RATIO - TYPE ESTIMATORS IN DOUBLE SAMPLING FOR TWO - STAGE DESIGNS

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CHAPTER - I

INTRODUCTION

The simplest method of sampling is to select certain number of units say n out of the total number of units in the population randomly such that each unit in the population has an equal probability of being selected. This procedure of sampling called ' Simple Random Sampling' is easy to conduct and estimation procedure requires little more than simple addition. But simplicity is not often the criterion for adopting a particular mode of sampling. More often, procedures other than simple random sampling are adopted, in practice for operational convenience, for keeping costs minimum or for increasing the overall precision of the estimates for a given cost.

Besides the use of different sampling designs, refinements in the method of estimation are also commonly used in obtaining more efficient estimates compared to estimates based upon the simple mean. Many a time, information on some auxiliary character related to the character under study is available and the use of this information is one of the ways of getting estimates of the population values with improved precision. It may be used at the stage of planning as in the case of stratification or at the stage of selection (selection with varying probability) or at the stage of estimation.

Ratio and regression methods of estimation are well known examples where the auxiliary information is used at the estimation stage. Ratio-estimates are frequently employed in sample surveys when estimating the population mean $\overline{Y}_N$ of the character under study y with the help of known population mean $\overline{X}_N$ of a variable x that is positively correlated with y.

Two types of ratio-estimators are in common use to estimate the population mean. One is equal to $\frac{\bar{y}_n}{\bar{x}_n} \bar{X}_N$ and the other is given by $\bar{r}_n \bar{X}_N$ where $\bar{y}_n$, $\bar{x}_n$ are the sample means for y and x respectively, $\bar{r}_n$, the mean of the individual ratios $r_i = y_i/x_i$ in the sample and $\bar{X}_N$ is the known population mean of the auxiliary variable x. Both the estimators are known to be biased. The ratio-estimator $\frac{\bar{y}_n}{\bar{x}_n} \bar{X}_N$ is known to be more efficient than the simple mean estimator $\bar{y}_n$, provided ρ, the correlation coefficient between x and y is positive and is greater than $\frac{1}{2}\, c_x/c_y$ where c_x, c_y are the coefficients of variation of x and y respectively. The fact that the ordinary ratio-estimator is a biased estimator and the bias may be substantial in certain cases, diminishes its value as a good estimator. Moreover, it is also not known in advance, how large sample should be, so that the bias in the estimate of the population mean becomes negligible. So in recent years, many research workers have investigated the means of developing ratio-type estimators that are unbiased.

Hartley and Ross (1954) have been the pioneers who tried to obtain an unbiased estimator of the population mean. In the case of simple random sampling without replacement, they have given an elegant expression for the bias in $\bar{r}_n \bar{X}_N$ and developed an unbiased version of this ratio-type estimator as equal to

$$\bar{r}_n \bar{X}_N + \frac{n(N-1)}{N(n-1)} (\bar{y}_n - \bar{r}_n \bar{x}_n).$$

The second term of this expression is the correction for the bias of the first term as an estimate of the population mean.

Robson (1957) has derived the exact formula for its variance when the finite population correction factor (i.e. fpc) is not negligible. He also obtained the unbiased estimate of the variance. When the fpc is negligible, the formula for variance of the estimated mean was given by Goodman and Hartley (1958), who also gave an unbiased sample estimate of variance of the unbiased estimate. They also discussed the relative efficiencies of the different estimators and pointed out that under certain conditions, the unbiased estimator is more precise to $\bar{r}_n \bar{X}_N$ in large samples.

B.V. Sukhatme (1962) presented several ratio type estimators of the population mean in the case of two phase sampling. They are given by $\frac{\bar{y}_n}{\bar{x}_n} \bar{x}_{n'}$, $\bar{r}_n \bar{x}_{n'}$ and $\bar{r}_n \bar{x}_{n'} + \frac{n(n'-1)}{n'(n-1)} (\bar{y}_n - \bar{r}_n \bar{x}_n)$ respectively where n' denotes the sample selected out of N units in the population to observe the variate x and n is a sub-sample selected out of n' to observe the variate y, the character under study. The third estimator is an unbiased estimator of the population mean and follows direct from the one given by Hartley and Ross (1954) for single phase sampling. He obtained the variance of each of these estimators for the same order of approximation and gave conditions under which the unbiased estimator is more efficient than the biased estimator. He indicated when it is so, the unbiased estimator is certainly to be preferred to the biased estimator because its variance is smaller than that of $\bar{r}_n \bar{x}_{n'}$. He pointed out, however, if it is not so, the estimator $\bar{r}_n \bar{x}_{n'}$ is not necessarily preferred to the unbiased estimator, since it is biased and the bias will have to be taken into account except in the case when $\frac{1}{N} \sum_{i=1}^{N} y_i/x_i = \frac{\bar{Y}_N}{\bar{X}_N}$. He also gave a numerical illustration and discussed the efficiency of two phase sampling with respect to single phase sampling using a simple cost function. The cost function which he considered was $C_o = c_1 n + c_2 n'$ where C_o denotes the total cost, c_1, the cost per unit of observation on the variable y, c_2, the cost per unit of observation on the variable x and n and n' are the same as defined earlier. He then, determined the optimum values of n and n' so that for a fixed cost C_o, the variance of the estimate was minimum.

In the present investigation, some ratio-type estimators of the population mean in the case of two-stage sampling have been presented. Two cases have been considered (i) when the mean of the auxiliary variable in the population is known, (ii) when it is not known and thereby resorting to double sampling. In each case, an unbiased estimate has been obtained. Also the technique of symmetric means developed by Tukey (1956) and extended by Robson (1957) has been further extended in the present work for use in two-stage designs. The new technique developed will hereafter be referred to as the "Extended Method of Symmetric Means " to two-stage designs. This has been utilised for constructing unbiased estimators and as will be seen in chapter III and IV, the results obtained here will reduce to the unbiased estimators obtained by Hartley and Ross (1954) and B.V. Sukhatme (1962) when the first-stage units are completely enumerated.

The exact expressions for the variances of the different estimators obtained, have been worked out and presented in the symmetrica mean notation. Afterwards assuming N and M to be large, the variances of each of these estimators for the same order of approximation have been obtained and presented in the standard notations. The conditions under which the unbiased estimators will be more efficient than the biased estimators have also been obtained for the two situations considered in chapter III and IV viz., (i) when single phase sampling in two stages is adopted, (ii) when two phase sampling in two stages is adopted. The estimates of variance as well as unbiased estimator of variance of the estimate for the (ii) situation have also been worked out for the case when the number of primary units and number of secondary units in the population are large.

The optimum values of the number of first-stage units and second-stage units to be sampled for attaining a given precision, the total cost being fixed, have been worked out using a cost function - both when single phase and two phase sampling schemes in two stages are adopted.

CHAPTER II

CONCEPT OF SYMMETRIC MEAN, MULTI-VARIATE SYMMETRIC MEAN AND THE EXTENDED METHOD OF SYMMETRIC MEAN TO TWO-STAGE DESIGNS

2.1. Symmetric Mean

Let y_i denotes the value of y, the character under study for the i th unit ($i = 1, 2, \ldots, N$) in the population and let a simple random sample of size n be drawn from this population.

A symmetric mean is then defined to be a polynomial of the type

$$\frac{1}{M} \sum_{i \neq j \neq \ldots \neq m} x_i^a x_j^b \ldots\ldots\ldots x_m^e$$, where the subscripts are summed from 1 to n (for samples) or from 1 to N (for population), the exponents are positive integers and M is the number of terms in the summation. When it is for the sample $M = (n)_m = n(n-1) \ldots\ldots (n-m+1)$, when it is for the population $M = (N)_m = N(N-1) \ldots\ldots (N-m+1)$. When the sample or population size is given, the symmetric mean is specified by the exponents and so is abbreviated by writing the exponents within brackets, as in

$$\langle abc \rangle = \frac{1}{n(n-1)(n-2)} \sum_{i \neq j \neq k}^{n} x_i^a x_j^b x_k^c \quad \text{over the sample and}$$

$$\langle abc \rangle' = \frac{1}{N(N-1)(N-2)} \sum_{i \neq j \neq k}^{N} x_i^a x_j^b x_k^c \quad \text{over the population.}$$

From these general expressions, symmetric means of degree one, and two over the sample as well as over the population can be easily written.

It is to be noted that symmetric means are inherited on the average. An expression is said to be inherited on the average if for a given sample size and population size, E(symmetric sample mean) is equal to the population mean where E denotes the operation of taking the expectation. Tukey(1956) has developed this concept for the uni-variate estimation problems. Sometimes multi-variate estimation problems do arise in the field of survey sampling where each sample element is measured for a variety of characteristics. For such problems, Robson (1957) proposed the concept of multi-variate symmetric means, a brief account of which, together with a multiplication formula for the product of two symmetric means, is given below.

2.2. Multi-variate Symmetric Means

The polynomial

$$\frac{1}{(n)_r} \sum_{j_1 \neq j_2 \neq \cdots \neq j_r} (x_{1j_1}^{a_{11}} x_{2j_1}^{a_{21}} \cdots x_{mj_1}^{a_{m1}}) \cdots (x_{1j_r}^{a_{1r}} x_{2j_r}^{a_{2r}} \cdots x_{mj_r}^{a_{mr}}) \qquad \ldots (2.1)$$

in the mn variates x_{ij}, $i = 1, \ldots, m$; $j = 1, \ldots, n$, is called a symmetric mean and is denoted by

$$\langle (a_r) \rangle = \langle (a_1^*)(a_2^*) \ldots (a_r^*) \rangle$$

where (a_i^*) is the vector $(a_{1i}, \ldots, a_{mi})$. Consider now two symmetric means

$$\langle (a_r) \rangle = \langle (a_1^*)(a_2^*) \ldots (a_r^*) \rangle \text{ and}$$

$$\langle (\beta_s) \rangle = \langle (b_1^*)(b_2^*) \ldots (b_s^*) \rangle$$

Let

$$\rho_\gamma(a_r \beta_s) = [(a_{i_1} + b_{j_1}), \ldots, (a_{i_\gamma} + b_{j_\gamma}), (a_{i_{\gamma+1}}), \ldots, (a_{i_r}), (b_{j_{\gamma+1}}), \ldots, (b_{j_s})]$$

be obtained by paring and adding γ elements of (a_r) with γ elements of (β_s). Let us denote by $R_\gamma(a_r \beta_s) = [\rho_\gamma(a_r \beta_s)]$ the set of all possible $\gamma!\binom{r}{\gamma}\binom{s}{\gamma}$ possible sets $\rho_\gamma(a_r \beta_s)$. Then the product of two symmetric means $\langle (a_r) \rangle$ and $\langle (\beta_s) \rangle$ is given by the formula

$$\langle(\alpha_r)\rangle \langle(\beta_s)\rangle = \frac{1}{(n)_r(n)_s} \sum_{\gamma=0}^{r} (n)_{r+s-\gamma} \sum_{R_\gamma} \langle \rho_\gamma(\alpha_r\beta_s)\rangle, \; r \leq s. \quad \ldots(2.2)$$

It can be seen that if these mn variates represent a simple random sample of n observations from an m-dimensional finite population of size N, the expected value of the statistics $\langle(\alpha_r)\rangle$ taken over $\binom{N}{n}$ possible samples is the corresponding symmetric mean of the population. Conversely, if a population statistic is written as a linear combination of symmetric means, then an unbiased estimate of the population symmetric mean is simply got by replacing each symmetric mean by the corresponding sample mean.

2.3. Extended Method of Symmetric Mean to Two-stage Designs

Let the population be composed of N first-stage units of M second-stage units each and assume that a simple random sample of n first-stage units be drawn from N and from each selected first-stage unit, a sample of m second-stage units be drawn by the method of simple random sampling. Let y_{ij}, denotes the value of the j th second-stage unit in the i th first-stage unit when y is the character under study. Then symmetric means of order one and two for the sample will be

$$\frac{1}{nm} \sum_{i}^{n} \sum_{j}^{m} y_{ij} = \frac{1}{n} \sum_{i}^{n} \langle 1 \rangle_i$$

$$\frac{1}{nm} \sum_{i}^{n} \sum_{j}^{m} y_{ij}^2 = \frac{1}{n} \sum_{i}^{n} \langle 2 \rangle_i$$

$$\frac{1}{nm(m-1)} \sum_{i}^{n} \sum_{j \neq j'}^{m} y_{ij}\, y_{ij'} = \frac{1}{n} \sum_{i}^{n} \langle 11 \rangle_i$$

and for the population will be

$$\frac{1}{NM} \sum_{i}^{N} \sum_{j}^{M} y_{ij} = \frac{1}{N} \sum_{i}^{N} \langle 1 \rangle'_i = \langle 1 \rangle''$$

$$\frac{1}{NM} \sum_{i}^{N} \sum_{j}^{M} y_{ij}^2 = \frac{1}{N} \sum_{i}^{N} \langle 2 \rangle'_i = \langle 2 \rangle''$$

$$\frac{1}{NM(M-1)} \sum_{i}^{N} \sum_{j \neq j'}^{M} y_{ij}\, y_{ij'} = \frac{1}{N} \sum_{i}^{N} \langle 11 \rangle'_i = \langle 11 \rangle''$$

An angle bracket with a single prime $\langle \; \rangle'_i$ denotes the symmetric mean of the sub-population consisting of 'M' units of the i th first-stage unit while $\langle \; \rangle''$ with a double prime denotes a symmetric population mean. $\langle \rangle_i$ without the prime denotes a sample symmetric mean consisting of 'm' units of the i th selected first-stage unit.

It is to be noted that since the sample is selected in two-stages, the expected value can be appropriately worked out in two stages, first over all the second-stage units in the i th first-stage unit and then over all the first-stage units, by using the extended method. This is the case when we have one variable at our disposal. Many a time, we can have another variable x which is highly correlated with y, the character under study. When information on x is available, the ratio-type estimators will under certain conditions, provide more efficient estimates than the simple mean estimator. Thus if every individual in the population is considered to be tri-variate taking values y_{ij}, x_{ij} and r_{ij} where y_{ij} is the value of the j th second-stage unit in the i th first-stage unit for the character under study, x_{ij} that of the auxiliary character x and $r_{ij} = y_{ij} / x_{ij}$, then the mean

$\frac{1}{nm}\sum_{i}^{n}\sum_{j}^{m} y_{ij}$ can be denoted in symmetric mean notation by $\frac{1}{n}\sum_{i}^{n}\langle olo\rangle_i$ and its expected value will be $\langle olo\rangle$. Similarly terms $\frac{1}{n}\sum_{i}^{n}\langle ool\rangle_i$ and $\frac{1}{n}\sum_{i}^{n}\langle loo\rangle_i$ will denote $\bar{r}_{nm}$ and $\bar{x}_{nm}$ respectively. The extended method of symmetric means can advantageously be adopted for deriving classical results about variances and their unbiased estimates etc. In chapter III and IV, this extended method of symmetric means will be used for deriving the various results presented there.

CHAPTER III

RATIO-TYPE ESTIMATORS WHEN THE POPULATION MEAN OF THE AUXILIARY VARIABLE IS KNOWN

In this chapter, the problem of obtaining unbiased ratio-type estimators for the two-stage design when the information on the auxiliary variable is completely known, while in the next chapter, the case when this is not known and double sampling is resorted to, will be discussed.

Let a finite population consist of N first-stage units of 'M' second-stage units each. Let y_{ij} denotes the value of the j th second-stage unit in the i th first-stage unit for the character under study y and x_{ij} that of the auxiliary variable x which is highly and positively correlated to y. Further let $r_{ij} = y_{ij}/x_{ij}$. We shall consider the problem of estimating the population mean $\bar{Y}_{NM}$. For this purpose, assume that a simple random sample of size n is drawn from the N first-stage units and 'm' be the number of second-stage units to be drawn from each selected first-stage unit by the method of simple random sampling.

Let

$\bar{Y}_{NM} = \langle 010 \rangle'' = \langle 101 \rangle''$

= the mean per element of y in the population

$\bar{X}_{NM} = \langle 100 \rangle''$

= the mean per element of x in the population

$\bar{r}_{NM} = \langle 001 \rangle''$

= the mean per element of the ratio of y to x in the population

$\bar{Y}_{i.} = \langle 010 \rangle'_i$

= the mean per element of y in the i th primary unit in the population

$\bar{X}_{i.} = \langle 100 \rangle'_i$

= the mean per element of x in the i th primary unit in the population

$\bar{r}_{iM} = \langle 001 \rangle_i$

= the mean per element of the ratio of y to x in the i th primary unit in the population

$\bar{y}_{nm} = \frac{1}{n} \sum_i^n \langle 010 \rangle_i$

= the mean per element of y in the sample

$\bar{x}_{nm} = \frac{1}{n} \sum_i^n \langle 100 \rangle_i$

= the mean per element of x in the sample

$\bar{r}_{nm} = \frac{1}{n} \sum_i^n \langle 001 \rangle_i$

= the mean per element of the ratio of y to x in the sample.

$\bar{y}_{im} = \langle 010 \rangle_i$

the mean per element of y in the i th primary unit in the sample

$\bar{x}_{im} = \langle 100 \rangle_i$

= the mean per element of x in the i th prinary unit in the sample

$\bar{r}_{im} = \langle 001 \rangle_i$

= the mean per element of the ratio of y to x in the i th primary unit in the sample.

Then the ratio type estimator of $\bar{Y}_{NM}$ is defined as :

$$T_1 = \bar{r}_{nm} \bar{X}_{NM} \qquad \ldots (3.1)$$

Since the sample is selected in two stages, the expected value is also worked out in two stages as shown below.

$$E(T_1) = E_1 [E_2 (\bar{r}_{nm} \bar{X}_{NM})] = \bar{r}_{NM} \bar{X}_{NM} \neq \bar{Y}_{NM} \quad \ldots (3.2)$$

where E_2 denotes the operation of taking expected value over all possible samples of 'm' from each of 'n' first-stage units and E_1 over all possible samples of n.

It is seen from (3.2) that T_1 is a biased estimator of the population mean. Let us consider the amount of bias in T_1.

Bias in T_1 = $E(\bar{y}_{nm}\bar{\bar{X}}_{NM}) - \bar{Y}_{NM}$ (by definition)

$$= E\left[\left(\frac{1}{n}\sum_i^n\langle 001\rangle_i\right)\left(\frac{1}{N}\sum_i^N\langle 100\rangle'_i\right)\right] - \langle 010\rangle''$$

$$= \left(\frac{1}{N}\sum_i^N\langle 001\rangle'_i\right)\left(\frac{1}{N}\sum_i^N\langle 100\rangle'_i\right) - \langle 010\rangle''$$

$$= \frac{1}{N^2}\left[\sum_i^N\langle 001\rangle'_i\langle 100\rangle'_i + \sum_{i\neq i'}^N\langle 001\rangle'_i\langle 100\rangle'_{i'}\right] - \langle 010\rangle''$$

Multiplying the symmetric means with the help of multiplication formula (2.2), we get

$$\text{Bias in } T_1 = \frac{1}{N^2}\left[\frac{1}{M}\sum_i^N\langle 101\rangle'_i + \frac{M-1}{M}\sum_i^N\langle(100)(001)\rangle'_i\right]$$

$$+\frac{1}{N^2}\left[\frac{1}{M}\sum_{i\neq i'}^N\langle 101\rangle'_{ii'} + \frac{M-1}{M}\sum_{i\neq i'}^N\langle(100)(001)\rangle'_{ii'}\right] - \langle 010\rangle''$$

$$= \frac{1}{NM}\langle 010\rangle'' + \frac{M-1}{NM}\langle(100)(001)\rangle''$$

$$+\frac{N-1}{NM}\left[\langle 010\rangle''_{ii'} + (M-1)\langle(100)(001)\rangle''_{ii'}\right] - \langle 010\rangle''$$

Hence the unbiased estimator of this bias is

$$= \frac{1}{NM}\frac{1}{n}\sum_i^n\langle 010\rangle_i + \frac{M-1}{NM}\cdot\frac{1}{n}\sum_i^n\langle(100)(001)\rangle_i$$

$$+\frac{N-1}{N}\left[\frac{1}{n(n-1)}\sum_{i\neq i'}^n\langle 100\rangle_{i'}\langle 001\rangle_i\right] - \frac{1}{n}\sum_i^n\langle 010\rangle_i$$

$$= \frac{1}{NM}\bar{y}_{nm} + \frac{M-1}{NM}\frac{1}{nm(m-1)}\sum_i^n\left[\sum_j^m r_{ij}\sum_j^m x_{ij} - \sum r_{ij}x_{ij}\right]$$

$$+\frac{N-1}{N}\cdot\frac{1}{n(n-1)m^2}\left[\sum_{i\neq i'}^n\sum_j^m r_{ij}x_{i'j} + \sum_{i\neq i'}^n\sum_{j\neq j'}^m r_{ij}x_{i'j'}\right] - \bar{y}_{nm}$$

$$= \frac{1}{NM}\bar{y}_{nm} + \frac{(M-1)m}{NMn(m-1)}\left(\sum_i^n\bar{r}_{im}\bar{x}_{im}\right) - \frac{M-1}{NM(m-1)}\bar{y}_{nm}$$

$$+ \frac{N-1}{Nn(n-1)m^2} \left[n^2m^2 \bar{r}_{nm} \bar{x}_{nm} - m^2 \sum_i^n \bar{r}_{im} \bar{x}_{im} \right] - \bar{y}_{nm}$$

$$= \frac{1}{NM} \left(1 - \frac{M-1}{m-1} \right) \bar{y}_{nm} + \left[\frac{(M-1)\, m}{NM\, n(m-1)} - \frac{N-1}{Nn(n-1)} \right] \sum_i^n \bar{r}_{im} \bar{x}_{im}$$

$$+ \frac{(N-1)n}{(n-1)N} \bar{r}_{nm} \bar{x}_{nm} - \bar{y}_{nm}$$

$$= -\bar{y}_{nm} + \bar{r}_{nm} \bar{x}_{nm} - \frac{N-n}{Nn(n-1)} \left[\sum_i^n \bar{r}_{im} \bar{x}_{im} - n\bar{r}_{nm} \bar{x}_{nm} \right]$$

$$- \frac{M-m}{NM(m-1)n} \left[\frac{1}{m} \sum_i^n \sum_j^m r_{ij} x_{ij} - \sum_i^n \bar{r}_{im} \bar{x}_{im} \right]$$

$$= -\bar{y}_{nm} + \bar{r}_{nm} \bar{x}_{nm} - \left(\frac{1}{n} - \frac{1}{N} \right) \frac{1}{n-1} \sum_i^n \left[(\bar{r}_{im} - \bar{r}_{nm})(\bar{x}_{im} - \bar{x}_{nm}) \right]$$

$$- \frac{1}{nN} \left(\frac{1}{m} - \frac{1}{M} \right) \frac{1}{m-1} \sum_i^n \sum_j^m \left(r_{ij} x_{ij} - \bar{r}_{im} \bar{x}_{im} \right)$$

$$= \left[-\bar{y}_{nm} + \bar{r}_{nm} \bar{x}_{nm} - \left(\frac{1}{n} - \frac{1}{N} \right) s_{brx} - \left(\frac{1}{m} - \frac{1}{M} \right) \frac{1}{N} \frac{1}{n} \sum_i^n s_{irx} \right] \quad ..(3.3)$$

where

$$s_{brx} = \frac{1}{n-1} \left[\sum_i^n (\bar{r}_{im} - \bar{r}_{nm})(\bar{x}_{im} - \bar{x}_{nm}) \right]$$

$$s_{irx} = \frac{1}{m-1} \left[\sum_j^m (r_{ij} - \bar{r}_{im})(x_{ij} - \bar{x}_{im}) \right]$$

From (3.3), it can be seen that

$$E \left[T_1 + (\bar{y}_{nm} - \bar{r}_{nm} \bar{x}_{nm}) + \left(\frac{1}{n} - \frac{1}{N} \right) s_{brx} + \left(\frac{1}{m} - \frac{1}{M} \right) \frac{1}{Nn} \sum_i^n s_{irx} \right] = \bar{Y}_{NM}$$

which shows that

$$T_2 = T_1 + (\bar{y}_{nm} - \bar{r}_{nm} \bar{x}_{nm}) + \left(\frac{1}{n} - \frac{1}{N} \right) s_{brx} + \left(\frac{1}{m} - \frac{1}{M} \right) \frac{1}{Nn} \sum_i^n s_{irx} \quad ..(3.4)$$

is an unbiased estimator of the population mean $\bar{Y}_{NM}$. This can also be written as

$$T_2 = \bar{r}_{nm} \bar{X}_{NM} + \frac{n(N-1)}{N(n-1)} \left[\bar{y}_{nm} - \bar{r}_{nm} \bar{\bar{x}}_{nm} \right]$$

$$+ \left[\left(\frac{1}{m} - \frac{1}{M}\right) \frac{1}{N} - \frac{(N-n)(m-1)}{Nm(n-1)} \right] \frac{1}{n} \sum_{i}^{n} s_{irx} \qquad \ldots (3.5)$$

If in (3.5), we suppose that the selected first-stage units have been completely enumerated i.e. if $m = M$, then T_2 reduces to

$$T_2 = \bar{r}_n \bar{X}_N + \frac{n(N-1)}{N(n-1)} (\bar{y}_n - \bar{r}_n \bar{x}_n) \qquad \ldots (3.6)$$

which is the same estimate as obtained by Hartley and Ross (1954) in the case of simple random sampling without replacement for a single phase design. We shall consider these two estimators T_1 and T_2 and find the condition under which T_2 is more efficient than T_1.

