SOME STUDIES IN TWO STAGE SUCCESSIVE SAMPLING

By

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CONTENTS

I. INTRODUCTION

For a dynamic population a single survey on a particular occasion provides information about population characteristics for that occasion only and does not give any information about the changes which occur in the population. But often, the experimenter apart from estimating the value of the population character for the most recent occasion is interested in estimating the change in the value of population character under study from one occasion to the other. He may also be interested in estimating the average value of the population character on all occasions over a given period of time. Under these circumstances it is essential to repeat the survey on several occasions. Once the decision to study the population on successive occasions is made, several alternatives in designing the sampling plan would arise, viz., (i) choosing a new sample on each occasion ; (ii) retaining the same sample on all occasions ; and (iii) replacing a part of the sample on each occasion.

The relative advantages of the various types of selection procedures would depend on the extent of variability among the units and the variability of changes in these units as well as on the relative importance of information on the population means and the changes in these means. In a two stage design the various alternatives which arise are as follows:

(a) Retaining all the primary stage sampling units (psu's) from occasion to occasion but selecting each time a fresh sample of second-stage-units (ssu's) from the selected psu's.

(b) Retaining a fraction 'p' of the psu's along with the ssu's in those psu's from occasion to occasion and selecting a fraction 'q' of psu's afresh such that p+q =1.

(c) Retaining all the psu's from the preceding occasion but keeping only a fraction 'p' of the ssu's and selecting afresh a fraction 'q' of ssu's in each psu's.

(d) Retaining a fraction 'p' of psu's and from each such psu retaining only a fraction 'r' of the ssu's and selecting afresh a fraction 's' of the ssu's such that r+s =1.

The choice of adopting one of the above procedures would depend on the requirements and the availability of resources. Suppose one is free to alter or retain the composition of the sample and the total sample size is to be same on all occasions. According to Cochran (1963) if one intends to maximize the precision, the statements to be made about the replacement policy are:

(i) For estimating the average over all occasions it is best to draw a new sample on each occasion;

(ii) For estimating change, it is best to retain the same sample throughout all occasions, and is

(iii) For current estimates equal precision is obtained by keeping the same sample or by changing it on every occasion. Replacement of a part of the sample on each occasion may be better than the alternatives.

According to Yates (1960) there are two further points which must be borne in mind in connection with sampling on successive occasions. Firstly repeated survey of the same units may be inexpedient since resistance to the provision of the necessary information may be engendered and secondly repeated survey may result in modifications of these units relative to the rest of the population.

Various research workers have worked on the above lines and obtained results considering specific correlation models in their study. In the present investigation an attempt has been made to work out the expressions for the estimates for a two stage design under a general correlation model. This is considered to be of practical value to the workers engaged in sample surveys in the field of agriculture and animal husbandry sciences.

2. REVIEW OF LITERATURE

Jessen (1942) was the first to study the theory of sampling on successive occasions with partial replacement of units on each occasion. His study was confined to only two occasions. He obtained two independent estimates of the mean on the second occasion, one being the sample mean based on new units only and the other a regression estimate based on the units common to both occasions. These two estimates were weighted with inverse of their variances to get an estimate of the mean on the second occasion with minimum variance. In addition an overall sample mean was also obtained on the first occasion.

Yates (1949) gave a simplified method for estimating the values of the mean on successive occasions by treating each occasion separately. He considered two cases, (i) when the sample on the second occasion was a sub-sample of the original sample and (ii) when the sub-sample retained was supplemented with a fresh sample on the second occasion. Yates extended Jessen's results for the study of one character on two occasions to h occasions under the restrictive conditions of a constant sample size and a fixed replacement fraction at each occasion. He assumed the variability on different occasions and the correlation 'ρ' between consecutive occasions as constant. Assuming further, the correlation between the i th and j th occasion to be $\rho^{|i-j|}$, he obtained the relation,

$$\bar{Y}_h = (1-\phi_h)\left[\bar{Y}'_h + \rho(\bar{Y}_{h-1} - \bar{Y}'_{h-1})\right] + \phi_h \bar{Y}''_h \qquad \text{.. (2.1)}$$

where $\bar{Y}_h$: precise estimate obtained for h th occasion, utilising all the information upto and including the h th occasion

$\bar{Y}_{h-1}$: similar estimate for the previous i.e. (h-1)th occasion.

