It is not difficult to generalize the results as indicated in (C), only point is that they involve $\sum_i l_{ij}^2$'s of various $L_j$ contrasts. In order to make formula independent of $\sum_i l_{ij}^2$'s we apply the following device:

Taking successive differences in (C) we obtain

$$\frac{C^1{}_6}{R_1} + \frac{C^2{}_6}{R_2} = 1/10(1/R_1 - 1/R_2)$$ Here 10 is the number of treatments with replications $\sum_{k=2}^{6} S_k$.

$$\frac{C^2{}_6}{R_2} + \frac{C^3{}_6}{R_3} = -1/9(1/R_2 - 1/R_3)$$ Here 9 is the number of treatments with replications $\sum_{k=3}^{6} S_k$.

$$\frac{C^3{}_6}{R_3} + \frac{C^4{}_6}{R_4} = 1/7(1/R_3 - 1/R_4)$$ Here 7 is the number of treatments with replications $\sum_{k=4}^{6} S_k$. (D)

$$\frac{C^4{}_6}{R_4} + \frac{C^5{}_6}{R_5} = -1/5(1/R_4 - 1/R_5)$$ Here 5 is the number of treatments with replications $\sum_{k=5}^{6} S_k$.

These relations are independent of $\sum_i l_{ij}^2$'s, but depend only on the number of treatments. Actually each of the above relations connects $C_m^p$ with $C_m^{p+1}$ . Generalizing over all sets (like (D)) and connecting any $C_m^p$ with $C_m^{p+1}$ we obtain as mentioned in article 4, page 20 and on page 54,

$$\frac{C_m^p}{R_p} + \frac{C_m^{p+1}}{R_{p+1}} = \frac{(-1)^{m-p-1}}{\sum_{k=p+1}^{q} S_k}(1/R_p - 1/R_{p+1}), \quad p < m \leqslant q, \qquad (40b)$$

$p = 1$ to $(m-2)$

$m = 1$ to $q$ with $C_m^p = 0$ for $p > m$.

where $C_m^{m-1} = P_m/R_m$ being the starting point for each m, and then varying p from 1 to (m-2) for each such m.

This formula (40b) is completely independent of nature of contrasts (article 9 page 42). Thus with the help of these devices the solution of the normal equations in Graybill and Pruitt's designs can be obtained very easily and in a much simpler way than what has been given by Graybill and Pruitt.

## SUMMARY

Graybill and Pruitt (1958) introduced a class of designs which they called Staircase designs. These designs provide for the analysis of one more type of non-orthogonal data having only two types pf cell frequencies viz.,0 and 1. Such designs have the draw back that they do not provide for any block of size greater than the number of treatments. Both to remove such limitations as also to provide for the analysis of one more type of non-orthogonal data, a generalised definition of the Staircase designs has been given. The generalised staircase designs have in each block two types of frequencies which may change from block to block and need not be zero and unity. The number of treatments having one type of frequency need not be the same from block to block. Such designs are particularly suitable for plant breeding trials as also for experiments with animals as the experimental units. A complete method of analysis of such designs has been presented in the thesis in a simplified and systematic way. A much more simplified method of analysis of the staircase designs of Graybill and Pruitt has also been presented in the thesis.

In many investigations it becomes necessary to get subdivisions of the adjusted treatment sum of squares. No general and simplified method for obtaining the components of adjusted sum of squares having more than one degree of freedom seems to be available in literature. In the present thesis a general method of obtaining such subdivisions has been presented. It has been shown that in the case of staircase designs such subdivisions to suit some particular hypothesis can be obtained in a very simple way.

These investigations have been illustrated by means of two examples.

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