3.1. Variance of the Estimator T_1

$$T_1 = \bar{r}_{nm} \bar{X}_{NM} = \left[\frac{1}{n} \sum_{i}^{n} \langle 001 \rangle_i \right] \left[\langle 100 \rangle'' \right]$$

Then

$$V(T_1) = E(T_1^2) - \left[E(T_1) \right]^2 \qquad \ldots (3.7)$$

Now $T_1^2 = \left[\frac{1}{n} \sum_{i}^{n} \langle 001 \rangle_i \right]^2 \left[\langle 100 \rangle'' \right]^2$

$$= \frac{1}{n^2} \left[\sum_{i}^{n} \langle 001 \rangle_i \langle 001 \rangle_i + \sum_{i \neq i'}^{n} \langle 001 \rangle_i \langle 001 \rangle_{i'} \right] \left[\langle 100 \rangle'' \right]^2$$

Using multiplication formula (2.2), and taking expectation, we get:

$$E(T_1^2) = \frac{1}{n^2} E_1 \left[\frac{1}{m} \sum_{i}^{n} E_2 \langle 002 \rangle_i + \frac{m-1}{m} \sum_{i}^{n} E_2 \langle (001)(001) \rangle_i \right.$$

$$\left. + \sum_{i \neq i'}^{n} E_2 \langle 001 \rangle_i \langle 001 \rangle_{i'} \right] \left[\langle 100 \rangle'' \right]^2$$

since sampling from i th and i'th first-stage unit is carried out independently.

$$E_2 \left[\langle 001 \rangle_i \langle 001 \rangle_{i'} \right] = E_2 \langle 001 \rangle_i \; E_2 \langle 001 \rangle_{i'} = \langle 001 \rangle'_i \langle 001 \rangle'_{i'}$$

$$\therefore\ E(T_1^2) = \Big[\ \frac{1}{nm}\ \langle 002\rangle'' + \frac{m-1}{nm}\langle (001)(001)\rangle''$$

$$+ \frac{n-1}{nN(N-1)M}\Big(\sum_{i\neq i'}^{N}\langle 002\rangle'_{ii'} + (M-1)\sum_{i\neq i'}^{N}\langle (001)(001)\rangle'_{ii'}\Big)\Big]\Big[\langle 100\rangle''\Big]^2$$

converting these symmetric means to standard notations , we have

$$E(T_1^2) = \Big[\ \frac{1}{nm\,NM}\sum_{i}^{N}\sum_{j}^{M} r_{ij}^2 + \frac{m-1}{nm}\Big(\frac{M}{N(M-1)}\sum_{i}^{N} r_{iM}^2 - \frac{1}{NM(M-1)}\sum_{i}^{N}\sum_{j}^{M} r_{ij}^2\Big)$$

$$+ \frac{n-1}{nM}\Big(\frac{NM\,\bar{r}_{NM}^2}{N-1} - \frac{\sum_{i}^{N}\sum_{j}^{M} r_{ij}^2}{NM(N-1)} - \frac{M-1}{N-1}\langle (001)(001)\rangle''\Big)\Big]\bar{X}_{NM}^2$$

This on further simplication gives

$$E(T_1^2) = \Big(\frac{1}{m} - \frac{1}{M}\Big)\frac{1}{nN}\sum_{i}^{N}\frac{1}{M-1}\sum_{j}^{M}(r_{ij} - \bar{r}_{iM})^2\,\bar{X}_{NM}^2$$

$$+ \Big(\frac{1}{n} - \frac{1}{N}\Big)\frac{1}{N-1}\sum_{i}^{N}(\bar{r}_{iM} - \bar{r}_{NM})^2\,\bar{X}_{NM}^2 + \bar{r}_{NM}^2\,\bar{X}_{NM}^2 \quad ..(3.8)$$

$$\text{Also}\quad \big[E(T_1)\big]^2 = \bar{r}_{NM}^2\,\bar{X}_{NM}^2 \quad ..(3.9)$$

Substituting (3.8) and (3.9) in (3.7), we get

$$V(T_1) = \Big(\frac{1}{m} - \frac{1}{M}\Big)\frac{1}{nN}\sum_{i}^{N}\frac{1}{M-1}\sum_{j}^{M}(r_{ij} - \bar{r}_{iM})^2\,\bar{X}_{NM}^2$$

$$+ \Big(\frac{1}{n} - \frac{1}{N}\Big)\frac{1}{N-1}\sum_{i}^{N}(\bar{r}_{iM} - \bar{r}_{NM})^2\,\bar{X}_{NM}^2$$

$$= \Big[\Big(\frac{1}{m} - \frac{1}{M}\Big)\frac{1}{nN}\sum_{i}^{N} S_{ir}^2 + \Big(\frac{1}{n} - \frac{1}{N}\Big)S_{br}^2\Big]\bar{X}_{NM}^2 \quad ...(3.10)$$

where

$$S_{ir}^2 = \frac{1}{M-1}\sum_{j}^{M}(r_{ij} - \bar{r}_{iM})^2$$

and

$$S_{br}^2 = \frac{1}{N-1}\sum_{i}^{N}(\bar{r}_{iM} - \bar{r}_{NM})^2$$

3.2. Variance of the Unbiased Estimator T_2

$$T_2 = \bar{r}_{nm}\bar{X}_{NM} + (\bar{y}_{nm} - \bar{r}_{nm}\bar{x}_{nm}) + (\frac{1}{n} - \frac{1}{N})s_{brx} + \frac{1}{nN}(\frac{1}{m} - \frac{1}{M})\sum_i^n s_{irx}$$

which can also be written as

$$T_2 = \bar{r}_{nm}\bar{X}_{NM} + \left[1 + \frac{M-m}{NM(m-1)}\right]\bar{y}_{nm} - \frac{n(N-1)}{N(n-1)}\bar{r}_{nm}\bar{x}_{nm}$$

$$- \frac{1}{nN}\left[\frac{M-m}{M(m-1)} - \frac{N-n}{n-1}\right]\sum_i^n \bar{r}_{im}\bar{x}_{im}$$

Now $V(T_2) = E(T_2^2) - \left[E(T_2)\right]^2$ ... (3.11)

$$= E\left[\bar{r}_{nm}\bar{X}_{NM}\right]^2 + \left[1 + \frac{M-m}{NM(m-1)}\right]^2 E(\bar{y}_{nm})^2$$

$$+ \left[\frac{n(N-1)}{N(n-1)}\right]^2 E(\bar{r}_{nm}\bar{x}_{nm})^2 + \frac{1}{n^2N^2}\left[\frac{M-m}{M(m-1)}\frac{N-n}{n-1}\right]^2 \sum_i^n \bar{x}_{im}^2\bar{r}_{im}^2$$

$$+2\left[1 + \frac{M-m}{NM(m-1)}\right]E(\bar{y}_{nm}\bar{r}_{nm}\bar{X}_{NM}) - \frac{2n(N-1)}{N(n-1)}E(\bar{r}_{nm}^2\bar{x}_{nm}\bar{X}_{NM})$$

$$- \frac{2}{nN}\left[\frac{M-m}{M(m-1)} - \frac{N-n}{n-1}\right]E\left[\sum_i^n \bar{r}_{im}\bar{x}_{im}\bar{r}_{nm}\bar{X}_{NM}\right]$$

$$-2\left[1 + \frac{M-m}{NM(m-1)}\right]\frac{n(N-1)}{N(n-1)}E(\bar{y}_{nm}\bar{x}_{nm}\bar{r}_{nm})$$

$$-2\left[1 + \frac{M-m}{NM(m-1)}\right]\frac{1}{nN}\left[\frac{M-m}{M(m-1)} - \frac{N-n}{n-1}\right]E\left(\sum_i^n \bar{r}_{im}\bar{x}_{im}\bar{y}_{nm}\right)$$

$$+\frac{2(N-1)}{N^2(n-1)}\left[\frac{M-m}{M(m-1)} - \frac{N-n}{n-1}\right]E\left(\sum_i^n \bar{r}_{im}\bar{x}_{im}\bar{r}_{nm}\bar{x}_{nm}\right) - \bar{Y}_{NM}^2$$

.... (3.12)

where

$$\text{(i)}\ E(\bar{r}_{nm}\bar{X}_{NM})^2 = \left[\left(\frac{1}{nm} - \frac{n-1}{nM(N-1)}\right)\langle 002\rangle'' + \left(\frac{m-1}{nm} - \frac{(n-1)(M-1)}{nM(N-1)}\right)\langle(001)(001)\rangle'' + \frac{(n-1)N}{(N-1)n}\langle 001\rangle''\langle 001\rangle''\right]\left[\langle 100\rangle''\right]^2$$

$$\text{(ii)}\ E(\bar{y}_{nm}^2) = \left[\frac{1}{nm} - \frac{n-1}{nM(N-1)}\right]\langle 020\rangle'' + \left[\frac{(m-1)}{nm} - \frac{(n-1)(M-1)}{nM(N-1)}\right]\langle(010)(010)\rangle''$$

$$+ \frac{(n-1)N}{(N-1)n}\left[\langle 010\rangle''\right]^2$$

$$\text{(iii)}\quad E\left(\sum_i^n \bar{r}_{im}\bar{x}_{im}\bar{y}_{nm}\right) = \left[\frac{1}{m^2} - \frac{n-1}{mM(N-1)}\right]\langle 020\rangle''$$

$$+\left[\frac{(n-1)N}{m(N-1)} - \frac{(m-1)(n-1)N}{m(N-1)(M-1)}\right]\left[\langle 010\rangle''\right]^2 + \frac{(m-1)(n-1)M}{m(N-1)(M-1)}\sum_i^N \langle 100\rangle'_i \cdot \langle 001\rangle'_i \cdot \langle 010\rangle''$$

$$+\left[\frac{(m-1)}{m^2} - \frac{(n-1)(m-1)}{mM(N-1)}\right]\left[\langle(100)(011)\rangle'' + \langle(110)\ (001)\rangle''\right]$$

$$+\left[\frac{m-1}{m^2} - \frac{(n-1)\ (M-1)}{mM(N-1)}\right]\langle(010)\ \ (010)\rangle''$$

$$+\left[\frac{(m-1)\ (m-2)}{m^2} - \frac{(m-1)\ (n-1)\ (M-2)}{mM(\ N-1)}\right]\langle(100)\ (010)\ (001)\rangle''$$

$$\text{(iv)}\quad E(\bar{y}_{nm}\bar{x}_{nm}\bar{r}_{nm}) = \frac{1}{n^2}\left[\frac{1}{m^2} - \frac{3\ (n-1)}{mM(N-1)}\right]\langle 020\rangle''$$

$$+\frac{1}{n^2}\left[\frac{(n-1)N}{m(N-1)} - \frac{(m-1)(n-1)N}{m(N-1)(M-1)}\right]\left[\langle 010\rangle''\langle 010\rangle'' + \langle 011\rangle''\langle 100\rangle'' + \langle 110\rangle''\langle 001\rangle''\right]$$

$$+\frac{(m-1)(n-1)\ M}{n^2 m(N-1)(M-1)}\left[\sum_i^N\langle 100\rangle'_i\langle 001\rangle'_i\langle 010\rangle'' + \sum_i^N\langle 100\rangle'_i\langle 010\rangle'_i\langle 001\rangle'' + \sum_i^N\langle 010\rangle'_i\cdot\langle 001\rangle'_i\cdot\langle 100\rangle''\right]$$

$$+\frac{1}{n^2}\left[\frac{m-1}{m^2} - \frac{2(m-1)(n-1)}{mM(\ N-1)} - \frac{(n-1)(M-1)}{mM(N-1)}\right]\left[\langle(010)(101)\rangle'' + \langle(100)(011)\rangle'' + \langle(001)(110)\rangle''\right]$$

$$+\frac{1}{n^2}\left[\frac{(m-1)(m-2)}{m^2} - \frac{3(m-1)(n-1)(M-2)}{mM(\ N-1)}\right]\langle(100)\ (010)\ (001)\rangle''$$

$$+\frac{(n-1)(n-2)}{n^2(N-1)(N-2)}\left[N^2\langle 100\rangle''\langle 010\rangle''\langle 001\rangle'' - 3\sum_i^N\langle 100\rangle'_i\langle 001\rangle'_i\langle 010\rangle'' + \frac{2}{N}\sum_i^N\langle 100\rangle'_i\langle 001\rangle'_i\langle 010\rangle'_i\right]$$

$$\text{(v)}\quad E\left(\sum_i^n \bar{r}_{im}\bar{x}_{im}\right)^2 = \left[\frac{n}{m^3} - \frac{n(n-1)}{m^2M(N-1)}\right]\langle 020\rangle''$$

$$+\left[\frac{2n(m-1)}{m^3} - \frac{n(n-1)(M-1)}{m^2M(N-1)} - \frac{n(n-1)(m-1)^2}{m^2M(N-1)(M-1)}\right]\langle(010)\ (010)\rangle''$$

$$+\left[\frac{4n(m-1)(m-2)}{m^3}-\frac{2n(n-1)(m-1)(M-2)}{m^2M(N-1)}-\frac{2n(n-1)(m-1)^2(M-2)}{m^2M(N-1)(M-1)}\right]\langle(100)(010)(001)\rangle''$$

$$+\left[\frac{n(n-1)N}{m^2(N-1)}-\frac{2n(n-1)(m-1)N}{m^2(N-1)(M-1)}\right]\left[\langle 010\rangle''\right]^2$$

$$+\left[\frac{n(m-1)(m-2)(m-3)}{m^3}-\frac{n(n-1)(m-1)^2(M-2)(M-3)}{m^2M(N-1)(M-1)}\right]\langle(100)(001)(100)(001)\rangle''$$

$$+\left[\frac{2n(m-1)}{m^3}-\frac{2n(n-1)(m-1)}{m^2(N-1)M}\right]\left[\langle(110)(001)\rangle''+\langle(011)(100)\rangle''\right]$$

$$+\frac{2n(n-1)(m-1)M}{m^3(N-1)(M-1)}\sum_{i}^{n}\langle 100\rangle'_i\langle 001\rangle'_i\langle 010\rangle''+\frac{n(n-1)(m-1)^2N}{m^2(N-1)}\left[\langle(100)(001)\rangle''\right]^2$$

$$+\left[\frac{n(m-1)(m-2)}{m^3}-\frac{n(n-1)(m-1)^2(M-2)}{m^2M(N-1)(M-1)}\right]\left[\langle(100)(100)(002)\rangle''+\langle(001)(001)(200)\rangle''\right]$$

$$+\left[\frac{n(m-1)}{m^3}-\frac{n(n-1)(m-1)^2}{m^2(N-1)(M-1)M}\right]\langle(200)(002)\rangle''$$

(vi) $$E(\bar{y}_{nm}\bar{x}_{nm}\bar{X}_{NM})=\left[\frac{1}{nm}-\frac{n-1}{nM(N-1)}\right]\langle 011\rangle''\langle 100\rangle''$$

$$+\left[\frac{m-1}{nm}-\frac{(n-1)(M-1)}{nM(N-1)}\right]\langle(010)(001)\rangle''\langle 100\rangle''+\frac{(n-1)N}{(N-1)n}\langle 100\rangle''\langle 010\rangle''\langle 001\rangle''$$

(vii) $$E\left(\sum_{i}^{n}\bar{y}_{im}\bar{x}_{im}\bar{y}_{nm}\bar{X}_{NM}\right)=\left[\frac{1}{m^2}-\frac{n-1}{mM(N-1)}\right]\langle 011\rangle''\langle 100\rangle''$$

$$+\left[\frac{(n-1)N}{m(N-1)}-\frac{(m-1)(n-1)N}{m(N-1)(M-1)}\right]\langle 010\rangle''\langle 001\rangle''\langle 100\rangle''+\frac{(m-1)(n-1)N}{m(N-1)(M-1)}\sum_{i}^{N}\langle 100\rangle'_i\cdot\langle 001\rangle'_i\cdot\langle 100\rangle''\langle 001\rangle''$$

$$+\left[\frac{2(m-1)}{m^2}-\frac{(n-1)(M-1)}{mM(N-1)}-\frac{(m-1)(n-1)}{mM(N-1)}\right]\langle(001)(010)\rangle''\langle 100\rangle''$$

$$+\left[\frac{m-1}{m^2}-\frac{(m-1)(n-1)}{mM(N-1)}\right]\langle(100)(002)\rangle''\langle 100\rangle''$$

$$+\left[\frac{(m-1)(m-2)}{m^2}-\frac{(m-1)(n-1)(M-2)}{mM(N-1)}\right]\langle(100)(001)(001)\rangle''\langle 100\rangle''$$

(viii) $$E(\bar{Y}_{nm}^2 \bar{X}_{nm} \bar{X}_{NM}) = \frac{1}{n^2}\left[\frac{1}{m^2} - \frac{3(n-1)}{mM(N-1)}\right]\langle 011\rangle'' \langle 100\rangle''$$

$$+ \frac{2}{n^2}\left[\frac{(n-1)N}{m(N-1)} - \frac{(m-1)(n-1)N}{m(N-1)(M-1)}\right]\langle 010\rangle''\langle 001\rangle''\langle 100\rangle'' + \frac{2(m-1)(n-1)M}{n^2 m(N-1)(M-1)} \sum_i^N \langle 100\rangle'_i \cdot \langle 001\rangle'_i \cdot \langle 100\rangle'' \langle 001\rangle''$$

$$+ \frac{1}{n^2}\left[\frac{(n-1)N}{m(N-1)} - \frac{(m-1)(n-1)N}{m(N-1)(M-1)}\right]\langle 002\rangle''\langle 100\rangle''\langle 100\rangle'' + \frac{(n-1)(m-1)M}{n^2 m(N-1)(M-1)} \sum_i^N \langle 001\rangle'_i \langle 001\rangle'_i \langle 100\rangle'' \langle 100\rangle''$$

$$+ \frac{2}{n^2}\left[\frac{m-1}{m^2} - \frac{2(n-1)(m-1)}{mM(N-1)} - \frac{(n-1)(M-1)}{mM(N-1)}\right] \langle (001)\,(010)\rangle''\langle 100\rangle''$$

$$+ \frac{1}{n^2}\left[\frac{m-1}{m^2} - \frac{2(m-1)(n-1)}{mM(N-1)} - \frac{(n-1)(M-1)}{mM(N-1)}\right] \langle (100)\,(002)\rangle''\langle 100\rangle''$$

$$+ \frac{1}{n^2}\left[\frac{(m-1)(m-2)}{m^2} - \frac{3(m-1)(n-1)(M-2)}{mM(N-1)}\right]\langle (100)\,(001)\,(001)\rangle''\langle 100\rangle''$$

$$+ \frac{(n-1)(n-2)}{n^2(N-1)(N-2)}\left[N^2\langle 100\rangle''\langle 001\rangle''\langle 100\rangle''\langle 001\rangle'' - 3\sum_i^N \langle 100\rangle'_i\langle 001\rangle'_i\langle 100\rangle''\langle 001\rangle'' + \frac{2}{N}\sum_i^N \langle 100\rangle'_i\langle 001\rangle'_i\langle 001\rangle'_i\langle 100\rangle''\right]$$