$\bar{Y}'_h$: mean of units common to h th and (h-1) th occasions

$\bar{Y}'_{h-1}$: mean of units common to (h-1) th and its previous occasion

$\bar{Y}''_h$: mean of units taken afresh in h th occasion

ϕ_h depends on correlation (ρ), the fraction replaced 'q' on each occasion and the number of occasions 'h'. As h increases ϕ_h rapidly tends to a limiting value which depends on 'ρ' and 'q'. He also established the recurrence relationship between ϕ_h and ϕ_{h-1} as

$$\frac{\phi_h}{1-\phi_h} = \frac{q}{p}(1-\rho^2) + \rho^2(\phi_{h-1}) \qquad \text{..... (2.2)}$$

where $p+q = 1$

Patterson's (1950) approach to the problem of sampling was different. He obtained the estimate as a linear function of a set of variates and developed a set of conditions for his estimate to be the most efficient. Using these conditions he obtained an efficient estimate of the population mean on the h th occasion, which is the same as (2.1) worked out by Yates. The recurrence relationship between ϕ_h and ϕ_{h-1} as obtained by him was

$$1-\phi_h = \frac{p}{1 + (q-p)\rho^2 - p\rho^2(1-\phi_{h-1})}$$

which was the same as that established by Yates (vide 2.2). Patterson further put the recurrence relationship in another form as

$$(1-\phi_h)(1-\phi_{h-1}) - (\alpha + \beta)(1-\phi_h) + \alpha\beta = 0$$

where α and β are the roots of the quadratic equation obtained by putting $\phi_h = \phi_{h-1} = \phi$. He proved that with increasing h, $1-\phi_h$ tends numerically to the smallest root of the quadratic

$p\phi^2\rho^2 + \phi(1-\rho^2) - q(1-\rho^2) = 0$ and thus obtained the limiting value of ϕ as

$$\phi = \frac{-(1-\rho^2) + \sqrt{(1-\rho^2)[1-\rho^2(1-4pq)]}}{2p\rho^2}$$

where $p+q = 1$

Patterson also gave an efficient estimate of the difference between the mean on the h th occasion and that on (h-1)th occasion. The case where the sample size varies from occasion to occasion was also considered by him.

Tikkiwal (1953) studied the problem following a more general approach. He considered the correlations between units drawn on successive occasions to vary assuming that correlations follow a product model. According to him $\rho_{ij} = \prod_{t=i}^{j-1} \rho_{t,t+1}$

$1 \leq i < j \leq h$, where ρ_{ij} is the correlation between the same units on i th and j th occasion.

When correlations between consecutive occasions were assumed to be equal on all occasions, he proved that with limiting ρ, the replacement fraction to be effected on different occasions tends to half from above.

Eckler (1955) developed a method of rotation sampling to obtain a minimum variance estimate of the population value (mean and total) by suitably constructing a linear function of sampling values at different times.

Singh (1968) observed that for estimating the mean on the third occasion it would be preferable to repeat the same sample fraction from one occasion to the next, while for estimating the mean overall the occasions the sample-fraction repeated on the second occasion should not be repeated on the third occasion but in its place a sub-sample of the sample selected afresh on the second occasion should be retained.

Singh and Kathuria (1969) studied the problem of successive sampling with partial replacement of units in a multi-stage design. They obtained estimates of the population mean (i) on the second occasion and (ii) on the h th occasion under the following two systems of replacement:

(a) partially replacing psu's and keeping ssu's fixed,

(b) keeping psu's fixed and partially replacing ssu's.

In generalising the results for h occasions, the pattern of variability between psu's and ssu's was assumed to be constant on all occasions.

In the present investigation an attempt has been made to obtain the minimum variance linear unbiased estimates of (i) the population mean on the most recent occasion ; (ii) the change in the population mean from one occasion to another ; and (iii) an overall estimate of the population mean over all occasions for a two stage design. The study is confined to two cases viz.,

(a) partially replacing psu's and keeping ssu's fixed, and

(b) keeping psu's fixed and partially replacing ssu's for a fixed sample size 'n' and under the retention pattern in which 'np' units are retained over all occasions and 'nq' units selected afresh at each occasion (p+q) =1.

The entire investigation has been made under a general correlation pattern. The results obtained by Yates (1949), Patterson (1950) under unistage and by Singh (1968), Singh and Kathuria (1969) under two stage design follow as particular cases under the above retention pattern.

3. SAMPLING ON h OCCASIONS IN A TWO STAGE DESIGN RETAINING A CONSTANT FRACTION 'p' OF THE PRIMARY STAGE UNITS (PSU'S) AND KEEPING THE SECOND STAGE UNITS(SSU'S) FIXED.

3.1. Estimate of mean at the h th occasion

Consider a population consisting of 'N' psu's each containing 'M' ssu's. On the first occasion take a simple random sample (s.r.s) of 'n' psu's and select a s.r.s. of 'm' ssu's from each of the selected psu's, selection being without replacement at each stage. On the second occasion, retain from the first occasion a sub sample of size 'np' of psu's along with ssu's and select afresh 'nq' psu's from the units not selected on the first occasion, (p+q = 1). In each of the 'nq' psu's select 'm' ssu's following the selection procedure as in the first occasion. The psu's retained during the second occasion will remain fixed for the subsequent occasions also but the remaining 'nq' psu's will be selected afresh in each occasion. The sample size is kept constant on each occasion. This is considered keeping in view the operational convenience when actually adopted under field conditions.

Let the character under study be 'X'. Under the pattern of sampling of psu's as indicated below, one can build up an unbiased linear estimate for the population mean ($\bar{\bar{X}}_h$) on the h th occasion and work out its variance.

The pattern

Occasions								
1	**	****						
2	**		****					
3	**			****				
.	**				****			
.	**					****		
.	**						****	
h	**							****
Sampling fraction	np+	nq	nq	nq	nq	nq	nq	nq

\+ same on each occasion.

Let

$\bar{X}_t$: population mean at the t th occasion

$\bar{x}_t$: sample mean at the t th occasion.

$\bar{x}'_t$: mean per ssu on the t th occasion for the nmp units which are common to all the occasions.

$\bar{x}''_t$: mean per ssu on the t th occasion for the nmq units which are selected afresh in the t th occasion.

An estimate of the population mean for the h th occasion utilising all the information collected from first to h th occasion including that on the h th occasion can be written as,

$$\bar{\bar{x}}_h = a_1\bar{x}_1' + b_1\bar{x}_1'' + a_2\bar{x}_2' + b_2\bar{x}_2'' + \ldots + a_h\bar{x}_h' + b_h\bar{x}_h''$$

$$= \sum_{i=1}^{h} (a_i\bar{x}_i' + b_i\bar{x}_i'')$$

Since $E(\bar{x}_i') = E(\bar{x}_i'') = \bar{X}_i$

$$\therefore E(\bar{\bar{x}}_h) = \sum_{i=1}^{h} (a_i + b_i)\bar{X}_i$$

In order that $\bar{\bar{x}}_h$ may be an unbiased estimate of $\bar{X}_h$ the condition should be

$$a_i + b_i = 0 \quad \text{for } i = 1, 2, \ldots, h-1$$

and $$a_h + b_h = 1$$

Hence

$$\bar{\bar{x}}_h = \sum_{i=1}^{h} a_i(\bar{x}_i' - \bar{x}_i'') + \bar{x}_h'' \qquad \ldots (3.1.1)$$

Considering the population to be sufficiently large the terms of order $\frac{1}{N}$ and $\frac{1}{M}$ can be ignored and the variance of the estimate $\bar{\bar{x}}_h$ is given by

$$V(\bar{\bar{x}}_h) = \sum_{i=1}^{h} a_i^2 \left[\frac{Sb_i^2}{np} + \frac{Sw_i^2}{nmp} + \frac{Sb_i^2}{nq} + \frac{Sw_i^2}{nmq} \right]$$

$$+ 2\sum_{i<i'}^{h} a_i a_{i'} \left[\frac{\rho'_{ii'} Sb_i Sb_{i'}}{np} + \frac{\rho''_{ii'} Sw_i Sw_{i'}}{nmp} \right]$$

$$+ (1-2a_h)\left(\frac{Sb_h^2}{nq} + \frac{Sw_h^2}{nmq} \right) \qquad \ldots (3.1.2)$$

where $$Sb_t^2 = \frac{1}{N-1} \sum_{i=1}^{N} (\bar{X}_{it} - \bar{X}_t)^2$$

= Mean square between psu means in the population on the t th occasion (t = 1, 2, ... h)

$$Sw_t^2 = \frac{1}{N(M-1)} \sum_{i=1}^{N} \sum_{j=1}^{M} (x_{ijt} - \bar{X}_{it})^2$$