(ix) $$E\left(\sum_i^n Y_{im} X_{im} Y_{nm} X_{nm}\right) = \left(\frac{1}{nm^3} - \frac{n-1}{nm^2M(N-1)}\right)\langle 020\rangle''$$

$$+ \frac{1}{n}\left[\frac{2(m-1)}{m^3} - \frac{(n-1)(M-1)}{m^2M(N-1)} - \frac{(m-1)^2(n-1)}{m^2M(N-1)(M-1)} - \frac{4(m-1)(n-1)}{m^2M(N-1)}\right]\langle (010)\,(101)\rangle''$$

$$+ \frac{1}{n}\left[\frac{4(m-1)(m-2)}{m^3} - \frac{6(n-1)(m-1)(M-2)}{m^2M(N-1)} - \frac{2(m-1)^2(n-1)(M-2)}{m^2(N-1)M(M-1)}\right]\langle (100)(010)(001)\rangle''$$

$$+ \frac{1}{n}\left[\frac{(n-1)N}{m^2(N-1)} - \frac{2(n-1)(m-1)N}{m^2(N-1)(M-1)}\right]\left[\langle 010\rangle''\right]^2$$

$$+\left[\frac{(m-1)(m-2)(m-3)}{n\,m^3} - \frac{(m-1)^2(n-1)(M-2)(M-3)}{nm^2(N-1)M(M-1)}\right]\langle (100)(001)(100)(001)\rangle''$$

$$+\frac{1}{n}\left[\frac{-2(m-1)}{m^3}-\frac{5(n-1)(m-1)}{m^2M(N-1)}\right]\left[\langle(110)(001)\rangle''+\langle(100)(011)\rangle''\right]$$

$$+\left[\frac{2(n-1)(m-1)M}{nm^2(N-1)(M-1)}-\frac{3(n-1)(n-2)}{nm(N-1)(N-2)}+\frac{3(m-1)(n-1)(n-2)}{nm(N-1)(N-2)(M-1)}\right]\sum_i^N\langle 100\rangle'_i\langle 001\rangle'_i\langle 010\rangle''$$

$$+\frac{(m-1)^2(n-1)N}{nm^2(N-1)}\left[\langle(100)(001)\rangle''\right]^2$$

$$+\frac{1}{n}\left[\frac{-m-1}{m^3}-\frac{(m-1)^2(n-1)}{m^2(N-1)(M-1)}-\frac{2(n-1)(m-1)}{m^2M(N-1)}\right]\langle(200)(002)\rangle''$$

$$+\frac{1}{n}\left[\frac{-(m-1)(m-2)}{m^3}-\frac{(m-1)^2(n-1)(M-2)}{m^2(N-1)M(M-1)}-\frac{(m-1)(n-1)(M-2)}{m^2M(N-1)}\right]\left[\langle(001)(001)(200)\rangle''+\langle(100)(100)(002)\rangle''\right]$$

$$+\frac{(m-1)(m-2)(n-1)N}{nm^2(N-1)}\langle 100\rangle''\langle(100)(001)(001)\rangle''$$

$$-\frac{(m-1)(m-2)(n-1)}{nm^2N(N-1)}\sum_i^N\langle 100\rangle'_i\langle(100)(001)(001)\rangle'_i$$

$$+\frac{(m-1)(n-1)M}{nm^2(N-1)(M-1)}\left[\sum_i^N\langle 100\rangle'_i\langle 002\rangle'_i\langle 100\rangle''+2\sum_i^N\langle 010\rangle'_i\langle 001\rangle'_i\langle 100\rangle''+\sum_i^N\langle 001\rangle'_i\langle 200\rangle'_i\langle 001\rangle''+2\sum_i^N\langle 100\rangle'_i\langle 010\rangle'_i\langle 001\rangle''\right]$$

$$+\left[\frac{(n-1)N}{nm^2(N-1)}-\frac{3(n-1)(m-1)N}{nm^2(N-1)(M-1)}\right]\left[\langle 100\rangle''\langle 011\rangle''+\langle 110\rangle''\langle 001\rangle''\right]$$

$$-\frac{n-1}{nm^2N(N-1)}\left[\sum_i^N\langle 100\rangle'_i\langle 011\rangle'_i+\sum_i^N\langle 001\rangle'_i\langle 110\rangle'_i\right]$$

$$+\frac{(m-1)(m-2)(n-1)N}{nm^2(N-1)}\langle 001\rangle''\langle(100)(100)(001)\rangle''-\frac{(m-1)(m-2)(n-1)}{nm^2N(N-1)}\sum_i^N\langle(100)(100)(001)\rangle'_i\langle 001\rangle'_i$$

$$+\frac{(n-1)(n-2)N^2}{nm(N-1)(N-2)}\langle 100\rangle''\langle 010\rangle''\langle 001\rangle''$$

$$+\left[\frac{2(n-1)(n-2)}{nm(N-1)(N-2)} - \frac{2(m-1)(n-1)(n-2)}{nm(N-1)(N-2)(M-1)} \right] \frac{1}{N} \sum_i^N \langle 100\rangle_i' \langle 010\rangle_i' \langle 001\rangle_i'$$

$$+\frac{(m-1)(n-1)(n-2)}{nm(N-1)(N-2)} \left[N^2 \langle 001\rangle'' \langle 100\rangle'' \langle (100)(001)\rangle'' \right.$$

$$+\frac{2M}{N(M-1)} \sum_i^N \langle 100\rangle_i' \langle 001\rangle_i' \langle 100\rangle_i' \langle 001\rangle_i'$$

$$\left. -\frac{3M}{N(M-1)} \left(\sum_i^N \langle 100\rangle_i' \langle 001\rangle_i' \right)^2 \right]$$

(x) $$E(\bar{r}_{nm}^{\,2}\, \bar{x}_{nm}^{\,2}) = \frac{1}{n^3} \left[\frac{1}{m^3} - \frac{n-1}{m^2 M(N-1)} \right] \langle 020\rangle''$$

$$+\frac{1}{n^3}\left[\frac{2(m-1)}{m^3} - \frac{(n-1)(M-1)}{m^2 M(N-1)} - \frac{(m-1)^2(n-1)}{m^2(N-1)M(M-1)} - \frac{8(n-1)(m-1)}{m^2 M(N-1)} \right] \langle (010)(101)\rangle''$$

$$+\frac{1}{n^3}\left[\frac{4(m-1)(m-2)}{m^3} - \frac{10(n-1)(m-1)(M-2)}{m^2 M(N-1)} - \frac{2(m-1)^2(n-1)(M-2)}{m^2(N-1)M(M-1)} \right] \langle (100)(010)(001)\rangle''$$

$$+\frac{1}{n^3}\left[\frac{(n-1)N}{m^2(N-1)} - \frac{2(n-1)(m-1)N}{m^2(N-1)(M-1)} \right] \left[\langle 010\rangle'' \right]^2$$

$$+\frac{1}{n^3}\left[\frac{(m-1)(m-2)(m-3)}{m^3} - \frac{(m-1)^2(n-1)(M-2)(M-3)}{m^2(N-1)M(M-1)} \right] \langle (100)(001)(100)(001)\rangle''$$

$$+\frac{1}{n^3}\left[\frac{-2(m-1)}{m^3} - \frac{8(n-1)(m-1)}{m^2 M(N-1)} \right] \left[\langle (110)(001)\rangle'' + \langle (100)(011)\rangle'' \right]$$

$$+\frac{2(n-1)(m-1)M}{n^3 m^2(N-1)(M-1)} \sum_i^N \langle 100\rangle_i' \langle 010\rangle'' \langle 001\rangle_i' + \frac{(m-1)^2 N(n-1)}{n^3 m^2 (N-1)} \left[\langle (100)(001)\rangle'' \right]^2$$

$$+\frac{1}{n^3}\left[\frac{(m-1)(m-2)}{m^3} - \frac{(m-1)^2(n-1)(M-2)}{m^2(N-1)M(M-1)} - \frac{2(m-1)(n-1)(M-2)}{m^2 M(N-1)} \right]$$

$$\left[\langle (100)(100)(002)\rangle'' + \langle (001)(001)(200)\rangle'' \right]$$

$$+ \frac{1}{n^3} \Big[\frac{m-1}{m^3} - \frac{(m-1)^2(n-1)}{m^2(N-1)M(M-1)} - \frac{4(m-1)(n-1)}{m^2M(N-1)} \Big] \langle (200)(002) \rangle''$$

$$+ \frac{2(m-1)(m-2)(n-1)N}{n^3 m^2 (N-1)} \langle 100 \rangle'' \langle (100)(001)(001) \rangle''$$

$$- \frac{2(n-1)(m-1)(m-2)}{n^3 m^2 N(N-1)} \sum_i^N \langle 100 \rangle'_i \langle (100)(001)(001) \rangle'_i$$

$$+ \Big[\frac{2(n-1)N}{n^3 m^2 (N-1)} - \frac{6(m-1)(n-1)N}{n^3 m^2 (N-1)(M-1)} \Big] \Big[\langle 100 \rangle'' \langle 011 \rangle'' + \langle 001 \rangle'' \langle 110 \rangle'' \Big]$$

$$+ \frac{2(m-1)(n-1)M}{n^3 m^2 (N-1)(M-1)} \Big[\sum_i^N \langle 100 \rangle'_i \langle 002 \rangle'_i \langle 100 \rangle'' + 2\sum_i^N \langle 010 \rangle'_i \langle 001 \rangle'_i \langle 100 \rangle''$$
$$+ \sum_i^N \langle 001 \rangle'_i \langle 200 \rangle'_i \langle 001 \rangle'' + 2\sum_i^N \langle 100 \rangle'_i \langle 010 \rangle'_i \langle 001 \rangle'' \Big]$$

$$- \frac{2(n-1)}{n^3 m^2 (N-1)N} \Big[\sum_i^N \langle 100 \rangle'_i \langle 011 \rangle'_i + \sum_i^N \langle 100 \rangle'_i \langle 110 \rangle'_i \Big]$$

$$+ \frac{2(m-1)(m-2)(n-1)N}{n^3 m^2 (N-1)} \langle 001 \rangle'' \langle (100)(100)(001) \rangle''$$

$$- \frac{2(m-1)(m-2)(n-1)}{n^3 m^2 N(N-1)} \sum_i^N \langle 001 \rangle'_i \langle (100)(100)(001) \rangle'_i$$

$$+ \frac{4(n-1)(n-2)}{n^3 m(N-1)(N-2)} \Big[N^2 \langle 100 \rangle'' \langle 010 \rangle'' \langle 001 \rangle'' - 3 \sum_i^N \langle 100 \rangle'_i \langle 001 \rangle'_i \langle 010 \rangle''$$
$$+ \frac{2}{N} \sum_i^N \langle 100 \rangle'_i \langle 010 \rangle'_i \langle 001 \rangle'_i \Big]$$

$$+ \frac{(n-1)(n-2)(m-1)}{n^3 m(N-1)(N-2)} \Big[\frac{4}{N} \sum_{i \neq i' \neq i''}^N \langle (100)(001) \rangle'_i \langle 001 \rangle'_{i'} \langle 100 \rangle'_{i''}$$
$$+ \frac{1}{N} \sum_{i \neq i' \neq i''}^N \langle (100)(100) \rangle'_{i'} \langle 001 \rangle'_i \langle 001 \rangle'_{i''}$$
$$+ \frac{1}{N} \sum_{i \neq i' \neq i''}^N \langle (001)(001) \rangle'_i \langle 100 \rangle'_{i'} \langle 100 \rangle'_{i''} \Big]$$

$$+ \frac{(n-1)(n-2)}{n^3 m(N-1)(M-1)} \Big[N^2 \langle 200\rangle'' \langle 001\rangle'' \langle 001\rangle'' - 3 \sum_i^N \langle 001\rangle'_i \langle 001\rangle'_i \langle 200\rangle'' + \frac{2}{N} \sum_i^N \langle 001\rangle'_i \langle 001\rangle'_i \langle 200\rangle'_i \Big]$$

$$+ \frac{(n-1)(n-2)}{n^3 m(N-1)(N-2)} \Big[N^2 \langle 002\rangle'' \langle 100\rangle'' \langle 100\rangle'' - 3 \sum_i^N \langle 100\rangle'_i \langle 100\rangle'_i \langle 002\rangle'' + \frac{2}{N} \sum_i^N \langle 100\rangle'_i \langle 100\rangle'_i \langle 002\rangle''_i \Big]$$

$$+ \frac{(n-1)}{n^3 m^2 N(N-1)} \Big[\sum_i^N \langle 200\rangle'_i \sum_i^N \langle 002\rangle'_i - \sum_i^N \langle 200\rangle'_i \langle 002\rangle'_i \Big]$$

$$+ \frac{(n-1)(m-1)}{n^3 m^2 N(N-1)} \Big[\sum_i^N \langle (100)(100)\rangle'_i \sum_i^N \langle 002\rangle'_i - \sum_i^N \langle (100)(100)\rangle'_i \langle 002\rangle'_i \Big]$$

$$+ \frac{(n-1)(m-1)}{n^3 m^2 N(N-1)} \Big[\sum_i^N \langle (001)(001)\rangle'_i \sum_i^N \langle 200\rangle'_i - \sum_i^N \langle (001)(001)\rangle'_i \langle 200\rangle'_i \Big]$$

$$+ \frac{(n-1)(m-1)^2}{n^3 m^2 N(N-1)} \Big[\sum_i^N \langle (100)(100)\rangle_i \sum_i^N \langle (001)(001)\rangle'_i - \sum_i^N \langle (100)(100)\rangle'_i \langle (001)(001)\rangle'_i \Big]$$

$$+ \frac{n-1}{n^3 m^2 N(N-1)} \Big[\sum_i^N \langle 101\rangle'_i \sum_i^N \langle 101\rangle'_i - \sum_i^N \langle 101\rangle'_i \langle 101\rangle'_i \Big]$$

$$+ \frac{2(n-1)(m-1)}{n^3 m^2 N(N-1)} \Big[\sum_i^N \langle (100)(001)\rangle'_i \sum_i^N \langle 101\rangle'_i - \sum_i^N \langle (100)(001)\rangle'_i \langle 101\rangle'_i \Big]$$

$$+ \frac{(n-1)(m-1)^2}{n^3 m^2 N(N-1)} \Big[\sum_i^N \langle (100)(001)\rangle'_i \sum_i^N \langle (100)(001)\rangle'_i - \sum_i^N \langle (100)(001)\rangle'^2_i \Big]$$

$$+ \frac{(n-1)(n-2)(n-3)}{n^3 N(N-1)(N-2)(N-3)} \Big[\sum_{i \neq i' \neq i'' \neq i'''}^N \langle 100\rangle'_{i'} \langle 001\rangle'_i \langle 100\rangle'_{i''} \langle 001\rangle'_{i'''} \Big]$$

These values can be substituted in (3.12) to obtain the exact variance of T_2. In the following sections, the expressions for variance of T_1 and T_2 have been given for the case when N and M are large.

3.3. Variance of T_1 When both N and M are large

From (3.10), the variance of T_1 when both N and M are large, is given by

$$V(T_1) = \left[\frac{1}{nmN} \sum_{i}^{N} S_{ir}^2 + \frac{1}{n} S_{br}^2 \right] \bar{X}_{NM}^2 \qquad \ldots (3.13)$$

3.4. Variance of T_2 when both N and M are large

It is given by

$$V(T_2) = E(\bar{r}_{nm}\bar{X}_{NM})^2 + E(\bar{y}_{nm})^2 + \frac{n^2}{(n-1)^2} E(\bar{r}_{nm}\bar{x}_{nm})^2$$

$$+ \frac{1}{n^2(n-1)^2} E(\sum_{i}^{n} \bar{r}_{im}\bar{x}_{im})^2 + 2E(\bar{y}_{nm}\bar{r}_{nm}\bar{X}_{NM})$$

$$- \frac{2n}{n-1} E(\bar{r}_{nm}^2 \bar{x}_{nm}\bar{X}_{NM}) + \frac{2}{n(n-1)} E(\sum_{i}^{n} \bar{r}_{im}\bar{x}_{im}\bar{r}_{nm}\bar{X}_{NM})$$

$$- \frac{2n}{n-1} E(\bar{y}_{nm}\bar{x}_{nm}\bar{r}_{nm}) + \frac{2}{n(n-1)} E(\sum_{i}^{n} \bar{r}_{im}\bar{x}_{im}\bar{y}_{nm})$$

$$- \frac{2}{(n-1)^2} E(\sum_{i}^{n} \bar{r}_{im}\bar{x}_{im}\bar{r}_{nm}\bar{x}_{nm}) - \bar{\bar{Y}}_{NM}^2 \qquad \ldots (3.14)$$

where

(i) $$E(\bar{y}_{nm}^2) = \frac{1}{nm}\frac{1}{N}\sum_{i}^{N} S_{iy}^2 + \frac{1}{n} S_{by}^2 + \bar{\bar{Y}}_{NM}^2$$

(ii) $$E(\bar{r}_{nm}^2 \bar{X}_{NM}^2) = \left[\frac{1}{nmN}\sum_{i}^{N} S_{ir}^2 + \frac{1}{n} S_{br}^2 + \bar{\bar{r}}_{NM}^2 \right] \bar{\bar{X}}_{NM}^2$$

(iii) $$E(\sum_{i}^{n} \bar{r}_{im}\bar{x}_{im}\bar{y}_{nm}) = \frac{1}{mN}\sum_{i}^{N} S_{iy}^2 + \frac{1}{m} S_{by}^2 + \frac{n}{m} \bar{\bar{Y}}_{NM}^2$$

$$+ \frac{(m-1)(n-1)}{mN}\sum_{i}^{N} \bar{X}_{i.}\bar{r}_{iM}\bar{\bar{Y}}_{NM} - \frac{m-1}{m^2 N}\sum_{i}^{N} S_{irxy} + \frac{m-1}{mN}\sum_{i}^{N} \bar{r}_{iM}\bar{X}_{i.}\bar{Y}_{i.}$$

where $$S_{irxy} = \frac{1}{M-1}\sum_{j}^{M}(x_{ij} - \bar{X}_{i.})(y_{ij} - \bar{Y}_{i.})(r_{ij} - \bar{r}_{iM})$$

(iv) $$E(\bar{y}_{nm}\bar{x}_{nm}\bar{r}_{nm}) = \frac{1}{n^2 mN}\sum_{i}^{N} S_{iy}^2 + \frac{1}{n^2 m} S_{by}^2 + \frac{1}{nm}\bar{\bar{Y}}_{NM}^2$$

$$- \frac{m-1}{n^2m^2N}\sum_i^N S_{irxy} + \left[\frac{(n-1)(n-2)}{n^2} + \frac{2(m-1)(n-1)}{n^2m} \right] \bar{X}_{NM}\bar{Y}_{NM}\bar{r}_{NM}$$

$$+ \frac{n-1}{n^2m}\left[\frac{1}{NM}\sum_i^N\sum_j^M y_{ij}r_{ij}\bar{X}_{NM} + \frac{1}{NM}\sum_i^N\sum_j^M x_{ij}y_{ij}\bar{r}_{NM} \right]$$

$$- \frac{(m-1)(n-1)}{n^2m} S_{brxy} + \frac{m-1}{nmN}\cdot\sum_i^N \bar{X}_{i.}\bar{r}_{iM}\bar{Y}_{i.}$$

where $S_{brxy} = \frac{1}{N-1}\sum_i^N (\bar{X}_{i.} - \bar{X}_{NM})(\bar{Y}_{i.} - \bar{Y}_{NM})(\bar{r}_{iM} - \bar{r}_{NM})$

(v) $E(\bar{y}_{nm}\bar{r}_{nm}\bar{\bar{X}}_{NM}) = \left[\frac{1}{nmN}\sum_i^N S_{iry} + \frac{1}{n} S_{bry} + \bar{Y}_{NM}\bar{r}_{NM} \right]\bar{X}_{NM}$

(vi) $E(\sum_i^n \bar{r}_{im}\bar{x}_{im}\bar{r}_{nm}\bar{X}_{NM}) = \left[\frac{1}{m^2N}\sum_i^N S_{iry} + \frac{1}{m} S_{bry} + \frac{n}{m}\bar{Y}_{NM}\bar{r}_{NM} \right.$

$$+ \frac{(m-1)(n-1)}{mN}\sum_i^N \bar{X}_{i.}\bar{r}_{iM}\bar{r}_{NM} + \frac{m-1}{mN}\sum_i^N \bar{X}_{i.} S^2_{ir}$$

$$\left. + \frac{(m-1)^2}{m^2N}\sum_i^N \bar{X}^2_{i.}\bar{r}^2_{iM} + \frac{m-1}{m^2N}\sum_i^N \bar{Y}_{i.}\bar{r}_{iM} \right]\bar{X}_{NM}$$

(vii) $E(\sum_i^n \bar{r}_{im}\bar{x}_{im})^2 = \frac{n(2m-1)}{m^3N}\sum_i^N S^2_{iy} + \frac{n}{m^2} S^2_{by} + \frac{n^2}{m^2}\bar{Y}^2_{NM}$

$$+ \frac{(m-1)n}{m^3N}\sum_i^N \bar{Y}^2_{i.} + \frac{4n(m-1)^2}{m^3N}\sum_i^N \bar{X}_{i.}\bar{Y}_{i.}\bar{r}_{iM} - \frac{2n(m-1)}{m^3N}\sum_i^N S_{irxy}$$

$$+ \frac{n(m-1)(m-2)(m-3)}{m^3N}\sum_i^N \bar{X}^2_{i.}\bar{r}^2_{iM} + \frac{2n(n-1)(m-1)}{m^2N}\sum_i^N \bar{X}_{i.}\bar{r}_{iM}\bar{Y}_{NM}$$

$$+ \frac{n(m-1)(m-2)}{m^3}\left[\frac{1}{NM}\sum_i^N\sum_j^M x^2_{ij}\bar{r}^2_{iM} + \frac{1}{NM}\sum_i^N\sum_j^M r^2_{ij}\bar{X}^2_{i.} \right]$$

$$+ \frac{n(m-1)}{m^3N}\sum_i^N \langle 200\rangle'_i \langle 002\rangle'_i + \frac{(m-1)^2n(n-1)}{m^2}\left[\frac{1}{N}\sum_i^N \bar{X}_{i.}\bar{r}_{iM} \right]^2$$

(viii) $E(\bar{r}^2_{nm}\bar{x}_{nm}\bar{X}_{NM}) = \frac{1}{n^2m^2N}\sum_i^N S_{iry}\bar{X}_{NM} + \frac{2}{n^2m} S_{bry}\bar{X}_{NM}$

$$+\frac{2}{nm}\bar{X}_{NM}\bar{Y}_{NM}\bar{r}_{NM}-\frac{1}{n^2m^2N}\sum_i^N \bar{Y}_{i.}\bar{r}_{iM}\bar{X}_{NM}+\frac{n-1}{n^2mN}\sum_i^N S^2_{ir}\bar{X}^2_{NM}$$

$$+\frac{n-1}{n^2}S^2_{br}\bar{X}^2_{NM}+\frac{(n-1)^2}{n^2}\bar{r}^2_{NM}\bar{X}^2_{NM}+\frac{m-1}{n^2m^2N}\sum_i^N S^2_{ir}\bar{X}_{i.}\bar{X}_{NM}$$

$$+\frac{(m-1)^2}{n^2m^2N}\sum_i^N \bar{X}_{i.}\bar{r}^2_{iM}\bar{X}_{NM}+\frac{2(n-1)(m-1)}{n^2mN}\sum_i^N \bar{X}_{i.}\bar{r}_{iM}\bar{r}_{NM}\bar{X}_{NM}$$

$$\text{(ix)}\quad E\left(\sum_i^n \bar{r}_{im}\bar{x}_{im}\bar{r}_{nm}\bar{x}_{nm}\right)=\frac{(2m-1)}{nm^3N}\sum_i^N S^2_{iy}+\frac{1}{nm^2}S^2_{by}$$

$$+\frac{m-1}{nm^3N}\sum_i^N \bar{Y}^2_{i.}+\frac{1}{m^2}\bar{Y}^2_{NM}-\frac{2(n-1)(m-1)}{nm^2}S_{brxy}$$

$$+\left[\frac{4(m-1)(m-2)}{nm^3}+\frac{4(m-1)}{nm^3}+\frac{2(n-1)(m-1)}{nm^2}\right]\frac{1}{N}\sum_i^N \bar{X}_{i.}\bar{Y}_{i.}\bar{r}_{iM}$$

$$-\frac{2(m-1)}{nm^3N}\sum_i^N S_{irxy}+\left[\frac{(n-1)(n-2)}{nm}+\frac{4(n-1)(m-1)}{nm^2}\right]\bar{X}_{NM}\bar{Y}_{NM}\bar{r}_{NM}$$

$$+\frac{(m-1)(m-2)(m-3)}{nm^3N}\sum_i^N \bar{X}^2_{i.}\bar{r}^2_{iM}+\frac{(m-1)^2(n-1)}{nm^2}\left[\frac{1}{N}\sum_i^N \bar{X}_{i.}\bar{r}_{iM}\right]^2$$

$$+\frac{m-1}{nm^3N}\sum_i^N \langle 200\rangle'_i\langle 002\rangle'_i+\frac{(m-1)(m-2)}{nm^3N}\left[\sum_i^N \bar{r}^2_{iM}\langle 200\rangle'_i+\sum_i^N \bar{X}^2_{i.}\langle 002\rangle'_i\right]$$

$$+\frac{(m-1)(m-2)(n-1)}{nm^2}\left[\frac{1}{N}\sum_i^N \bar{X}_{i.}\bar{r}^2_{iM}\bar{X}_{NM}+\frac{1}{N}\sum_i \bar{r}_{iM}\bar{X}^2_{i.}\bar{r}_{NM}\right]$$

$$+\frac{(m-1)(n-1)}{nm^2}\left[\frac{1}{N}\sum_i^N \bar{X}_{i.}\langle 002\rangle'_i\bar{X}_{NM}+\frac{1}{N}\sum_i^N \bar{r}_{iM}\langle 200\rangle'_i\bar{r}_{NM}\right]$$

$$+\frac{n-1}{nm^2}\left[\frac{1}{NM}\sum_i^N\sum_j^M y_{ij}r_{ij}\bar{X}_{NM}+\frac{1}{NM}\sum_i^N\sum_j^M x_{ij}y_{ij}\bar{r}_{NM}\right]$$

$$+\frac{(m-1)(n-1)(n-2)}{nmN}\sum_i^N \bar{X}_{i.}\bar{r}_{iM}\bar{r}_{NM}\bar{X}_{NM}$$

$$\text{(x)}\quad E(\bar{r}_{nm}\bar{x}_{nm})^2=\frac{2(m)-1}{n^3m^3N}\sum_i^N S^2_{iy}+\frac{2}{n^3m^2}S^2_{by}-\frac{1}{n^3m^3N}\sum_i^N \bar{Y}^2_{i.}$$

$$+ \frac{2}{n^2 m^2} \bar{Y}^2_{NM} - \frac{2(m-1)}{n^3 m^3 N} \sum_i^N S_{ixxy} + \frac{4(m-1)^2 N}{n^3 m^3 N} \sum_i \bar{X}_{i.} \bar{Y}_{i.} \bar{r}_{iM}$$

$$+ \frac{(m-1)(m-2)(m-3)}{n^3 m^3 N} \sum_i^N \bar{X}^2_{i.} \bar{r}^2_{iM} + \frac{4(n-1)(m-1)}{n^3 m^2 N} \sum_i^N \bar{X}_{i.} \bar{r}_{iM} \bar{Y}_{NM}$$

$$+ \frac{(m-1)(m-2)}{n^3 m^3} \left[\frac{1}{N} \sum_i^N \bar{r}^2_{iM} \langle 200 \rangle'_i + \frac{1}{N} \sum_i^N \bar{X}^2_{i.} \langle 002 \rangle'_i \right]$$

$$+ \frac{2(m-1)^2 (n-1)}{n^3 m^2} \left[\frac{1}{N} \sum_i^N \bar{X}_{i.} \bar{r}_{iM} \right]^2 + \frac{m-1}{n^3 m^3 N} \sum_i^N \langle 200 \rangle'_i \langle 002 \rangle'_i$$

$$+ \frac{2(m-1)(m-2)(n-1)}{n^3 m^2} \left[\frac{1}{N} \sum_i^N \bar{X}_{i.} \bar{r}^2_{iM} \bar{X}_{NM} + \frac{1}{N} \sum_i^N \bar{r}_{iM} \bar{X}^2_{i.} \bar{r}_{NM} \right]$$

$$+ \frac{2(n-1)}{n^3 m^2} \left[\frac{1}{NM} \sum_i^N \sum_j^M y_{ij} r_{ij} \bar{X}_{NM} + \frac{1}{NM} \sum_i^N \sum_j^M x_{ij} y_{ij} \bar{r}_{NM} \right]$$

$$+ \frac{4(n-1)(n-2)}{n^3 m} \bar{X}_{NM} \bar{Y}_{NM} \bar{r}_{NM} + \frac{n-1}{n^3 m^2} \langle 200 \rangle'' \langle 002 \rangle''$$

$$+ \frac{(n-1)(m-1)}{n^3 m^2} \left[\frac{1}{N} \sum_i^N \bar{X}^2_{i.} \langle 002 \rangle'' + \frac{1}{N} \sum_i^N \bar{r}^2_{iM} \langle 200 \rangle'' \right] + \frac{(n-1)(m-1)^2}{n^3 m^2 N^2} \left(\sum_i^N \bar{X}^2_{i.} \right) \left(\sum_i^N \bar{r}^2_{iM} \right)$$

$$+ \frac{2(m-1)(n-1)}{n^3 m^2 N} \left[\sum_i^N \bar{X}_{i.} \langle 002 \rangle'_i \bar{X}_{NM} + 2 \sum_i^N \bar{Y}_{i.} \bar{r}_{iM} \bar{X}_{NM} + \sum_i^N \bar{r}_{iM} \langle 200 \rangle'_i \bar{r}_{NM} + 2 \sum_i^N \bar{X}_{i.} \bar{Y}_{i.} \bar{r}_{NM} \right]$$

$$+ \frac{(n-1)(n-2)(m-1)}{n^3 m N} \left[\sum_i^N \bar{X}^2_{i.} \bar{r}^2_{NM} + \sum_i^N \bar{r}^2_{iM} \bar{X}^2_{NM} + 4 \sum_i^N \bar{X}_{i.} \bar{r}_{iM} \bar{X}_{NM} \bar{r}_{NM} \right]$$

$$+ \frac{(n-1)(n-2)}{n^3 m} \left[\bar{r}^2_{NM} \langle 200 \rangle'' + \bar{X}^2_{NM} \langle 002 \rangle'' \right] + \frac{(n-1)(n-2)(n-3)}{n^3} \bar{r}^2_{NM} \bar{X}^2_{NM}$$

Substituting these values in (3.14) and simplifying, we get

$$V(T_2) = \left[Var(\bar{y}_{nm}) + \bar{r}^2_{NM} Var(\bar{x}_{nm}) - 2 \bar{r}_{NM} Cov.(\bar{y}_{nm}, \bar{x}_{nm}) \right]$$

$$+ \frac{n}{n-1} \left[\{ Cov(\bar{r}_{nm}, \bar{x}_{nm}) \}^2 + Var(\bar{r}_{nm}) Var(\bar{x}_{nm}) \right] \quad \ldots\ (3.15)$$

If in (3.15), we suppose that the selected first-stage units, have been completely enumerated viz., if m = M, then $V(T_2)$ reduces to

$$V(T_2) = \frac{1}{n}\left[S_y^2 + \bar{Y}_N^2 S_X^2 - 2\bar{Y}_N S_{yx} \right] + \frac{1}{n(n-1)}\left[S_y^2 S_x^2 + S_{yx}^2 \right] \quad \ldots(3.16)$$

where the symbols have their usual meaning. This expression was obtained by Goodman and Hartley (1958).

3.5. Comparison of the ratio-type Estimators T_1 and T_2

Using the expressions (3.15) and (3.13), we get

$$V(T_1) - V(T_2) = \mathrm{Var}(\bar{r}_{nm})\bar{X}_{NM}^2 - \mathrm{Var}(\bar{y}_{nm}) + \bar{r}_{NM}^2 \mathrm{Var}(\bar{x}_{nm}) + 2\bar{r}_{NM}\mathrm{Cov}(\bar{y}_{nm}, \bar{x}_{nm})$$

$$- \frac{n}{n-1}\left[\mathrm{Cov}(\bar{r}_{nm}, \bar{x}_{nm}) \right]^2 - \frac{n}{n-1}\mathrm{Var}(\bar{x}_{nm})\mathrm{Var}(\bar{r}_{nm}) \quad \ldots.\ (3.17)$$

From this comparison of the variance of (T_1) and (T_2) given above, it can be seen that the unbiased estimator T_2 will be more efficient than T_1 if the following condition is satisfied.

$$\mathrm{Var}(\bar{r}_{nm})\bar{X}_{NM}^2 + 2\bar{r}_{NM}\mathrm{Cov.}(\bar{y}_{nm}, \bar{x}_{nm}) > \mathrm{Var}(\bar{y}_{nm}) + \bar{r}_{NM}^2 \mathrm{Var}(\bar{x}_{nm})$$

$$+ \frac{n}{n-1}\left[\mathrm{Cov}(\bar{r}_{nm}, \bar{x}_{nm}) \right]^2 + \frac{n}{n-1}\mathrm{Var}(\bar{x}_{nm})\mathrm{Var}(\bar{r}_{nm}) \quad \ldots(3.18)$$

If (3.18) is satisfied, then T_2 is to be preferred to T_1 since it is unbiased and has got less variance. However, if $V(T_1)$ is less than $V(T_2)$, T_1 is not necessarily preferred to T_2 since it is biased and the extent of bias will have to be taken into account for choosing between the two estimators.

3.6. Comparison of the ratio-type Estimator T_2 with simple mean estimator

When no information is available concerning the auxiliary variable, the appropriate estimator for the population mean is $\bar{y}_{nm}$ and its variance when both N and M are large, is given by

$$V(\bar{y}_{nm}) = \frac{1}{n} S^2_{by} + \frac{1}{nm} \cdot \frac{1}{N} \sum_{i}^{N} S^2_{iy} \qquad \ldots \quad (3.19)$$

also we have obtained the variance of T_2 as (up to the order of first approx.)

$$V(T_2) = \frac{1}{n} \left[S^2_{by} + \bar{r}^2_{NM} S^2_{bx} - 2\bar{r}_{NM} S_{bxy} \right] + \frac{1}{nm} \left[\frac{1}{N} \sum_{i}^{N} S^2_{iy} + \bar{r}^2_{NM} \frac{1}{N} \sum_{i}^{N} S^2_{ix} - 2\bar{r}_{NM} \frac{1}{N} \sum_{i}^{N} S_{ixy} \right]$$

Hence ratio type estimator T_2 will be more efficient than the simple mean estimator $\bar{y}_{nm}$ if

$$\frac{1}{n} \left[S^2_{by} + \bar{r}^2_{NM} S^2_{bx} - 2\bar{r}_{NM} S_{bxy} \right] + \frac{1}{nm} \left[\frac{1}{N} \sum_{i}^{N} S^2_{iy} + \bar{r}^2_{NM} \frac{1}{N} \sum_{i}^{N} S^2_{ix} - 2\bar{r}_{NM} \frac{1}{N} \sum_{i}^{N} S_{ixy} \right] < \frac{1}{n} S^2_{by} + \frac{1}{nmN} \sum_{i}^{N} S^2_{iy}$$

or

$$\frac{1}{n} \bar{r}_{NM} \left[\bar{r}_{NM} S^2_{bx} - 2S_{bxy} \right] + \frac{1}{nm} \bar{r}_{NM} \left[\bar{r}_{NM} \frac{1}{N} \sum_{i}^{N} S^2_{ix} - \frac{2}{N} \sum_{i}^{N} S_{ixy} \right] < 0 \qquad \ldots . \quad (3.20)$$

If we define

$$\rho_b = \frac{S_{bxy}}{S_{bx} S_{by}} \quad \text{and}$$

$$\bar{\rho}_w = \frac{\frac{1}{N} \sum_{i}^{N} S_{ixy}}{\sqrt{\frac{1}{N} \sum_{i}^{N} S^2_{ix} \frac{1}{N} \sum_{i}^{N} S^2_{iy}}} = \frac{\bar{S}_{wxy}}{\bar{S}_{wx} \bar{S}_{wy}} \quad \text{say}$$

Then $V(T_2)$ is less than $V(\bar{y}_{nm})$ if

$$\rho_b > \frac{1}{2} \frac{S_{bx}}{S_{by}} \bar{r}_{NM} \text{ and } \bar{\rho}_w > \frac{1}{2} \frac{\bar{S}_{wx}}{\bar{S}_{wy}} \bar{r}_{NM} \qquad \ldots \quad (3.21)$$

Thus (3.21) gives the condition for the ratio-type estimator T_2 to be more efficient than the simple mean estimator $\bar{y}_{nm}$.

3.7. Optimum Allocation of the Sample Between Two Stages

The expressions for the variance of T_1 and T_2 in a two-stage sampling design show that the precision of a two-stage sample, apart from the values of S^2_{br}, S^2_{by}, $\frac{1}{N}\sum_i^N S^2_{ir}$, $\frac{1}{N}\sum_i^N S^2_{iy}$, etc., depends on the distribution of the sample between the two stages viz., on n and m individually. The cost of surveying a two-stage sample will therefore depends on the values of n and m. Here we shall consider the problem of determining n and m so that the variance of T_1 as well as that of T_2 is minimised subject to the restriction that the total cost of the survey is given and is fixed. Let the total cost C_o be represented by a function

$C_o = c_1 n + c_2 nm$ where the first component of cost is proportional to the number of primary units in the sample and second component is proportional to the total number of second-stage units in the sample. c_1 and c_2 are positive constants. First we shall determine the optimum values of n and m for the estimator T_1 and afterwards we shall consider the same for the estimator T_2.

3.7(a) Optimum Allocation of the Sample Considering the Estimator T_1

The cost function which we have considered is

$$C_o = c_1 n + c_2 nm \quad \dots \quad (3.22)$$

$$= c_1 n_1 + c_2 n_2 \text{ say} \quad \dots \quad (3.23)$$

where $n_1 = n$ and $n_2 = nm$.

Also $V(T_1) = \left[\frac{1}{n_1} S^2_{br} + \frac{1}{n_2 N} \sum_i^N S^2_{ir} \right] \bar{\bar{X}}^2_{NM}$

It can be put in the form as

$$V(T_1) = \frac{V_1}{n_1} + \frac{V_2}{n_2} \quad \dots \quad (3.24)$$

where

$$V_1 = S^2_{br}\ \bar{X}^2_{NM} \qquad \ldots. \qquad (3.25)$$

$$V_2 = \frac{1}{N}\sum_{i}^{N} S^2_{ir}\ \bar{X}^2_{NM} \qquad \ldots. \qquad (3.26)$$

We want to determine n_1 and n_2 such that (3.24) is minimised subject to the restriction that the total cost of the survey is fixed. To achieve this, we consider a function

$$\phi = \frac{V_1}{n_1} + \frac{V_2}{n_2} + \lambda\,(c_1 n_1 + c_2 n_2 - C_o) \qquad \ldots.. \qquad (3.27)$$

where λ is a constant.

Differentiating ϕ with respect to n_1 and n_2 and equating both the equations so obtained equal to zero, we have

$$\frac{\partial \phi}{\partial n_1} = -\frac{V_1}{n_1^2} + \lambda c_1 = 0 \qquad \ldots. \qquad (3.28)$$

$$\frac{\partial \phi}{\partial n_2} = -\frac{V_2}{n_2^2} + \lambda c_2 = 0 \qquad \ldots. \qquad (3.29)$$

From (3.28) and (3.29), we get

$$\frac{\sqrt{V_1}}{\sqrt{c_1}\ n_1} = \frac{\sqrt{V_2}}{\sqrt{c_2}\ n_2} = \sqrt{\lambda}$$ which is further equal to $\dfrac{\sqrt{c_1 V_1} + \sqrt{c_2 V_2}}{C_o}$

from this we have

$$n_1 = \frac{C_o\sqrt{V_1}}{\sqrt{c_1}\,\left[\sqrt{c_1 V_1} + \sqrt{c_2 V_2}\right]}$$

and

$$n_2 \doteq \frac{\sqrt{c_1 V_2}}{\sqrt{c_2 V_1}}\ n_1$$

but $n_1 = n$ and $n_2 = nm$. Therefore we have the optimum values of n and m as

$$n = \frac{C_o\sqrt{V_1}}{\sqrt{c_1}\,\left[\sqrt{c_1 V_1} + \sqrt{c_2 V_2}\right]} \qquad \ldots.. \qquad (3.30)$$

and $$m = \frac{\sqrt{c_1 V_2}}{\sqrt{c_2 V_1}} \qquad \ldots\ldots \quad (3.31)$$

where V_1 and V_2 are given by (3.25) and (3.26) respectively. Substituting these values in (3.24), the optimum variance of T_1 is obtained as

$$V(T_1)_{opt.} = \frac{(\sqrt{c_1 V_1} + \sqrt{c_2 V_2})^2}{C_o} \qquad \ldots\ldots \quad (3.32)$$

3.7 (b). <u>Optimum Allocation of the Sample Considering the Estimator T_2</u>

From (3.15), we have

$$V(T_2) = \frac{1}{n_1} S^2_{by} + \frac{1}{n_2} \frac{1}{N} \sum_i^N S^2_{iy} + \frac{1}{n_1} S^2_{bx} \bar{r}^2_{NM} + \frac{1}{n_2} \frac{1}{N} \sum_i^N S^2_{ix} \bar{r}^2_{NM}$$

$$- \frac{2}{n_1} S_{bxy} \bar{r}_{NM} - \frac{2}{n_2} \frac{1}{N_1} \sum_i^N S_{ixy} \bar{r}_{NM} + \left(\frac{n_1}{n_1 - 1} \right).$$

$$\left[\left(\frac{1}{n_1} S_{brx} + \frac{1}{n_2 N} \sum_i^N S_{irx} \right)^2 + \left(\frac{1}{n_1} S^2_{br} + \frac{1}{n_2 N} \sum_i^N S^2_{ir} \right) \left(\frac{1}{n_1} S^2_{bx} + \frac{1}{n_2 N} \sum_i^N S^2_{ix} \right) \right]$$

where $n_1 = n$; $n_2 = nm$.

This can be written as

$$V(T_2) = \frac{1}{n_1} \left[S^2_{by} + \bar{r}^2_{NM} S^2_{bx} - 2\bar{r}_{NM} S_{bxy} \right]$$

$$+ \frac{1}{n_2} \left[\frac{1}{N} \sum_i^N S^2_{iy} + \bar{r}^2_{NM} \frac{1}{N} \sum_i^N S^2_{ix} - 2\bar{r}_{NM} \frac{1}{N} \sum_i^N S_{ixy} \right]$$

$$+ \left[1 - \frac{1}{n_1} \right]^{-1} \left[\left(\frac{1}{n_1} S_{brx} + \frac{1}{n_2} \frac{1}{N} \sum_i^N S_{irx} \right)^2 + \left(\frac{1}{n_1} S^2_{br} + \frac{1}{n_2} \frac{1}{N} \sum_i^N S^2_{ir} \right) \right.$$

$$\left. \left(\frac{1}{n_1} S^2_{bx} + \frac{1}{n_2} \frac{1}{N} \sum_i^N S^2_{ix} \right) \right] \qquad \ldots\ldots \quad (3.33)$$

Since $/ \frac{1}{n_1} / < 1$, the expansion is valid and neglecting terms of order

$\frac{1}{n_1^2}$, $\frac{1}{n_2^2}$, $\frac{1}{n_1 n_2}$ and higher , we get

$$V(T_2) = \frac{1}{n_1} \left[S_{by}^2 + \bar{r}_{NM}^2 S_{bx}^2 - 2\bar{r}_{NM} S_{bxy} \right]$$

$$+ \frac{1}{n_2} \left[\frac{1}{N} \sum_i^N S_{iy}^2 + \bar{r}_{NM}^2 \frac{1}{N} \sum_i^N S_{ix}^2 - 2\bar{r}_{NM} \frac{1}{N} \sum_i^N S_{ixy} \right] \quad \dots (3.34)$$

(3.34) can be put in the form as

$$V(T_2) = \frac{V_1}{n_1} + \frac{V_2}{n_2} \quad \dots (3.35)$$

where

$$V_1 = S_{by}^2 + \bar{r}_{NM}^2 S_{bx}^2 - 2\bar{r}_{NM} S_{bxy} \quad \dots (3.36)$$

and

$$V_2 = \frac{1}{N} \sum_i^N S_{iy}^2 + \bar{r}_{NM}^2 \frac{1}{N} \sum_i^N S_{ix}^2 - 2\bar{r}_{NM} \frac{1}{N} \sum_i^N S_{ixy} \quad \dots (3.37)$$

Proceeding in the same manner as we did for the estimator T_1 to determine optimum values of n and m, we can show that the optimum variance of T_2 is given by

$$V(T_2)_{opt.} = \frac{(\sqrt{c_1 V_1} + \sqrt{c_2 V_2})^2}{C_o} \quad \dots (3.38)$$

where V_1 and V_2 are given by (3.36) and (3.37) respectively.