= Mean square between ssu's within psu's in the population on the t th occasion (t = 1, 2, ... h).

$$\rho'_{tt'} Sb_t Sb_{t'} = \frac{1}{N-1} \sum_{i=1}^{N} (\bar{X}_{it} - \bar{X}_t)(\bar{X}_{it'} - \bar{X}_{t'})$$

= True covariance between psu mean values on the t th and t' th occasion (t, t' = 1, 2, ..., h) $t \neq t'$

$$\rho''_{tt'} Sw_t Sw_{t'} = \frac{1}{N(M-1)} \sum_{i=1}^{N} \sum_{j=1}^{M} (x_{ijt} - \bar{X}_{it})(x_{ijt'} - \bar{X}_{it'})$$

= True covariance between ssu values within psu's on the t th and t' th occasion (t, t' = 1, 2, ..., h) $t \neq t'$

x_{ijt} = the observation at the t th occasion on j th ssu in the i th psu.

$$\bar{X}_{it} = \frac{1}{M} \sum_{j=1}^{M} x_{ijt}$$

Defining $$a_t = Sb_t^2 + \frac{Sw_t^2}{m}$$

and

$$\beta_{tt'} = \rho'_{tt'}\, Sb_t\, Sb_{t'} + (\rho''_{tt'}\, Sw_t\, Sw_{t'})/m$$

The $V(\bar{x}_h)$ given in (3.1.2) takes the form

$$npq\, V(\bar{x}_h) = \sum_{i=1}^{h} a_i^2 a_i + q \sum_{i \neq i'}^{h} a_i a_{i'} \beta_{ii'} + pa_h (1 - 2a_h)$$

$$= \sum_{i=1}^{h} \sum_{i'=1}^{h} a_i a_{i'} \gamma_{ii'} + pa_h (1 - 2a_h) \qquad \ldots\ldots (3.1.3)$$

where $\gamma_{ii'} = q\, \beta_{ii'}$ for $i \neq i'$

$= a_i$ for $i = i'$

Optimum values of a_i's $(i = 1, 2, \ldots, h)$ which will minimise the variance $V(\bar{x}_h)$ may be determined by solving the equations $\frac{d}{da_i} V(\bar{x}_h) = 0, \quad (i = 1, 2, \ldots, h)$, viz.

$$\gamma_{11}a_1 + \gamma_{12}a_2 + \gamma_{13}a_3 + \ldots + \gamma_{1i}a_i + \ldots + \gamma_{1h}a_h = 0$$

$$\gamma_{21}a_1 + \gamma_{22}a_2 + \gamma_{23}a_3 + \ldots + \gamma_{2i}a_i + \ldots + \gamma_{2h}a_h = 0$$

$$\gamma_{31}a_1 + \gamma_{32}a_2 + \gamma_{33}a_3 + \ldots + \gamma_{3i}a_i + \ldots + \gamma_{3h}a_h = 0$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots = 0$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots = 0$$

$$\gamma_{j1}a_1 + \gamma_{j2}a_2 + \gamma_{j3}a_3 + \ldots + \gamma_{ji}a_i + \ldots + \gamma_{jh}a_h = 0$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots = 0$$

$$\gamma_{h1}a_1 + \gamma_{h2}a_2 + \gamma_{h3}a_3 + \ldots + \gamma_{hi}a_i + \ldots + \gamma_{hh}a_h = pa_h$$

These h equations involving h unknowns can be put in the matrix notation

$$PA = B$$

where

$$P = \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} \cdots & Y_{1i} \cdots & Y_{1h} \\ Y_{21} & Y_{22} & Y_{23} \cdots & Y_{2i} \cdots & Y_{2h} \\ Y_{31} & Y_{32} & Y_{33} \cdots & Y_{3i} \cdots & Y_{3h} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ Y_{j1} & Y_{j2} & Y_{j3} \cdots & Y_{ji} \cdots & Y_{jh} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ Y_{h1} & Y_{h2} & Y_{h3} \cdots & Y_{hi} \cdots & Y_{hh} \end{bmatrix}$$

is a h x h matrix of coefficients

$$A' = [\, a_1 \;\; a_2 \;\; a_3 \cdots a_i \cdots a_h \,]$$ is a vector of h unknowns and

$$B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ \vdots \\ pa_h \end{bmatrix}$$ is a vector of h elements.