CHAPTER IV

RATIO-TYPE ESTIMATORS WHEN THE POPULATION MEAN OF THE AUXILIARY VARIABLE IS NOT KNOWN.

In the previous chapter, it was assumed that the population mean of the auxiliary variable is known. Sometimes it so happens that this information is not available. In such a situation, the method of double sampling is used. This technique consists in drawing a large preliminary sample of size n' from the population and observing the value of the character x. From this sample, an estimate of $\bar{X}_N$ is obtained. Then a secondary sample of size n is drawn out of the preliminary sample and the character under study y, is observed. In this chapter, two ratio-type estimators in the case of two-stage , two phase sampling are proposed.

Let as before, the finite population consist of N first-stage units of 'M' second-stage units each. Let us denote by y_{ij}, the value of the j th second-stage unit in the i th first-stage unit for the character y under study and x_{ij} that of the auxiliary variable x. Let $r_{ij} = y_{ij}/x_{ij}$.

Let a simple random sample of size n' be drawn from N first-stage units and from each selected first-stage unit , m' second-stage units be drawn for observing x. Again a sub-sample of size n be drawn from n' and from each first-stage unit thus selected, m units be drawn to observe y by the method of simple random sampling. Here we shall consider the problem of estimating the population mean $\bar{Y}_{NM}$.

Let

$$\bar{Y}_{NM} = \langle 010 \rangle =$$
= the mean per element of y in the population

$$\bar{X}_{NM} = \langle 100 \rangle =$$
= the mean per element of x in the population

$\bar{r}_{NM}$ = $\langle 001 \rangle'$
= the mean per element of the ratio of y to x in the population

$\bar{Y}_{i.}$ = $\langle 010 \rangle'_i$
= the mean per element of y in the i th primary unit in the population

$\bar{X}_{i.}$ = $\langle 100 \rangle'_i$
= the mean per element of x in the i th primary unit in the population

$\bar{r}_{iM}$ = $\langle 001 \rangle'_i$
= the mean per element of the ratio of y to x in the i th primary unit in the population

$\bar{y}_{nm} = \frac{1}{n} \sum_{i}^{n} \langle 010 \rangle_i$
= the mean per element of y in the sample

$\bar{x}_{n'm'} = \frac{1}{n'} \sum_{i}^{n'} \langle 100 \rangle_i^o$
= the mean per element of x in the first sample

$\bar{x}_{nm} = \frac{1}{n} \sum_{i}^{n} \langle 100 \rangle_i$
= the mean per element of x in the second sample

$\bar{r}_{nm} = \frac{1}{n} \sum_{i}^{n} \langle 001 \rangle_i$
= the mean per element of the ratio of y to x in the sample

$\bar{y}_{im}$ = $\langle 010 \rangle_i$
= the mean per element of y in the i th primary unit in the sample.

$\bar{x}_{im'}$ = $\langle 100 \rangle_i^o$
= the mean per element of x in the i th primary unit in the first sample.

$\bar{x}_{im}$ = $\langle 100 \rangle_i$
= the mean per element of x in the i th primary unit in the second sample.

Then the ratio-type estimator of $\bar{Y}_{NM}$ in the case of double sampling in two-stages, is defined as

$$T_3 = \bar{r}_{nm} \, \bar{x}_{n'm'} \qquad \ldots\ldots \quad (4.1)$$

It can be seen that

$$E(T_3) = E_1(E_2(\bar{r}_{nm}\bar{x}_{n'm'})) = E_1(\bar{r}_{n'm'}, \bar{x}_{n'm'})$$

$$= Cov(\bar{r}_{nm}, \bar{x}_{n'm'}) + \bar{r}_{NM}\bar{X}_{NM}$$

$$= \bar{Y}_{NM} - \left[\bar{Y}_{NM} - \left(\frac{1}{n'} - \frac{1}{N}\right) S_{brx} - \left(\frac{1}{m'} - \frac{1}{M}\right)\frac{1}{n'N}\sum_i^N S_{irx} - \bar{r}_{NM}\bar{X}_{NM} \right] \quad ..(4.2)$$

where

$$S_{brx} = \frac{1}{N-1}\sum_i^N (\bar{r}_{iM} - \bar{r}_{NM})(\bar{X}_{i.} - \bar{X}_{NM})$$

$$S_{irx} = \frac{1}{M-1}\sum_j^M (r_{ij} - \bar{r}_{iM})(x_{ij} - \bar{X}_{i.})$$

from (4.2), it follows that T_3 is a biased estimator of the population mean $\bar{Y}_{NM}$, the amount of bias being equal to

$$\bar{Y}_{NM} - \left(\frac{1}{n'} - \frac{1}{N}\right) S_{brx} - \left(\frac{1}{m'} - \frac{1}{M}\right)\frac{1}{n'}\frac{1}{N}\sum_i^N S_{irx} - \bar{r}_{NM}\bar{X}_{NM}$$

4.1. Unbiased Estimator of the Population Mean

The unbiased estimator in this case can be easily obtained by replacing N and M by n' and m' in (3.4) of chapter III. It is thus given by

$$T_4 = \bar{r}_{nm}\bar{x}_{n'm'} + (\bar{y}_{nm} - \bar{r}_{nm}\bar{x}_{nm}) + \left(\frac{1}{n} - \frac{1}{n'}\right) s_{brx}$$
$$+ \frac{1}{nn'}\left(\frac{1}{m} - \frac{1}{m'}\right)\sum_i^n s_{irx} \quad ... (4.3)$$

This can also be written as

$$T_4 = \bar{r}_{nm}\bar{x}_{n'm'} + \frac{n(n'-1)}{n'(n-1)}\left[\bar{y}_{nm} - \bar{r}_{nm}\bar{x}_{nm}\right]$$
$$+ \left[\left(\frac{1}{m} - \frac{1}{m'}\right)\cdot\frac{1}{n'} - \frac{(n'-n)(m-1)}{n'm(n-1)}\right]\frac{1}{n}\sum_i^n s_{irx} \quad ... (4.4)$$

If in (4.4), we suppose that the selected first-stage units are completely enumerated viz., if $m = m' = M$, then T_4 reduces to

$$T_4 = \bar{y}_n \bar{x}_{n'} + \frac{n(n'-1)}{n'(n-1)} (\bar{y}_n - \bar{y}_{nm} \bar{x}_{nm}) \quad \ldots\ldots (4.5)$$

which is identical to the estimate obtained by Sukhatme (1962). We shall consider these two estimators T_3 and T_4 and find the condition under which T_4 is more efficient than T_3.

4.2. Variance of the estimator T_3

$$T_3 = \bar{y}_{nm} \bar{x}_{n'm'}$$

$$V(T_3) = E_1 \left[E_2(T_3^2)/n' \right] - \left[E_1 E_2 (T_3/n') \right]^2 \ldots \quad (4.6)$$

Let us work out each factor separately.

$$E_2(T_3^2)/n' = E_2 \left[\frac{1}{n} \sum_i^n \langle 001 \rangle_i \times \frac{1}{n} \sum_i^n \langle 001 \rangle_i \right] \left[\frac{1}{n'} \sum_i^{n'} \langle 100 \rangle_i^* \right]^2$$

$$= E_2 \left[\frac{1}{n^2} \sum_i^n \langle 001 \rangle_i \langle 001 \rangle_i + \frac{1}{n^2} \sum_{i \neq i'}^n \langle 001 \rangle_i \langle 001 \rangle_{i'} \right] \times$$

$$\left[\frac{1}{n'^2} \sum_i^{n'} \langle 100 \rangle_i^* \langle 100 \rangle_i^* + \frac{1}{n'^2} \sum_{i \neq i'}^{n'} \langle 100 \rangle_i^* \langle 100 \rangle_{i'}^* \right]$$

using multiplication formula (2.2) and taking expectation, we get

$$E_2(T_3^2)/n' = \left[\frac{1}{nm} \frac{1}{n'} \sum_i^{n'} \langle 002 \rangle_i^* + \frac{m-1}{nm} \cdot \frac{1}{n'} \sum_i^{n'} \langle (001)(001) \rangle_i^* \right.$$

$$\left. + \frac{n-1}{nn'(n'-1)} \sum_{i \neq i'}^{n'} \langle 001 \rangle_i^* \langle 001 \rangle_{i'}^* \right] \times$$

$$\left[\frac{1}{n'^2} \sum_i^{n'} \langle 100 \rangle_i^* \langle 100 \rangle_i^* + \frac{1}{n'^2} \sum_{i \neq i'}^{n'} \langle 100 \rangle_i^* \langle 100 \rangle_{i'}^* \right]$$

Again multiplying, we get

$$E_2(T_3^2)/n' = \frac{1}{nm\, n'^3 m'^2} \sum_i^{n'} \left[(m'-1)(m'-2) \langle (100)(100)(002) \rangle_i^* \right.$$

$$+ (m'-1) \left\{ \langle (002)(200) \rangle_i^* + 2 \langle (100)(011) \rangle_i^* \right\} + \langle 020 \rangle_i$$

$$+\frac{1}{nm\,n'^3m'}\Big[\sum_{i\neq i'}^{n'}\langle 200\rangle_i^0\langle 002\rangle_{i'}^0+(m'-1)\sum_{i\neq i'}^{n'}\langle(100)(100)\rangle_i^0\langle 002\rangle_{i'}^0\Big]$$

$$+\frac{(m-1)}{nmn'^3m'^2}\sum_{i}^{n'}\Big[(m'-2)(m'-3)\langle(100)(100)(001)(001)\rangle_i^0$$

$$+(m'-2)\Big\{4\langle(100)(010)(001)\rangle_i^0+\langle(001)(001)(200)\rangle_i^0\Big\}$$

$$+2\langle(010)(010)\rangle_i^0+2\langle(001)(110)\rangle_i^0\Big]$$

$$+\frac{(m-1)}{nmn'^3m'}\sum_{i\neq i'}^{n'}\Big[\langle 200\rangle_i^0\langle(001)(001)\rangle_{i'}^0+(m'-1)\langle(100)(100)\rangle_i^0\langle(001)(001)\rangle_{i'}^{*}\Big]$$

$$+\frac{2(n-1)}{nn'^3(n'-1)m'^2}\sum_{i\neq i'}^{n'}\Big[(m'-1)(m'-2)\langle(100)(100)(001)\rangle_i^0\langle 001\rangle_{i'}^0$$

$$+(m'-1)\Big\{\langle(001)(200)\rangle_i^0\langle 001\rangle_{i'}^0+2\langle(100)(010)\rangle_i^0\langle 001\rangle_{i'}^0\Big\}$$

$$+\langle 001\rangle_{i'}^0\langle 110\rangle_i^0\Big]$$

$$+\frac{n-1}{nn'^3(n'-1)m'}\Big[\sum_{i\neq i'\neq i''}^{n'}\langle 001\rangle_i^0\langle 001\rangle_{i'}^0\langle 200\rangle_{i''}^0$$

$$+(m'-1)\sum_{i\neq i'\neq i''}^{n'}\langle 001\rangle_i^0\langle 001\rangle_{i'}^0\langle(100)(100)\rangle_{i''}^0$$

$$+\frac{2}{nmn'^3m'}\Big[\sum_{i\neq i'}^{n'}\langle 011\rangle_i^0\langle 100\rangle_{i'}^0+(m'-1)\sum_{i\neq i'}^{n'}\langle 100\rangle_{i'}^0\langle(100)(002)\rangle_i^0\Big]$$

$$+\frac{1}{nmn'^3}\sum_{i\neq i'\neq i''}^{n'}\langle 100\rangle_i^0\langle 100\rangle_{i'}^0\langle 002\rangle_{i''}^0$$

$$+\frac{2(m-1)}{nmn'^3m'}\sum_{i\neq i'}^{n'}\Big[(m'-2)\langle(100)(001)(001)\rangle_i^0\langle 100\rangle_{i'}^0$$

$$+2\langle(001)(101)\rangle_i^0\langle 100\rangle_{i'}^0\Big]$$

$$+\frac{m-1}{nm\,n'^3}\sum_{i\neq i'\neq i''}^{n'}\langle 100\rangle_i^0\langle 100\rangle_{i'}^0\langle(001)(001)\rangle_{i''}^0$$

$$+\frac{(n-1)}{nn'^3(n'-1)}\Big[2\sum_{i\neq i'}^{n'}\langle 001\rangle_i^0\langle 001\rangle_{i'}^0\langle 100\rangle_i^0\langle 100\rangle_{i'}^0$$

$$+4 \sum_{i\neq i'\neq i''}^{n'} \langle 001\rangle_i^{o} \langle 001\rangle_{i'}^{o} \langle 100\rangle_i^{o} \langle 100\rangle_{i''}^{o}$$

$$+ \sum_{i\neq i'\neq i''\neq i'''}^{n'} \langle 001\rangle_i^{o} \langle 001\rangle_{i'}^{o} \langle 100\rangle_{i''}^{o} \langle 100\rangle_{i'''}^{o} \Big]$$

Taking expectation again, we get:

$$E_1\Big[E_2(T_3^2 \mid /n)\Big] = \frac{1}{nmn'^2m'^2}\Big[(m'-1)(m'-2)\langle(100)(100)(002)\rangle''$$

$$+ (m'-1)\Big\{\langle(002)(200)\rangle'' + 2\langle(100)(011)\rangle''\Big\} + \langle 020\rangle''\Big]$$

$$+ \frac{n'-1}{nmn'^2m'(N-1)N}\Big[\sum_{i\neq i'}^{N} \langle 200\rangle_i' \langle 002\rangle_{i'}' + (m'-1)\sum_{i\neq i'}^{N}\langle 002\rangle_{i'}'\langle(100)(100)\rangle_i'\Big]$$

$$+ \frac{(m-1)}{nmn'^2m'^2}\Big[(m'-2)(m'-3)\langle(100)(100)(001)(001)\rangle''$$

$$+(m'-2)\Big\{4\langle(100)(010)(001)\rangle'' + \langle(001)(001)(200)\rangle''\Big\} + 2\langle(010)(010)\rangle'' + 2\langle(001)(110)\rangle''\Big]$$

$$+ \frac{(m-1)(n'-1)}{nmn'^2m'N(N-1)} \sum_{i\neq i'}^{N} \Big[\langle 200\rangle_i'\langle(001)(001)\rangle_{i'}' + (m'-1)\langle(100)(100)\rangle_i'\langle(001)(001)\rangle_{i'}'$$

$$+ \frac{2(n-1)}{nn'^2m'^2N(N-1)} \sum_{i\neq i'}^{N}\Big[(m'-1)(m'-2)\langle(100)(100)(001)\rangle_i'\langle 001\rangle_{i'}'$$

$$+ (m'-1)\Big\{\langle(001)(200)\rangle_i'\langle 001\rangle_{i'}' + 2\langle(100)(010)\rangle_i'\langle 001\rangle_{i'}'\Big\}$$

$$+ \langle 001\rangle_{i'}'\langle 110\rangle_i'\Big]$$

$$+ \frac{(n-1)(n'-2)}{nn'^2m'N(N-1)(N-2)}\Big[\sum_{i\neq i'\neq i''}^{N} \langle 001\rangle_i' \langle 001\rangle_{i'}' \langle 200\rangle_{i''}'$$

$$+ (m'-1)\sum_{i\neq i'\neq i''}^{N} \langle 001\rangle_i'\langle 001\rangle_{i'}' \langle(100)(100)\rangle_{i''}'\Big]$$

$$+ \frac{2(n'-1)}{nmn'^2m'N(N-1)} \sum_{i\neq i'}^{N}\Big[\langle 011\rangle_i'\langle 100\rangle_{i'}' + (m'-1)\langle 100\rangle_{i'}'\langle(100)(002)\rangle_i'\Big]$$

$$+ \frac{(n'-1)(n'-2)}{nmn'^2N(N-1)(N-2)} \sum_{i\neq i'\neq i''}^{N} \langle 100\rangle_i' \langle 100\rangle_{i'}' \langle 002\rangle_{i''}'$$

$$+\frac{2(m-1)(n'-1)}{nmn'^2m'N(N-1)}\Big[\sum_{i\neq i'}^{N}(m'-2)\langle(100)(001)(001)\rangle'_i\langle 100\rangle'_{i'} + 2\sum_{i\neq i'}^{N}\langle(001)(010)\rangle'_i\langle 100\rangle'_{i'}\Big]$$

$$+\frac{(m-1)(n'-1)(n'-2)}{nmn'^2N(N-1)(N-2)}\sum_{i\neq i'\neq i''}^{N}\langle 100\rangle'_i\langle 100\rangle'_{i'}\langle(001)(001)\rangle'_{i''}$$

$$+\frac{2(n-1)}{nn'^2m'^2(N-1)N}\sum_{i\neq i'}^{N}\Big[\langle 010\rangle'_i\langle 010\rangle'_{i'} + 2(m'-1)\langle(001)(100)\rangle'_i\langle 010\rangle'_{i'} + (m'-1)^2\langle(001)(100)\rangle'_i\langle(100)(001)\rangle'_{i'}\Big]$$

$$+\frac{4(n-1)(n'-2)}{mm'n'^2N(N-1)(N-2)}\Big[\sum_{i\neq i'\neq i''}^{N}\langle 001\rangle'_{i'}\langle 100\rangle'_{i''}\langle 101\rangle'_i + (m'-1)\sum_{i\neq i'\neq i''}^{N}\langle 001\rangle'_{i'}\langle 100\rangle'_{i''}\langle(100)(001)\rangle'_i\Big]$$

$$+\frac{(n-1)(n'-2)(n'-3)}{nn'^2N(N-1)(N-2)(N-3)}\sum_{i\neq i'\neq i''\neq i'''}^{N}\langle 001\rangle'_i\langle 001\rangle'_{i'}\langle 100\rangle'_{i''}\langle 100\rangle'_{i'''} \quad . \qquad \ldots (4.7)$$

Also

$$\Big[E_1E_2(T_3/n')\Big]^2 = \frac{1}{n'^2m'^2}\langle 010\rangle''\langle 010\rangle''$$

$$+\frac{(m'-1)^2}{n'^2m'^2}\langle(100)(001)\rangle''\langle(100)(001)\rangle'' + \frac{2(m'-1)}{m'^2n'^2}\langle 010\rangle''\langle(100)(001)\rangle''$$

$$+\frac{(n'-1)^2}{n'^2}\Big[\frac{1}{N(N-1)}\sum_{i\neq i'}^{N}\langle 001\rangle'_i\langle 100\rangle'_{i'}\Big]^2 + \frac{2(n'-1)}{n'^2m'N(N-1)}\langle 010\rangle''\sum_{i\neq i'}^{N}\langle 001\rangle'_i\langle 100\rangle'_{i'}$$

$$+\frac{2(n'-1)(m'-1)}{n'^2m'N(N-1)}\langle(100)(001)\rangle''\sum_{i\neq i'}^{N}\langle 001\rangle'_i\langle 100\rangle'_{i'} . \qquad \ldots (4.8)$$

Substituting (4.7) and (4.8) in (4.6), we can get the exact variance of T_3.

4.3. Variance of the Unbiased Estimator T_4

$$T_4 = \bar{r}_{nm}\bar{x}_{n'm'} + (\bar{y}_{nm} - \bar{r}_{nm}\bar{x}_{nm}) + (\frac{1}{n} - \frac{1}{n'})\, e_{brx}$$

$$+ \frac{1}{nn'}(\frac{1}{m} - \frac{1}{m'})\sum_{i}^{n} e_{irx}$$

This can also be written as

$$T_4 = \bar{r}_{nm}\bar{x}_{n'm'} + (1 + \frac{m'-m}{n'm'(m-1)})\bar{y}_{nm} - \frac{n(n'-1)}{n'(n-1)}\bar{r}_{nm}\bar{x}_{nm}$$

$$- \frac{1}{nn'}(\frac{m'-m}{m'(m-1)} - \frac{n'-n}{n-1})\sum_{i}^{n}\bar{r}_{im}\bar{x}_{im} \quad \ldots. (4.9)$$

Now $\quad V(T_4) = E(T_4^2) - [E(T_4)]^2 \quad \ldots. (4.10)$

$$V(T_4) = E(\bar{r}_{nm}\bar{x}_{n'm'})^2 + [1 + \frac{m'-m}{n'm'(m-1)}]^2 E(\bar{y}_{nm})^2 + \frac{n^2(n'-1)^2}{n'^2(n-1)^2}E(\bar{r}_{nm}\bar{x}_{nm})^2$$

$$+ \frac{1}{n^2n'^2}(\frac{m'-m}{m'(m-1)} - \frac{n'-n}{n-1})^2 E(\sum_{i}^{n}\bar{r}_{im}\bar{x}_{im})^2$$

$$+2(1 + \frac{m'-m}{n'm'(m-1)})E(\bar{y}_{nm}\bar{r}_{nm}\bar{x}_{n'm'}) - \frac{2(n'-1)n}{(n-1)n'}E(\bar{r}_{nm}^2\bar{x}_{nm}\bar{x}_{n'm'})$$

$$- \frac{2}{nn'}(\frac{m'-m}{m'(m-1)} - \frac{n'-n}{n-1})E(\sum_{i}^{n}\bar{r}_{im}\bar{x}_{im}\bar{r}_{nm}\bar{x}_{n'm'})$$

$$- \frac{2(n'-1)n}{(n-1)n'}(1 + \frac{m'-m}{n'm'(m-1)})E(\bar{y}_{nm}\bar{x}_{nm}\bar{r}_{nm})$$

$$- \frac{2}{nn'}(\frac{m'-m}{m'(m-1)} - \frac{n'-n}{n-1})(1 + \frac{m'-m}{n'm'(m-1)})E(\sum_{i}^{n}\bar{r}_{im}\bar{x}_{im}\bar{y}_{nm})$$

$$+ \frac{2(n'-1)}{n'^2(n-1)}(\frac{m'-m}{m'(m-1)} - \frac{n'-n}{n-1})E(\sum_{i}^{n}\bar{r}_{im}\bar{x}_{im}\bar{r}_{nm}\bar{x}_{nm}) - \bar{Y}_{NM}^2$$

$\ldots. (4.11)$

where

$$(i)\ E(\bar{y}_{nm}\bar{r}_{nm}\bar{x}_{n'm'}) = \frac{1}{nn'}[\frac{1}{mm'} - \frac{n'-1}{mM(N-1)} - \frac{2(n-1)}{m'M(N-1)}]\langle 020\rangle''$$

$$+ \frac{1}{nn'}[\frac{(n'-1)N}{m'(N-1)} - \frac{(m-1)(n'-1)N}{m(N-1)(M-1)}]\langle 100\rangle''\langle 011\rangle''$$

$$+\frac{1}{nn'}\left[\frac{(n-1)N}{m'(N-1)}-\frac{(n-1)(m'-1)N}{m')(N-1)(M-1)}\right]\left[\langle 110\rangle''\langle 001\rangle''+\langle 101\rangle''\langle 010\rangle''\right]$$

$$+\frac{(m-1)(n'-1)M}{nn'm(N-1)(M-1)}\sum_{i}^{N}\langle 010\rangle'_i\langle 001\rangle'_i\langle 100\rangle''$$

$$+\frac{(n-1)(m'-1)M}{nn'm'(N-1)(M-1)}\left[\sum_{i}^{N}\langle 100\rangle'_i\langle 010\rangle'_i\langle 001\rangle''+\sum_{i}^{N}\langle 100\rangle'_i\langle 001\rangle'_i\langle 010\rangle''\right]$$

$$+\frac{1}{nn'}\left[\frac{m'-1}{mm'}-\frac{(n'-1)(M-1)}{mM(N-1)}-\frac{2(n-1)(m'-1)}{m'M(N-1)}\right]\langle (100)(011)\rangle''$$

$$+\frac{1}{nn'}\left[\frac{m-1}{mm'}-\frac{(m-1)(n'-1)}{mM(N-1)}-\frac{(n-1)(m'-1)}{m'M(N-1)}-\frac{(n-1)(M-1)}{m'M(N-1)}\right]\langle (101)(010)\rangle''$$

$$+\frac{1}{nn'}\left[\frac{m-1}{mm'}-\frac{(m-1)(n'-1)}{mM(N-1)}-\frac{(n-1)(m'-1)}{m'M(N-1)}-\frac{(n-1)(M-1)}{m'M(N-1)}\right]\langle (110)(001)\rangle''$$

$$+\frac{1}{nn'}\left[\frac{(m-1)(m'-2)}{mm'}-\frac{(m-1)(n'-1)(M-2)}{mM(N-1)}\frac{2(n-1)(m'-1)(M-2)}{m'M(N-1)}\right]\langle (100)(010)(001)\rangle''$$

$$+\frac{(n-1)(n'-12)}{nn'(N-1)(N-2)}\left[N^2\langle 100\rangle''\langle 010\rangle''\langle 001\rangle''-3\sum_{i}^{N}\langle 100\rangle'_i\langle 001\rangle'_i\langle 010\rangle''+\frac{2}{N}\sum_{i}^{N}\langle 001\rangle'_i\langle 100\rangle'_i\langle 010\rangle'_i\right]$$

(ii) $$E\left(\sum_{i}^{n}\bar{r}_{im}\bar{x}_{im}\bar{r}_{nm}\bar{x}_{n'm'}\right)=\frac{(m-1)(m-2)}{m^2n'm'}\left[(m'-3)\langle (100)(001)(100)(001)\rangle''+2\langle (100)(001)(010)\rangle''+\langle (001)(001)(200)\rangle''\right]$$

$$+\frac{(m-1)(m-2)(n'-1)N}{m^2n'(N-1)}\langle (100)(001)(001)\rangle''\langle 100\rangle''$$

$$-\frac{(m-1)(m-2)(n'-1)}{m^2n'N(N-1)}\sum_{i}^{N}\langle (100)(001)(001)\rangle'_i\langle 100\rangle'_i$$

$$+\frac{(m-1)(n'-1)N}{m^2n'(N-1)}\langle (100)(002)\rangle''\langle 100\rangle''-\frac{(m-1)(n'-1)}{m^2n'N(N-1)}\sum_{i}^{N}\langle (100)(002)\rangle'_i\langle 100\rangle'_i$$

$$+\frac{m-1}{m^2n'm'}\left[(m'-2)\langle(100)(100)(002)\rangle''+\langle(100)(011)\rangle''+\langle(200)(002)\rangle''\right]$$

$$+\frac{2(m-1)}{m^2n'm'}\left[(m'-2)\langle(100)(010)(001)\rangle''+\langle(001)(110)\rangle''+\langle(101)(010)\rangle''\right]$$

$$+\frac{2(m-1)(n'-1)N}{m^2n'(N-1)}\langle(001)(010)\rangle''\langle 100\rangle''-\frac{2(m-1)(n'-1)}{m^2n'N(N-1)}\sum_i^N\langle(001)(010)\rangle'_i\langle 100\rangle'_i$$

$$+\frac{1}{m^2n'm'}\left[(m'-1)\langle(100)(011)\rangle''+\langle 020\rangle''\right]+\frac{(n'-1)N}{m^2n'(N-1)}\langle 100\rangle''\langle 011\rangle''$$

$$-\frac{(n'-1)}{m^2n'N(N-1)}\sum_i^N\langle 100\rangle'_i\langle 011\rangle'_i+\frac{(n-1)N}{mn'm'(N-1)}\langle 110\rangle''\langle 001\rangle''$$

$$-\frac{n-1}{mn'm'N(N-1)}\sum_i^N\langle 110\rangle'_i\langle 001\rangle'_i+\frac{(n-1)(m'-1)N}{mn'm'(N-1)}\langle(100)(010)\rangle''\langle 001\rangle''$$

$$-\frac{(n-1)(m'-1)}{mn'm'N(N-1)}\sum_i^N\langle(100)(010)\rangle'_i\langle 001\rangle'_i+\frac{(n-1)N}{mn'm'(N-1)}\langle 010\rangle''\langle 010\rangle''$$

$$-\frac{(n-1)}{mn'm'N(N-1)}\sum_i^N\langle 101\rangle'_i\langle 010\rangle'_i+\frac{(n-1)(m'-1)N}{mn'm'(N-1)}\langle(100)(001)\rangle''\langle 010\rangle''$$

$$-\frac{(n-1)(m'-1)}{mn'm'N(N-1)}\sum_i^N\langle(100)(001)\rangle'_i\langle(010)\rangle'_i+\frac{(m-1)(n-1)(m'-2)N}{mn'm'(N-1)}\langle(100)(100)(001)\rangle''\langle 001\rangle''$$

$$+\frac{(n-1)(n'-2)}{mn'(N-1)(N-2)}\left[N^2\langle 100\rangle''\langle 010\rangle''\langle 001\rangle''-3\sum_i^N\langle 100\rangle'_i\langle 001\rangle'_i\langle 010\rangle''+\frac{2}{N}\sum_i^N\langle 100\rangle'_i\langle 001\rangle'_i\langle 010\rangle'_i\right]$$

$$-\frac{(m-1)(n-1)(m'-2)}{mn'm'N(N-1)}\sum_i^N\langle(100)(100)(001)\rangle'_i\langle 001\rangle'_i$$

$$+\frac{(m-1)(n-1)N}{mn'm'(N-1)}\langle(100)(010)\rangle''\langle 001\rangle''-\frac{(m-1)(n-1)}{mn'm'N(N-1)}\sum_i^N\langle(100)(010)\rangle'_i\langle 001\rangle'_i$$

$$+\frac{(m-1)(n-1)N}{mn'm'(N-1)}\langle(001)(200)\rangle''\langle 001\rangle''-\frac{(m-1)(n-1)}{mn'm'N(N-1)}\sum_i^N\langle(001)(200)\rangle'_i\langle 001\rangle'_i$$

$$+\frac{(m-1)(n-1)(Mm'-1)N}{mn'm'(N-1)}\left[\langle(100)(001)\rangle''\right]^2-\frac{(m-1)(n-1)(m'-1)}{mn'm'N(N-1)}\sum_i^N\left[\langle(100)(001)\rangle'_i\right]^2$$

$$+\frac{(m-1)(n-1)N}{mn'm'(N-1)}\langle(100)(001)\rangle''\langle 010\rangle''-\frac{(m-1)(n-1)}{mn'm'N(N-1)}\sum_i^N\langle(100)(001)\rangle'_i\langle 010\rangle'_i$$

$$+\frac{(m-1)(n-1)(n'-2)}{mn'(N-1)(N-2)}\Big[N^2\langle 100\rangle''\langle 001\rangle''\langle(100)(001)\rangle''-\frac{3M}{(M-1)N}\Big(\sum_i^N\langle 100\rangle'_i\langle 001\rangle'_i\Big)^2$$

$$+\frac{3}{M-1}\sum_i^N\langle 100\rangle'_i\langle 001\rangle'_i\langle 010\rangle''+\frac{2M}{N(M-1)}\sum_i^N\langle 100\rangle'_i\langle 001\rangle'_i\langle 100\rangle'_i\langle 001\rangle'_i$$

$$-\frac{2}{(M-1)N}\sum_i^N\langle 100\rangle'_i\langle 010\rangle'_i\langle 001\rangle'_i\Big]$$

(iii) $$E(\overline{F}^2_{nm}\,\overline{x}_{nm}\,\overline{x}_{n'm'}) = \frac{(m-1)(m-2)}{n^2m^2n'm'}\Big[(m'-3)\langle(100)(001)(100)(001)\rangle''$$

$$+2\langle(100)(010)(001)\rangle''+\langle(001)(001)(200)\rangle''\Big]$$

$$+\frac{(m-1)(m-2)(n'-1)N}{n^2m^2n'(N-1)}\langle(100)(001)(001)\rangle''\langle 100\rangle''-\frac{(m-1)(m-2)(n'-1)}{n^2m^2n'N(N-1)}\sum_i^N\langle(100)(001)(001)\rangle'_i\langle 100\rangle'_i$$

$$+\frac{(m-1)(n'-1)N}{n^2m^2n'(N-1)}\langle(100)(002)\rangle''\langle 100\rangle''-\frac{(m-1)(n'-1)}{n^2m^2n'N(N-1)}\sum_i^N\langle(100)(002)\rangle'_i\langle 100\rangle'_i$$

$$+\frac{(m-1)}{n^2m^2n'm'}\Big[(m'-2)\langle(100)(100)(002)\rangle''+\langle(100)(011)\rangle''+\langle(200)(002)\rangle''\Big]$$

$$+\frac{2(m-1)}{n^2m^2n'm'}\Big[(m'-2)\langle(100)(010)(001)\rangle''+\langle(001)(110)\rangle''+\langle(010)(101)\rangle''\Big]$$

$$+\frac{2(m-1)(n'-1)N}{n^2m^2n'(N-1)}\langle(001)(010)\rangle''\langle 100\rangle''-\frac{2(m-1)(n'-1)}{n^2m^2n'N(N-1)}\sum_i^N\langle(001)(010)\rangle'_i\langle 100\rangle'_i$$

$$+\frac{1}{n^2m^2n'm'}\Big[(m'-1)\langle(100)(011)\rangle''+\langle 020\rangle''\Big]+\frac{(n'-1)N}{n^2m^2n'(N-1)}\langle 100\rangle''\langle 011\rangle''$$

$$+\frac{2(n-1)N}{n^2mn'm'N-1}\langle 110\rangle''\langle 001\rangle''-\frac{(n'-1)}{n^2m^2n'N(N-1)}\sum_i^N\langle 100\rangle'_i\langle 011\rangle'_i$$

$$-\frac{2(n-1)}{n^2mn'm'N(N-1)}\sum_i^N\langle 110\rangle'_i\langle 001\rangle'_i$$

$$+\frac{2(n-1)(m'-1)\,N}{n^2mn'm'(N-1)}\langle 001\rangle''\langle(100)\,(010)\rangle''-\frac{2(n-1)(m'-1)}{n^2mn'm'N(N-1)}\sum_i^N\langle 001\rangle_i'\langle(100)(010)\rangle_i'$$

$$+\frac{2(n-1)\,N}{n^2mn'm'(N-1)}\langle 010\rangle''\langle 010\rangle''-\frac{2(n-1)}{n^2mn'm'N(N-1)}\sum_i^N\langle 010\rangle_i'\langle 101\rangle_i'$$

$$+\frac{2(n-1)(m'-1)\,N}{n^2mn'm'(N-1)}\langle 010\rangle''\langle(100)(001)\rangle''-\frac{2(n-1)(m'-1)}{n^2mn'm'N(N-1)}\sum_i^N\langle 010\rangle_i'\langle(100)(001)\rangle_i'$$

$$+\frac{2(n-1)(n'-2)}{n^2mn'(N-1)(N-2)}\Big[N^2\langle 100\rangle''\langle 010\rangle''\langle 001\rangle''-3\sum_i^N\langle 100\rangle_i'\langle 001\rangle_i'\langle 010\rangle''+\frac{2}{N}\sum_i^N\langle 100\rangle_i'\langle 010\rangle_i'\langle 001\rangle_i'\Big]$$

$$+\frac{2(m-1)(n-1)(m'-2)\,N}{n^2mn'm'(N-1)}\langle 001\rangle''\langle(100)(100)(001)\rangle''$$

$$-\frac{2(n-1)(m-1)(m'-2)}{n^2mn'm'N(N-1)}\sum_i^N\langle 001\rangle_i'\langle(100)(100)(001)\rangle_i'$$

$$+\frac{2(n-1)(m-1)\,N}{n^2mn'm'(N-1)}\langle 001\rangle''\langle(100)(010)\rangle''-\frac{2(n-1)(m-1)}{n^2mn'm'N(N-1)}\sum_i^N\langle 001\rangle_i'\langle(100)(010)\rangle_i'$$

$$+\frac{2(n-1)(m-1)\,N}{n^2mn'm'(N-1)}\langle 001\rangle''\langle(001)\,(200)\rangle''-\frac{2(n-1)(m-1)}{n^2mn'm'N(N-1)}\sum_i^N\langle(001)(200)\rangle_i'\langle 001\rangle_i'$$

$$+\frac{2(n-1)(m-1)\,N}{n^2mn'm'(N-1)}\langle 010\rangle''\langle(100)(001)\rangle''-\frac{2(n-1)(m-1)}{n^2mn'm'N(N-1)}\sum_i^N\langle 010\rangle_i'\langle(100)(001)\rangle_i'$$

$$+\frac{2(n-1)(m-1)(m'-1)N}{n^2mn'm'(N-1)}\Big[\langle(100)(001)\rangle''\Big]^2-\frac{2(n-1)(m-1)(m'-1)}{n^2mn'm'N(N-1)}\sum_i^N\big(\langle(100)(001)\rangle_i'\big)^2$$

$$+\frac{2(n-1)(m-1)(n'-2)}{n^2mn'(N-1)(N-2)}\Big[N^2\langle 100\rangle''\langle 001\rangle''\langle(100)(001)\rangle''-\frac{3M}{(M-1)N}\Big(\sum_i^N\langle 100\rangle_i'\langle 001\rangle_i'\Big)^2+\frac{3}{M-1}\sum_i^N\langle 100\rangle_i'\langle 001\rangle_i'\langle 010\rangle''+\frac{2M}{(M-1)N}\sum_i^N\langle 100\rangle_i'\langle 001\rangle_i'\langle 100\rangle_i'\langle 001\rangle_i'-\frac{2}{(M-1)N}\sum_i^N\langle 100\rangle_i'\langle 010\rangle_i'\langle 001\rangle_i'\Big]$$

$$+\frac{(n-1)\ N}{n^2mn'm'(N-1)}\langle 100\rangle''\langle 011\rangle'' - \frac{(n-1)}{n^2mn'm'N(N-1)}\sum_i^N \langle 100\rangle'_i\langle 011\rangle'_i$$

$$+\frac{(n-1)(m'-1)\ N}{n^2mn'm'(N-1)}\langle 100\rangle''\langle(100)\ (002)\rangle'' - \frac{(n-1)\ (m'-1)}{n^2mn'm'N(N-1)}\sum_i^N \langle 100\rangle'_i\langle(100)\ (002)\rangle'_i$$

$$+\frac{(n-1)N}{n^2mn'm'(N-1)}\langle 200\rangle''\langle 002\rangle'' - \frac{(n-1)}{n^2mn'm'N(N-1)}\sum_i^N \langle 200\rangle'_i\langle 002\rangle'_i$$

$$+\frac{(n-1)(m'-1)\ N}{n^2mn'm'(N-1)}\langle 002\rangle''\langle(100)\ (100)\rangle'' - \frac{(n-1)\ (m'-1)}{n^2mn'm'N(N-1)}\sum_i^N \langle 002\rangle'_i\langle(100)(100)\rangle'_i$$

$$+\frac{(n-1)\ (n'-2\ )}{n^2mn'N(N-1)(N-2)}\sum_{i\neq i'\neq i''}^N \langle 002\rangle'_i\langle 100\rangle'_{i'}\ \langle 100\rangle'_{i''}$$

$$+\frac{(n-1)(m-1)(n'-2\ )}{n^2mn'N(N-1)(N-2)}\sum_{i\neq i'\neq i''}^N \langle(001)(001)\rangle'_i\langle 100\rangle'_{i'}\ \ \langle 100\rangle'_{i''}$$

$$+\frac{(n-1)(m-1)(m'-2)\ \ N}{n^2mn'm'\ (N-1)}\langle 100\rangle''\langle(100)(001)(001)\rangle'' - \frac{(n-1)(m-1)(m'-2)}{n^2mn'm'N(N-1)}$$

$$\sum_i^N \langle 100\rangle'_i\langle(100)(001)(001)\rangle'_i$$

$$+\frac{2(n-1)(m-1)\ \ N}{n^2mn'm'(N-1)}\langle 100\rangle''\langle(001)(010)\rangle'' - \frac{2(n-1)(m-1\ )}{n^2mn'm'N(N-1)}\sum_i^N \langle 100\rangle'_i\langle(001)(010)\rangle'_i$$

$$+\frac{(n-1)(m-1)\ N}{n^2mn'm'(N-1)}\langle 200\rangle''\langle(001)\ (001)\rangle'' - \frac{(n-1)(m-1)}{n^2mn'm'N(N-1)}\sum_i^N \langle 200\rangle'_i\langle(001)(001)\rangle'_i$$

$$+\frac{(n-1)(m-1)(m'-1)N}{n^2mn'm'(N-1)}\langle(100)(100)\rangle''\langle(001)\ (001)\rangle''$$

$$-\frac{(n-1)(m-1)(m'-1)}{n^2mn'm'N(N-1)}\sum_i^N \langle(100)\ (100)\rangle'_i\langle(001)(001)\rangle'_i$$

$$+\frac{2(n-1)(\text{[illegible]}\ n-2)}{n^2n'm'N(N-1)(N-2)}\left[\sum_{i\neq i'\neq i''}^N \langle 010\rangle'_i\ \langle 100\rangle'_{i'}\ \langle 001\rangle'_{i''}\right]$$

$$+ \frac{2(n-1)(n-2)(m'-1)}{n^2 n' m' N(N-1)(N-2)} \left[\sum_{i \neq i' \neq i''}^{N} \langle (100)(001) \rangle_i' \langle 100 \rangle_{i'}' \langle 001 \rangle_{i''}' \right]$$

$$+ \frac{(n-1)(n-2)}{n^2 n' m' N(N-1)(N-2)} \left[\sum_{i \neq i' = i''}^{N} \langle 001 \rangle_i' \langle 200 \rangle_{i'}' \langle 001 \rangle_{i''}' \right]$$

$$+ \frac{(n-1)(n-2)(m'-1)}{n^2 n' m' N(N-1)(N-2)} \left[\sum_{i \neq i' \neq i''}^{N} \langle 001 \rangle_i' \langle (100)(100) \rangle_{i'}' \langle 001 \rangle_{i''}' \right]$$

$$+ \frac{(n-1)(n-2)(n'-3)}{n^2 n' N(N-1)(N-2)(N-3)} \left[\sum_{i \neq i' \neq i'' \neq i'''}^{N} \langle 001 \rangle_i' \langle 100 \rangle_{i'}' \langle 001 \rangle_{i''}' \langle 100 \rangle_{i'''}' \right]$$

$E(\bar{y}_{nm} \bar{\bar{x}}_{n'm'})^2$ has already been found out while determining the variance of the estimator T_3 and $E(\bar{y}_{nm})^2$, $E(\bar{y}_{nm} \bar{\bar{x}}_{nm})^2$, $E(\sum_i^n \bar{y}_{im} \bar{\bar{x}}_{im})^2$,

$E(\bar{y}_{nm} \bar{\bar{x}}_{nm} \bar{y}_{nm})$, $E(\sum_i^n \bar{y}_{im} \bar{\bar{x}}_{im} \bar{y}_{nm})$ and $(E(\sum_i^n \bar{y}_{im} \bar{\bar{x}}_{im} \bar{y}_{nm} \bar{\bar{x}}_{nm})$ are the same as given in chapter III. These values can be substituted in (4.11) and the exact variance of the estimator $\bar{T}_4$ can be worked out.

The case when both N and M are large, will be considered and the results so obtained will be presented in the standard notations.

4.4. Variance of T_3 When both N and M are Large

From (4.6), after simplification, we get the variance of T_3 as

$$V(T_3) = \frac{1}{nm n'^2 m'^2} \frac{1}{N} \sum_i^N S_{iy}^2 + \frac{2}{nn'^2 m'^2} S_{by}^2 + \frac{1}{n'^2 m'^2} \bar{Y}_{NM}^2$$

$$- \frac{1}{nmn'^2 m'^2 N} \sum_i^N \bar{Y}_{i.}^2 + \frac{(m'-1)(m'-2)}{nmn'^2 m'^2 NM} \sum_i^N \sum_j^M x_{ij}^2 \bar{X}_{i.}^2$$

$$+ \frac{(m'-1)}{nmn'^2 m'^2} \left[\frac{1}{N} \sum_i^N \langle 200 \rangle_i' \langle 002 \rangle_i' + \frac{2}{NM} \sum_i^N \sum_j^M y_{ij} x_{ij} \bar{X}_{i.} \right]$$

$$+ \frac{(n'-1)}{nmn'^2 m'} \left[\frac{1}{NM} \sum_i^N \sum_j^M x_{ij}^2 \right] \left[\frac{1}{NM} \sum_i^N \sum_j^M x_{ij}^2 \right]$$

$$+\frac{(n'-1)(m'-1)}{nmn'^2m'}\left(\frac{1}{N}\sum_{i}^{N}\bar{X}_{i.}^{2}\right)\left(\frac{1}{NM}\sum_{i}^{N}\sum_{j}^{M}y_{ij}^{2}\right)+\frac{(m-1)(n'-1)}{nmn'^2m'}\left(\frac{1}{N}\sum_{i}^{N}\bar{y}_{iM}^{2}\right)\left(\frac{1}{NM}\sum_{i}\sum_{j}^{NM}x_{ij}^{2}\right)$$

$$+\frac{(m-1)(m'-2)(m'-3)}{nmn'^2m'^2N}\sum_{i}^{N}\bar{X}_{i.}^{2}\,\bar{y}_{iM}^{2}+\frac{2(m-1)}{nmn'^2m'^2NM}\sum_{i}^{N}\sum_{j}^{M}x_{ij}\,y_{ij}\,\bar{y}_{iM}$$

$$+\frac{(m-1)(m'-2)}{nm\,n'^2m'^2}\left[\frac{4}{N}\sum_{i}^{N}\bar{X}_{i.}\,\bar{Y}_{i.}\,\bar{y}_{iM}+\frac{1}{NM}\sum_{i}^{N}\sum_{j}^{M}x_{ij}^{2}\,\bar{y}_{iM}^{2}\right]$$

$$+\frac{(m-1)(n'-1)(m'-1)}{nmn'^2m'}\left[\frac{1}{N}\sum_{i}^{N}\bar{X}_{i.}^{2}\right]\left[\frac{1}{N}\sum_{i}^{N}\bar{y}_{iM}^{2}\right]$$

$$+\frac{2(n-1)(m'-1)(m'-2)}{nn'^2m'^2\,N}\sum_{i}^{N}\bar{X}_{i.}^{2}\,\bar{y}_{iM}\,\bar{y}_{NM}+\frac{2(n-1)}{nn'^2m'^2}\frac{1}{NM}\sum_{i}^{N}\sum_{j}^{M}x_{ij}\,y_{ij}\,\bar{y}_{NM}$$

$$+\frac{2(n-1)(m'-1)}{nn'^2m'^2}\left[\frac{1}{NM}\sum_{i}^{N}\sum_{j}^{M}x_{ij}^{2}\,\bar{y}_{iM}\,\bar{y}_{NM}+\frac{2}{N}\sum_{i}^{N}\bar{X}_{i.}\,\bar{Y}_{i.}\,\bar{y}_{NM}\right]$$

$$+\frac{(n-1)(n'-2)}{nn'^2m'NM}\sum_{i}\sum_{j}^{NM}x_{ij}^{2}\,\bar{y}_{NM}^{2}+\frac{(n-1)(n'-2)(m'-1)}{nn'^2m'N}\sum_{i}^{N}\bar{X}_{i.}^{2}\,\bar{y}_{NM}^{2}$$

$$+\frac{2(n'-1)}{nmn'^2m'NM}\sum_{i}^{N}\sum_{j}^{M}y_{ij}\,x_{ij}\,\bar{X}_{NM}+\frac{2(n'-1)(m'-1)}{nmn'^2m'NM}\sum_{i}^{N}\sum_{j}^{M}y_{ij}^{2}\,\bar{X}_{i.}\,\bar{X}_{NM}$$

$$+\frac{(n'-1)(n'-2)}{nmn'^2NM}\sum_{i}\sum_{j}^{NM}y_{ij}^{2}\,\bar{X}_{NM}^{2}+\frac{4(m-1)(n'-1)}{nmn'^2m'N}\sum_{i}^{N}\bar{y}_{iM}\,\bar{Y}_{i.}\,\bar{X}_{NM}$$

$$+\frac{2(m-1)(n'-1)(m'-2)}{nmn'^2m'\,N}\sum_{i}^{N}\bar{X}_{i.}\,\bar{y}_{iM}^{2}\,\bar{X}_{NM}+\frac{(m-1)(n'-1)(n'-2)}{nmn'^2N}\sum_{i}^{N}\bar{y}_{iM}^{2}\,\bar{X}_{NM}^{2}$$

$$+\left[\frac{4(n-1)(m'-1)}{nm'^2n'^2}-\frac{2(m'-1)}{n'^2m'^2}\right]\frac{1}{N}\sum_{i}^{N}\bar{X}_{i.}\,\bar{y}_{iM}\,\bar{Y}_{NM}$$

$$+\left[\frac{-2(n-1)(m'-1)^2}{nn'^2m'^2}-\frac{(m'-1)^2}{n'^2m'^2}\right]\left[\frac{1}{N}\sum_{i}^{N}\bar{X}_{i.}\,\bar{y}_{iM}\right]^{2}$$

$$+\left[\frac{4(n-1)(n'-2)}{nm'n^2}-\frac{2(n'-1)}{n'^2m'}\right]\bar{X}_{NM}\,\bar{Y}_{NM}\,\bar{y}_{NM}$$

$$+\left[\frac{4(n-1)(n'-2)(m'-1)}{nm'\,n'^2}-\frac{2(n'-1)(n'-1)}{n'^2m'}\right]\frac{1}{N}\sum_{i}^{N}\bar{X}_{i.}\,\bar{y}_{iM}\,\bar{X}_{NM}\,\bar{y}_{NM}$$

$$+\left[\frac{(n-1)(n'-2)(n'-3)}{nn'^2}-\frac{(n'-1)^2}{n'^2}\right]\bar{y}_{NM}^{2}\,\bar{X}_{NM}^{2}$$

If we retain the terms of the order of $\frac{1}{nm}$ or $\frac{1}{n'm'}$ in (4.12) and neglect others, we get after simplifying

$$V(T_3) = Var(\bar{y}_{nm})\bar{X}^2_{NM} + Var(\bar{x}_{n'm'})\bar{Y}^2_{NM} + 2\bar{Y}_{NM}\bar{X}_{NM} Cov(\bar{y}_{nm}, \bar{x}_{n'm'}) \quad \ldots. (4.13)$$

4.5. Variance of T_4 When both N and M are large

From (4.11), we have

$$V(T_4) = E(\bar{y}_{nm}\bar{x}_{n'm'})^2 + \left[1 + \frac{m'-m}{m'n'(m-1)}\right]^2 E(\bar{y}_{nm})^2$$

$$+ \frac{n^2(n'-1)^2}{n'^2(n-1)^2} E(\bar{y}_{nm}\bar{x}_{nm})^2 - \frac{2(n'-1)n}{(n-1)n'} E(\bar{y}^2_{nm}\bar{x}_{nm}\bar{x}_{n'm'})$$

$$+ \frac{1}{n^2n'^2}\left[\frac{m'-m}{m'(m-1)} - \frac{n'-n}{n-1}\right]^2 E\left(\sum_i^n y_{im}x_{im}\right)^2 + 2\left[1+\frac{m'-m}{n'm'(m-1)}\right]E(\bar{y}_{nm}\bar{y}_{nm}\bar{x}_{n'm'})$$

$$- \frac{2}{nn'}\left[\frac{m'-m}{m'(m-1)} - \frac{n'-n}{n-1}\right] E\left(\sum_i^n y_{im}x_{im}\bar{y}_{nm}\bar{x}_{n'm'}\right)$$

$$- \frac{2(n'-1)n}{(n-1)n'}\left[1+\frac{m'-m}{n'm'(m-1)}\right] E(\bar{x}_{nm}\bar{y}_{nm}\bar{y}_{nm})$$

$$- \frac{2}{nn'}\left[\frac{m'-m}{m'(m-1)} - \frac{n'-n}{n-1}\right]\left[1+\frac{m'-m}{n'm'(m-1)}\right]E\left(\sum_i^n y_{im}x_{im}\bar{y}_{nm}\right)$$

$$+ \frac{2(n'-1)}{n'^2(n-1)}\left[\frac{m'-m}{m'(m-1)} - \frac{n'-n}{n-1}\right]E\left(\sum_i^n y_{im}x_{im}\bar{y}_{nm}\bar{x}_{nm}\right) - \bar{Y}^2_{NM}$$

where

$$(i)\quad E(\bar{y}_{nm}\bar{y}_{nm}\bar{x}_{n'm'}) = \frac{1}{nm}\cdot\frac{1}{n'm'}\cdot\frac{1}{N}\sum_i^N S^2_{iy} + \frac{1}{nn'm'}S^2_{by} + \frac{1}{n'm'}\bar{Y}^2_{NM}$$

$$+ \frac{n-1}{nn'm'NM}\sum_i^N\sum_j^M x_{ij}y_{ij}\bar{y}_{NM} + \frac{n'-1}{nmn'NM}\sum_i^N\sum_j^M y_{ij}x_{ij}\bar{X}_{NM}$$

$$+ \frac{(m-1)(n'-1)}{nmn'N}\sum_i^N \bar{Y}_{i.}\bar{y}_{iM}\bar{X}_{NM} + \frac{(n-1)(m'-1)}{nn'm'N}\left[\sum_i^N \bar{X}_{i.}\bar{Y}_{i.}\bar{y}_{NM} + \sum_i^N \bar{X}_{i.}\bar{y}_{iM}\bar{Y}_{NM}\right]$$

$$+\frac{(m'-1)}{nmn'm'NM}\sum_i^N\sum_j^M y_{ij}\,z_{ij}\,\bar{X}_{i.}+\frac{(m-1)}{nmn'm'NM}\sum_i^N\sum_j^M x_{ij}\,y_{ij}\,\bar{z}_{iM}$$

$$+\frac{(m-1)(m'-2)}{nmn'm'N}\sum_i^N\bar{X}_{i.}\,\bar{Y}_{i.}\,\bar{z}_{iM}+\frac{(n-1)(n'-2)}{nn'}\bar{X}_{NM}\,\bar{Y}_{NM}\,\bar{z}_{NM}$$

$$(ii)\ E\left(\sum_i^n\bar{z}_{im}\bar{x}_{im}\bar{z}_{nm}\bar{x}_{n'm'}\right)=\frac{1}{m^2n'm'N}\sum_i^N S^2_{iy}+\frac{1}{mn'm'}S^2_{by}+\frac{n}{mn'm'}\bar{Y}^2_{NM}$$

$$+\frac{m-1}{m^2n'm'N}\sum_i^N\bar{Y}^2_{i.}+\frac{(m-1)(m-2)(m'-3)}{m^2n'm'N}\sum_i^N\bar{X}^2_{i.}\bar{z}^2_{iM}$$

$$+\left[\frac{2((m-1)(m-2)}{m^2n'm'}+\frac{2(m^2-1)(m'-2)}{m^2n'm'}\right]\frac{1}{N}\sum_i^N\bar{X}_{i.}\,\bar{Y}_{i.}\,\bar{z}_{iM}$$

$$+\frac{(m-1)(m-2)}{m^2n'm'N}\sum_i^N\bar{z}_{iM}\,\langle 200\rangle'_i+\frac{(m-1)(m'-2)}{m^2n'm'N}\sum_i^N\bar{X}^2_{i.}\,\langle 002\rangle'_i$$

$$+\frac{(m-1)(m-2)(n'-1)}{m^2n'N}\sum_i^N\bar{X}_{i.}\,\bar{z}_{iM}\,\bar{X}_{NM}$$

$$+\frac{(m-1)(n-1)(m'-2)}{mn'm'N}\sum_i^N\bar{X}^2_{i.}\bar{z}_{iM}\,\bar{z}_{NM}+\frac{(m-1)}{m^2n'm'N}\sum_i^N\langle 200\rangle'_i\langle 002\rangle'_i$$

$$+\left[\frac{(m-1)}{m^2n'm'}+\frac{m'-1}{m^2n'm'}\right]\frac{1}{NM}\sum_i^N\sum_j^M y_{ij}\,z_{ij}\,\bar{X}_{i.}$$

$$+\frac{2(m-1)}{m^2n'm'NM}\sum_i^N\sum_j^M x_{ij}y_{ij}\bar{z}_{iM}+\frac{(m-1)((n'-1)}{m^2n'N}\sum_i^N\bar{X}_{i.}\,\langle 002\rangle'_i\,\bar{X}_{NM}$$

$$+\frac{(m-1)(n-1)}{mn'm'N}\sum_i^N\bar{z}_{iM}\langle 200\rangle'_i\,\bar{z}_{NM}+\frac{2(m-1)(n'-1)}{m^2n'N}\sum_i^N\bar{Y}_{i.}\,\bar{z}_{iM}\bar{X}_{NM}$$

$$+\left[\frac{(n-1)(m'-1)}{mn'm'}+\frac{(m-1)(n-1)}{mn'm'}\right]\left[\frac{1}{N}\sum_i^N\bar{X}_{i.}\,\bar{Y}_{i.}\,\bar{z}_{NM}+\frac{1}{N}\sum_i^N\bar{X}_{i.}\,\bar{z}_{iM}\bar{Y}_{NM}\right]$$

$$+\frac{(n-1)(n'-2)}{n'm}\bar{X}_{NM}\bar{Y}_{NM}\bar{z}_{NM}+\frac{n'-1}{m^2n'NM}\sum_i^N\sum_j^M y_{ij}\,z_{ij}\,\bar{X}_{NM}$$

$$+ \frac{n-1}{mn'm'NM} \sum_{i}^{NM}\sum_{j} x_{ij} y_{ij} \bar{y}_{NM} + \frac{(m-1)(n-1)(m'-1)}{mn'm'} \left[\frac{1}{N} \sum_{i}^{N} \bar{x}_{i.} \bar{y}_{iM} \right]^2$$

$$+ \frac{(m-1)(n-1)(n'-2)}{mn'N} \sum_{i}^{N} \bar{x}_{i.} \bar{y}_{iM} \bar{x}_{NM} \bar{y}_{NM} .$$

$$(iii)\ E(\bar{y}^2_{nm} \bar{x}_{nm} \bar{x}_{n'm'}) = \frac{1}{n^2m^2n'm'N} \sum_{i}^{N} S^2_{iy} + \frac{2}{n^2mn'm'} S^2_{by} + \frac{2}{nmn'm'} \bar{Y}^2_{NM}$$

$$- \frac{1}{n^2m^2n'm'N} \sum_{i}^{N} \bar{Y}^2_{i.} + \frac{(m-1)(m-2)(m'-3)}{n^2m^2n'm'N} \sum_{i}^{N} \bar{X}^2_{i.} \bar{y}^2_{iM}$$

$$+ \left[\frac{2(m-1)(m-2)}{n^2m^2n'm'} + \frac{2(m-1)(m'-2)}{n^2m^2n'm'} \right] \frac{1}{N} \sum_{i}^{N} \bar{X}_{i.} \bar{Y}_{i.} \bar{y}_{iM}$$

$$+ \frac{(m-1)(m-2)}{n^2m^2n'm'N} \sum_{i}^{N} \bar{y}^2_{iM} \langle 200\rangle'_i + \frac{(m-1)(m'-2)}{n^2m^2n'm'N} \sum_{i}^{N} \bar{X}^2_{i.} \langle 002\rangle'_i$$

$$+ \left[\frac{(m-1)(m-2)(n'-1)}{n^2m^2n'} + \frac{(n-1)(m-1)(m'-2)}{n^2mn'm'} \right] \frac{1}{N} \sum_{i}^{N} \bar{y}^2_{iM} \bar{X}_{i.} \bar{X}_{NM}$$

$$+ \left[\frac{(m-1)(n'-1)}{n^2m^2n'} + \frac{(n-1)(m'-1)}{n^2mn'm'} \right] \frac{1}{N} \sum_{i}^{N} \bar{X}_{i.} \langle 002\rangle'_i \bar{X}_{NM}$$

$$+ \left[\frac{m-1}{n^2m^2n'm'} + \frac{m'-1}{n^2m^2n'm'} \right] \frac{1}{NM} \sum_{i}^{NM}\sum_{j} y_{ij} x_{ij} \bar{X}_{i.}$$

$$+ \frac{m-1}{n^2m^2n'm'N} \sum_{i}^{N} \langle 200\rangle'_i \langle 002\rangle'_i + \frac{(n-1)}{n^2mn'm'} \langle 200\rangle'' \langle 002\rangle'' + \frac{2(m-1)}{n^2m^2n'm'NM} \sum_{i}^{NM}\sum_{j} x_{ij} y_{ij} \bar{y}_{iM}$$

$$+ \left[\frac{2(m-1)(n'-1)}{n^2m^2n'} + \frac{2(n-1)(m-1)}{n^2mn'm'} \right] \frac{1}{N} \sum_{i}^{N} \bar{Y}_{i.} \bar{y}_{iM} \bar{X}_{NM}$$

$$+ \left[\frac{n'-1}{n^2m^2n'} + \frac{n-1}{n^2mn'm'} \right] \frac{1}{NM} \sum_{i}^{NM}\sum_{j} y_{ij} x_{ij} \bar{X}_{NM} + \frac{2(n-1)}{n^2mn'm'NM} \sum_{i}^{MN}\sum_{j} x_{ij} y_{ij} \bar{y}_{NM}$$

$$+ \left[\frac{2(n-1)(m'-1)}{n^2 mn'm'} + \frac{2(n-1)(m-1)}{n^2 mn'm'} \right] \left[\frac{1}{N} \sum_i^N \bar{X}_{i.} \bar{Y}_{i.} \bar{r}_{NM} + \frac{1}{N} \sum_i^N \bar{X}_{i.} \bar{r}_{iM} \bar{Y}_{NM} \right]$$

$$+ \left[\frac{2(n-1)(n'-2)}{n^2 mn'} + \frac{2(n-1)(n-2)}{n^2 m'n'} \right] \bar{X}_{NM} \bar{Y}_{NM} \bar{r}_{NM}$$

$$+ \frac{2(n-1)(m-1)(m'-2)}{n^2 mn'm'N} \sum_i^N \bar{X}_{i.} \bar{r}_{iM} \bar{r}_{NM} + \frac{2(n-1)(m-1)}{n^2 mn'm'N} \sum_i^N \bar{r}_{iM} \langle 200 \rangle_i' \bar{r}_{NM}$$

$$+ \frac{2(n-1)(m-1)(m'-1)}{n^2 mn'm'} \left[\frac{1}{N} \sum_i^N \bar{X}_{i.} \bar{r}_{iM} \right]^2 + \frac{(n-1)(m'-1)}{n^2 mn'm'N} \sum_i^N \bar{X}_{i.}^2 \langle 002 \rangle''$$

$$+ \left[\frac{2(n-1)(m-1)(n'-2)}{n^2 mn'} + \frac{2(n-1)(n-2)(m'-1)}{n^2 m'n'} \right] \frac{1}{N} \sum_i^N \bar{X}_{i.} \bar{r}_{iM} \bar{X}_{NM} \bar{r}_{NM}$$

$$+ \frac{(n-1)(n'-2)}{n^2 mn'NM} \sum_i^{NM} \sum_j r_{ij}^2 \bar{X}_{NM}^2 + \frac{(n-1)(m-1)(n'-2)}{n^2 mn'N} \sum_i^N \bar{r}_{iM}^2 \bar{X}_{NM}^2$$

$$+ \frac{(n-1)(m-1)}{n^2 mn'm'N} \sum_i^N \bar{r}_{iM}^2 \langle 200 \rangle'' + \frac{(n-1)(m-1)(m'-1)}{n^2 mn'm'} \left(\frac{1}{N} \sum_i^N \bar{X}_{i.}^2 \right) \left(\frac{1}{N} \sum_i^N \bar{r}_{iM}^2 \right)$$

$$+ \frac{(n-1)(n-2)}{n^2 m'n'NM} \sum_i^{NM} \sum_j x_{ij}^2 \bar{r}_{NM}^2 + \frac{(n-1)(n-2)(m'-1)}{n^2 n'm'N} \sum_i^N \bar{X}_{i.}^2 \bar{r}_{NM}^2$$

$$+ \frac{(n-1)(n-2)(n'-3)}{n^2 n'} \bar{r}_{NM}^2 \bar{X}_{NM}^2 .$$

The root terms have already been worked out. Substituting these values in (4.11) and retaining terms of the order of $\frac{1}{nm}$ or $\frac{1}{n'm'}$ and simplifying, we get

$$V(T_4) = Var(\bar{y}_{nm}) + Var(\bar{\bar{x}}_{nm}) \bar{r}_{NM}^2 - Var(\bar{\bar{x}}_{n'm'}) \bar{r}_{NM}^2$$

$$- 2\bar{r}_{NM} Cov(\bar{y}_{nm}, \bar{\bar{x}}_{nm}) + 2\bar{r}_{NM} Cov.(\bar{y}_{nm}, \bar{\bar{x}}_{n'm'}) \quad \ldots \quad (4.14)$$

Thus the variance of T_3 and T_4 are respectively given by (4.13) and (4.14).

4.6. Comparison of the Ratio-Type Estimators T_3 and T_4

From (4.13) and (4.14), we get

$$V(T_3) - V(T_4) = \text{Var}(\bar{r}_{nm})\bar{X}^2_{NM} + \text{Var}(\bar{x}_{n'm'})\bar{r}^2_{NM}$$

$$+2\,\bar{r}_{NM}\bar{X}_{NM}\,\text{Cov}(\bar{r}_{nm}, \bar{x}_{n'm'}) - \text{Var}(\bar{y}_{nm}) - \text{Var}(\bar{x}_{nm})\bar{r}^2_{NM}$$

$$+\text{Var}(\bar{x}_{n'm'})\bar{r}^2_{NM} + 2\bar{r}_{NM}\,\text{Cov}(\bar{y}_{nm}, \bar{x}_{nm}) - 2\,\bar{r}_{NM}\,\text{Cov}(\bar{y}_{nm}, \bar{x}_{n'm'})$$

$$= \Big[\text{Var}(\bar{r}_{nm})\bar{X}^2_{NM} + 2\,\text{Var}(\bar{x}_{n'm'})\bar{r}^2_{NM}$$

$$+ 2\,\bar{r}_{NM}\bar{X}_{NM}\,\text{Cov}(\bar{r}_{nm}, \bar{x}_{n'm'}) + 2\,\bar{r}_{NM}\,\text{Cov.}(\bar{y}_{nm}, \bar{x}_{nm})\Big]$$

$$-\Big[\text{Var}(\bar{y}_{nm}) + \text{Var}(\bar{x}_{nm})\bar{r}^2_{NM} + 2\,\bar{r}_{NM}\,\text{Cov}(\bar{y}_{nm}, \bar{x}_{n'm'})\Big] \quad \ldots \quad (4.15)$$

From the comparison of the variance of T_3 and T_4 given above, it can be seen that the unbiased estimator T_4 will be more efficient than T_3 provided

$$\text{Var}(\bar{r}_{nm})\bar{X}^2_{NM} + 2\,\text{Var}(\bar{x}_{n'm'})\bar{r}^2_{NM} + 2\bar{r}_{NM}\,\text{Cov.}(\bar{y}_{nm}, \bar{x}_{nm})$$

$$+ 2\bar{r}_{NM}\bar{X}_{NM}\,\text{Cov}(\bar{r}_{nm}, \bar{x}_{n'm'}) > \text{Var}(\bar{y}_{nm}) + \text{Var}(\bar{x}_{nm})\bar{r}^2_{NM}$$

$$+2\bar{r}_{NM}\,\text{Cov}(\bar{y}_{nm}, \bar{x}_{n'm'}) \quad \ldots (4.16)$$

If (4.16) is satisfied then T_4 is to be preferred to T_3 since it is unbiased and has got less variance. But if however, this is not the case, T_3 is not necessarily preferred to T_4 since it is biased and the bias will have to be taken into account except in the case when $\bar{r}_{NM} = \bar{Y}_{NM} / \bar{X}_{NM}$.

4.7. Estimates of Variance

From (4.13)

$$V(T_3) = \text{Var}(\bar{r}_{nm})\bar{X}^2_{NM} + \text{Var}(\bar{x}_{n'm'})\bar{r}^2_{NM} + 2\bar{r}_{NM}\bar{X}_{NM}\,\text{Cov.}(\bar{r}_{nm}, \bar{x}_{n'm'})$$

Consistent estimate of this variance can be obtained by replacing the population values by the corresponding sample values. Therefore

$$\hat{V}(T_3) = \hat{Var}(\bar{r}_{nm})\bar{x}^2_{nm} + \hat{Var}(\bar{x}_{n'm'})\bar{r}^2_{nm} + 2\bar{r}_{nm}\bar{x}_{nm}\hat{Cov}.(\bar{r}_{nm},\bar{x}_{n'm'}) \quad \ldots. \quad (4.17)$$

where

$$\hat{Var}(\bar{r}_{nm}) = \frac{1}{n} s^2_{br}$$

$$= \frac{1}{n}\frac{1}{n-1}\left[\sum_i^n \bar{r}^2_{im} - n\bar{r}^2_{nm}\right]$$

$$\hat{Var}(\bar{x}_{n'm'}) = \frac{1}{n'} s^2_{bx'}$$

$$= \frac{1}{n'}\frac{1}{n'-1}\left[\sum_i^{n'} \bar{x}^2_{im'} - n'\bar{x}^2_{n'm'}\right]$$

$$\hat{Cov}(\bar{r}_{nm},\bar{x}_{n'm'}) = \frac{1}{n'} s_{brx} + \frac{m}{m-1}\left(\frac{1}{n'm'} - \frac{1}{n'm}\right)\left(\bar{y}_{nm} - \frac{1}{n}\sum_i^n \bar{r}_{im}\bar{x}_{im}\right)$$

$$\text{where } s_{brx} = \frac{1}{n-1}\left[\sum_i^n \bar{r}_{im}\bar{x}_{im} - n\bar{r}_{nm}\bar{x}_{nm}\right]$$

Similarly from (4.14), we have

$$\hat{V}(T_4) = \hat{Var}(\bar{y}_{nm}) + \hat{Var}(\bar{x}_{nm})\bar{r}^2_{nm} + 2\bar{r}_{nm}\hat{Cov}.(\bar{y}_{nm},\bar{x}_{n'm'})$$

$$- \hat{Var}(\bar{x}_{n'm'})\bar{r}^2_{nm} - 2\bar{r}_{nm}\hat{Cov}(\bar{y}_{nm},\bar{x}_{nm}) \quad \ldots.(4.18)$$

where

$$\hat{Var}(\bar{y}_{nm}) = \frac{1}{n} s^2_{by}$$

$$= \frac{1}{n}\frac{1}{n-1}\left[\sum_i^n \bar{y}^2_{im} - n\bar{y}^2_{nm}\right]$$

$$\hat{Var}(\bar{x}_{nm}) = \frac{1}{n} s^2_{bx}$$

$$= \frac{1}{n}\frac{1}{n-1}\left[\sum_i^n \bar{x}^2_{im} - n\bar{x}^2_{nm}\right]$$

$$\hat{Cov}(\bar{y}_{nm},\bar{x}_{nm}) = \frac{1}{n} s_{bxy}$$

$$= \frac{1}{n}\frac{1}{n-1}\left[\sum_i^n \bar{x}_{im}\bar{y}_{im} - n\bar{x}_{nm}\bar{y}_{nm}\right]$$

$$\hat{Cov}(\bar{y}_{nm},\bar{x}_{n'm'}) = \frac{1}{n'} s_{bxy} + \frac{m}{m-1}\left(\frac{1}{n'm'} - \frac{1}{n'm}\right)$$

$$\left(\frac{1}{nm}\sum_i\sum_j^{nm} x_{ij}y_{ij} - \frac{1}{n}\sum_i^n \bar{x}_{im}\bar{y}_{im}\right)$$

where

$$s_{bxy} = \frac{1}{n-1} \left[\sum_{i}^{n} \bar{x}_{im} \bar{y}_{im} - n\bar{x}_{nm} \bar{y}_{nm} \right]$$

Thus the estimates of variance of T_3 and T_4 are respectively given by (4.17) and (4.18).

4.8. Optimum Allocation of the Sample

In this section, we shall consider the optimum allocation of the sample size, when double sampling is adopted in a two-stage design. Let C_o denotes the total cost and suppose that it is fixed. This C_o can be expressed as

$$C_o = c_1 n + c_2 nm + c_3 n' + c_4 n'm'$$

where the first component of cost is proportional to the number of primary units in the second sample, second component is proportional to the total number of second-stage units in the second sample, third component is proportional to the number of primary units in the first sample and fourth component is proportional to the total number of second-stage units in the first sample. c_1, c_2, c_3 and c_4 are positive constants. First we shall determine the optimum values of n, m, n' and m' for the estimator T_3 and afterwards we shall consider the same for the estimator T_4.

4.8(a) Optimum Allocation of the Sample Considering the Estimator T_3

The cost function which we have considered is

$$C_o = c_1 n + c_2 nm + c_3 n' + c_4 n'm' \qquad (4.19)$$

$$= c_1 n_1 + c_2 n_2 + c_3 n_3 + c_4 n_4$$

$$= \sum_{i}^{4} c_i n_i \qquad (4.20)$$

where $n_1 = n$; $n_2 = nm$; $n_3 = n'$; $n_4 = n'm'$.

From (4.13), we have

$$V(T_3) = \Big[\frac{1}{n_1} S^2_{br} + \frac{1}{n_2 N} \sum_i^N S^2_{ir} \Big] \bar{X}^2_{NM}$$

$$+ \Big[\frac{1}{n_3} (S^2_{bx} \bar{r}^2_{NM} + 2 \bar{r}_{NM} \bar{X}_{NM} S_{brx})$$

$$+ \frac{1}{n_4} (\frac{1}{N} \sum_i^N S^2_{ix} \bar{r}^2_{NM} + \frac{2}{N} \sum_i^N S_{irx} \bar{r}_{NM} \bar{X}_{NM}) \Big]$$

It can be put in the form as

$$V(T_3) = \frac{V_1}{n_1} + \frac{V_2}{n_2} + \frac{V_3}{n_3} + \frac{V_4}{n_4}$$

$$= \sum_i^4 \frac{V_i}{n_i} \qquad \ldots. \quad (4.21)$$

where

$$V_1 = S^2_{br} \bar{X}^2_{NM} \qquad \ldots \quad (4.22)$$

$$V_2 = \frac{1}{N} \sum_i^N \bar{X}^2_{NM} S^2_{ir} \qquad \ldots \quad (4.23)$$

$$V_3 = S^2_{bx} \bar{r}^2_{NM} + 2 S_{brx} \bar{r}_{NM} \bar{X}_{NM} \qquad \ldots \quad (4.24)$$

$$V_4 = \frac{1}{N} \sum_i^N S^2_{ix} \bar{r}^2_{NM} + \frac{2}{N} \sum_i^N S_{irx} \bar{r}_{NM} \bar{X}_{NM} \qquad \ldots \quad (4.25)$$

We want to determine n, m, n' and m' such that (4.21) is minimised subject to the restriction that the total cost of the survey is fixed. To achieve this, we consider a function

$$\phi = \sum_i^4 \frac{V_i}{n_i} + \lambda \, (\sum_i^4 c_i n_i - C_o) \qquad \ldots \quad (4.26)$$

where λ is a constant.

Differentiating ϕ with respect to n_1, n_2, n_3 and n_4 and equating all the equations thus obtained equal to zero, we have

$$\frac{\partial \phi}{\partial n_1} = -\frac{V_1}{n_1^2} + \lambda c_1 = 0 \quad \ldots \quad (4.27)$$

$$\frac{\partial \phi}{\partial n_2} = -\frac{V_2}{n_2^2} + \lambda c_2 = 0 \quad \ldots \quad (4.28)$$

$$\frac{\partial \phi}{\partial n_3} = -\frac{V_3}{n_3^2} + \lambda c_3 = 0 \quad \ldots \quad (4.29)$$

$$\frac{\partial \phi}{\partial n_4} = -\frac{V_4}{n_4^2} + \lambda c_4 = 0 \quad \ldots \quad (4.30)$$

From (4.27), (4.28), (4.29) and (4.30), we get

$$\frac{\sqrt{V_1}}{\sqrt{c_1}\, n_1} = \frac{\sqrt{V_2}}{\sqrt{c_2}\, n_2} = \frac{\sqrt{V_3}}{\sqrt{c_3}\, n_3} = \frac{\sqrt{V_4}}{\sqrt{c_4}\, n_4} = \sqrt{\lambda}$$

which is further equal to

$$\frac{\sqrt{c_1 V_1} + \sqrt{c_2 V_2} + \sqrt{c_3 V_3} + \sqrt{c_4 V_4}}{C_o}$$

from this , we have the optimum values of n, m, n' and m' as

$$n = \frac{C_o \sqrt{V_1}}{\sqrt{c_1}\,\left[\sqrt{c_1 V_1} + \sqrt{c_2 V_2} + \sqrt{c_3 V_3} + \sqrt{c_4 V_4}\right]} \quad \ldots \quad (4.31)$$

$$m = \frac{\sqrt{c_1 V_2}}{\sqrt{c_2 V_1}} \quad \ldots \quad (4.32)$$

$$n' = \frac{C_o \sqrt{V_3}}{\sqrt{c_3}\,\left[\sqrt{c_1 V_1} + \sqrt{c_2 V_2} + \sqrt{c_3 V_3} + \sqrt{c_4 V_4}\right]} \quad \ldots \quad (4.33)$$

$$m' = \frac{\sqrt{c_3 V_4}}{\sqrt{c_4 V_3}} \quad \ldots \quad (4.34)$$

Substituting these values in (4.21), the optimum variance of T_3 is obtained as

$$V(T_3)_{opt.} = \frac{\left[\sqrt{c_1 V_1} + \sqrt{c_2 V_2} + \sqrt{c_3 V_3} + \sqrt{c_4 V_4}\right]^2}{C_o} \quad \ldots \quad (4.35)$$

where V_1, V_2, V_3 and V_4 are respectively given by (4.22), (4.23), (4.24) and (4.25).

4.8(b). Optimum Allocation of the Sample Considering the Estimator T_4

From (4.14), we have

$$V(T_4) = \left[\frac{1}{n_1} \left(S_{by}^2 + S_{bx}^2 \bar{Y}_{NM}^2 - 2\bar{Y}_{NM} S_{bxy} \right) + \frac{1}{n_2} \left(\frac{1}{N} \sum_i^N S_{iy}^2 + \frac{1}{N} \sum_i^N S_{ix}^2 \bar{Y}_{NM}^2 - 2\bar{Y}_{NM} \frac{1}{N} \sum_i^N S_{ixy} \right) \right]$$

$$+ \left[\frac{1}{n_3} \left(2\bar{Y}_{NM} S_{bxy} - S_{bx}^2 \bar{Y}_{NM}^2 \right) + \frac{1}{n_4} \left(\frac{2}{N} \bar{Y}_{NM} \sum_i^N S_{ixy} - \frac{1}{N} \sum_i^N S_{ix}^2 \bar{Y}_{NM}^2 \right) \right]$$

It can be put in the form as

$$V(T_4) = \frac{V_1}{n_1} + \frac{V_2}{n_2} + \frac{V_3}{n_3} + \frac{V_4}{n_4} = \sum_i^4 \frac{V_i}{n_i} \quad \ldots (4.36)$$

where

$$V_1 = S_{by}^2 + S_{bx}^2 \bar{Y}_{NM}^2 - 2\bar{Y}_{NM} S_{bxy} \quad \ldots (4.37)$$

$$V_2 = \frac{1}{N} \sum_i^N S_{iy}^2 + \frac{1}{N} \sum_i^N S_{ix}^2 \bar{Y}_{NM}^2 - 2\bar{Y}_{NM} \frac{1}{N} \sum_i^N S_{ixy} \quad \ldots (4.38)$$

$$V_3 = 2\bar{Y}_{NM} S_{bxy} - S_{bx}^2 \bar{Y}_{NM}^2 \quad \ldots (4.39)$$

$$V_4 = \frac{2}{N} \bar{Y}_{NM} \sum_i^N S_{ixy} - \frac{1}{N} \sum_i^N S_{ix}^2 \bar{Y}_{NM}^2 \quad \ldots (4.40)$$

Proceeding in the same manner as we did for the estimator T_3 to determine optimum values of n, m, n' and m', we can show that the optimum variance of T_4 is given by

$$V(T_4)_{opt.} = \frac{\left[\sqrt{c_1 V_1} + \sqrt{c_2 V_2} + \sqrt{c_3 V_3} + \sqrt{c_4 V_4} \right]^2}{C_o} \quad \ldots (4.41)$$

where V_1, V_2, V_3 and V_4 are respectively given by (4.37),(4.38),(4.39) and (4.40).

4.9. Numerical Illustration

The results which we have obtained will now be illustrated with reference to the data on wheat crop, collected in Patna district of Bihar State during Rabi season 1966-67 under the scheme operated by Institute of Agricultural Research Statistics for evolving suitable sampling procedure for obtaining reliable estimates of yield rates of principal cereal crops at Community Development Block Level. The design of the survey was one of two-phase, two-stage random sampling in which the first-stage units were villages and second-stage units were fields. In the first-phase of sampling about 50 villages were selected at random in each of the Community Development Blocks in the district and in each selected villages, 4 fields were selected at random for obtaining pre-harvest estimates of yield through eye-estimation of the yield of fields. In the second-phase of sampling, a sub-sample of about 25 villages were selected from the first-phase sample and in each of these 25 villages, a random sub-sample of 2 fields was selected from the first-phase sample of 4 fields. For the fields selected in the second-phase, the yield of the crop was estimated by crop-cutting experiments.

The data collected in the survey have been utilised to estimate the yield of wheat crop in Kilogram per hectare for each block in the district. The results obtained are presented in Tables I and II. The estimates obtained are presented along with their estimated variance and percentage standard errors and have been compared with simple estimates based upon crop-cutting experiments alone.

Three type of estimators have been considered:

(i) Simple estimate based upon crop-cutting experiment.

(ii) Biased ratio type estimator (T_3)

(iii) Unbiased ratio-type estimator (T_4)

From tables I and II, it is to be seen that the estimate T_3 is always more than T_4 in all blocks of the district except Masaurhi. This explains that T_3 is positively biased and bias is substantial in some blocks. Also the variance of T_4 is smaller than variance of T_3 in every block of the district except Asthawan. Hence for this type of data, T_4 is certainly to be preferred to T_3.

If no use is made of supplementary information, the estimate of the population mean is simply given by $\bar{y}_{nm}$. This is tabulated in Column (4) of tables I and II respectively. From the comparison of the estimated variances of T_4 and $\bar{y}_{nm}$, it is to be seen that variance of T_4 is less than variance of $\bar{y}_{nm}$ in every block of the district except Masaurhi and Bakhtiarpur. That the variance of T_4 is more than variance of $\bar{y}_{nm}$ in these two blocks, may be due to the fact that the correlation between the crop-cutting values and eye estimated value between primary units and within primary units were low for these blocks.

Table I

State : Bihar, District: Patna, Crop - Wheat, Year and Season 1966-67 (Rabi)

Block wise estimate of average dry yield of wheat (kg/hectare)

Block	$\bar{r}_{nm}$	Mean of the eye estimated yield for fields selected for crop-cutting $\bar{x}_{nm}$	Mean of crop-cutting yield $\bar{y}_{nm}$	Mean of eye-estimated yields for all fields $\bar{x}_{n'm'}$	Estimated yield (biased) T_3	Variance of the estimated yield $V(T_3)$	% S.E. (T_3)	No. of villages (psu) selected in the second phase n	No. of villages selected in the first phase n'	$V(\bar{y}_{nm})$
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
1. Giriak	1.28	729.26	846.04	795.07	1022.50	12692.21	11.01	17	28	11163.1
2. Asthawan	0.99	662.69	635.55	672.05	666.61	1530.56	5.86	25	54	3997.2
3. Rahui	1.06	612.59	578.55	554.86	591.15	3872.51	10.52	21	30	2283.8
4. Dhanarua	1.04	736.44	717.51	689.11	723.55	5889.14	10.60	22	36	2658.9
5. Masaurhi	1.09	1005.93	922.96	937.04	1022.81	9584.81	9.57	19	37	4737.9
6. Chandi	0.83	846.48	704.59	790.86	657.71	8803.63	14.26	19	22	4089.8
7. Paliganj	1.04	644.00	621.23	640.29	666.24	10974.05	15.72	13	26	7665.5
8. Pandarak	1.01	540.47	488.71	446.68	454.50	2914.57	11.87	25	40	1392.6
9. Bakhtiarpur	0.81	698.39	505.67	727.15	594.94	11963.03	18.38	20	33	10121.3

Table II

State: Bihar, District : Patna, Crop - Wheat, Year and Season 1966-67 (Rabi)

Block wise estimate of average dry yield of wheat (kg/hectare)

Block	$\bar{r}_{nm}$	Mean of the eye estimated yield for fields selected for crop-cutting $\bar{x}_{nm}$	Mean of crop-cutting yield $\bar{y}_{nm}$	Mean of eye-estimated yields for all fields $\bar{x}_{n'm'}$	Estimated yield (unbiased) T_4	Variance of the estimated yield $V(T_4)$	o/o S.E. (T_4)	No. of villages (psu) selected in the second phase n	No. of villages selected in the first phase n'	$V(\bar{y}_{nm})$
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
1. Giriak	1.28	729.26	846.04	795.07	929.45	8290.45	9.79	17	28	11163.1
2. Asthawan	0.99	662.69	635.55	672.05	644.34	2496.24	7.75	25	54	3997.2
3. Rahui	1.06	612.59	578.55	554.86	516.41	1619.78	7.79	21	30	2283.8
4. Dhanarua	104	736.44	717.51	689.11	667.34	1498.90	5.80	22	36	2658.9
5. Masaurhi	1.09	1005.93	922.96	937.04	843.90	5326.20	8.64	19	37	4737.9
6. Chandi	0.83	846.48	704.59	790.86	658.46	3001.33	8.32	19	22	4089.8
7. Paliganj	1.04	644.00	621.23	640.29	616.32	1040.82	5.23	13	26	7565.5
8. Pandarak	1.01	540.47	488.71	446.68	392.64	1790.11	10.77	25	40	1292.6
9. Bakhtiarpur	0.81	698.39	505.67	727.15	527.80	11209.48	20.05	20	33	10121.3

CHAPTER V

SUMMARY

In the case of simple random sampling without replacement, unbiased ratio-type estimator was first developed by Hartley and Ross(1954). They assumed that the information on the auxiliary character in the population was completely available. But sometimes, it happens that this information is lacking. Under this situation, the problem was considered by Sukhatme(1962) who resorted to the technique of single stage, double sampling and presented several ratio-type estimators out of which one was an unbiased estimate of the population mean.

In the present investigation, unbiased ratio-type estimators for two-stage sampling design have been presented when the population mean of the auxiliary variable is known or unknown. For obtaining unbiased ratio-type estimators, a technique called "The extended method of symmetric means" to two-stage designs has been developed which is an extension to the technique developed by Tukey (1956) and generalised for the multi-variate case by Robson (1957). By using this extended method, variances of the several estimators considered, for the same order of approximation, have been worked out. While simplifying the expressions for variances, we have assumed that the number of primary units and secondary units in the population are large. The conditions under which unbiased ratio-type estimators were more efficient than the corresponding biased estimators for the two situations viz. when single phase sampling in two stages is adopted and when two-phase sampling in two-stages is adopted, have been obtained. The estimates of variance in double sampling for two-stage designs have been worked out. Using a cost function, the optimum variances for a fixed cost have also been obtained. The results obtained in chapter IV when double sampling is resorted to in two stages, have been illustrated with the help of data on wheat crop , collected in Patna district of Bihar State during Rabi season 1966-67 under the scheme operated by I.A.R.S. for evolving suitable sampling procedure for obtaining reliable estimates of yield rates of principal cereal crops at Community Development Block Level.

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