If P is a non-singular matrix, then,

$$A = P^{-1} B$$

$$A = \begin{bmatrix} \gamma^{11} & \gamma^{12} & \gamma^{13} \ldots & \gamma^{1i} \ldots & \gamma^{1h} \\ \gamma^{21} & \gamma^{22} & \gamma^{23} \ldots & \gamma^{2i} \ldots & \gamma^{2h} \\ \gamma^{31} & \gamma^{32} & \gamma^{33} \ldots & \gamma^{3i} \ldots & \gamma^{3h} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \gamma^{j1} & \gamma^{j2} & \gamma^{j3} \ldots & \gamma^{ji} \ldots & \gamma^{jh} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \gamma^{h1} & \gamma^{h2} & \gamma^{h3} \ldots & \gamma^{hi} \ldots & \gamma^{hh} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ \vdots \\ p\alpha_h \end{bmatrix}$$

where $\gamma^{ij} = \dfrac{\Delta_{ji}}{\Delta_h}$

Δ_h is the determinant of the matrix P i.e.

$$\Delta_h = \begin{vmatrix} \alpha_1 & q\beta_{12} & q\beta_{13} \ldots\ldots & q\beta_{1h} \\ q\beta_{12} & \alpha_2 & q\beta_{23} \ldots\ldots & q\beta_{2h} \\ q\beta_{13} & q\beta_{23} & \alpha_3 \ldots\ldots & q\beta_{3h} \\ \vdots & \vdots & \vdots & \vdots \\ q\beta_{1h} & q\beta_{2h} & q\beta_{3h} \ldots\ldots & \alpha_h \end{vmatrix}$$

and Δ_{ji} is the cofactor of the element in the j th row and the i th column of Δ_h.

Hence

$$a_i = p\alpha_h \gamma^{ih} = p\alpha_h (\Delta_{hi} / \Delta_h), \; (i = 1, 2, \ldots h)$$

Substituting these values, the equation (3.1.1) would be

$$\bar{x}_h = \sum_{i=1}^{h} pa_h \frac{\Delta_{hi}}{\Delta_h} (\bar{x}'_i - \bar{x}''_i) + \bar{x}''_h \quad \ldots\ldots \quad (3.1.4)$$

and (3.1.3) would be

$$npq\ V(\bar{x}_h) = \sum_{i=1}^{h} a_i \sum_{i'=1}^{h} pa_h \frac{\Delta_{hi'}\ \gamma_{ii'}}{\Delta_h} - 2p\, a_h a_h + pa_h$$

Now it is known that $\sum_{i'=1}^{h} \gamma_{ii'} \Delta_{hi'} = \Delta_h$ for $i = h$

$= 0$ " for $i \neq h$

Therefore,

$$V(\bar{x}_h) = (1/npq)(pa_h a_h - 2pa_h a_h + pa_h)$$

$$= \frac{a_h}{nq} (1 - a_h) \quad \ldots \quad (3.1.5)$$

$$= \frac{a_h}{nq} \left(1 - \frac{pa_h\ \Delta_{hh}}{\Delta_h} \right) \quad \ldots \quad (3.1.6)$$

Defining $\delta_{ij} = \dfrac{\beta_{ij}}{\sqrt{a_i a_j}}$ one can see that

$$\Delta_h = \left(\prod_{i=1}^{h} a_i \right) \Delta_h^*$$

where

$$\Delta^{*}_{h} = \begin{vmatrix} 1 & q\delta_{12} & q\delta_{13} & \cdots & q\delta_{1h} \\ q\delta_{12} & 1 & q\delta_{23} & \cdots & q\delta_{2h} \\ q\delta_{13} & q\delta_{23} & 1 & \cdots & q\delta_{3h} \\ \vdots & \vdots & \vdots & & \vdots \\ q\delta_{1h} & q\delta_{2h} & q\delta_{3h} & \cdots & q\delta_{hh} \end{vmatrix}$$

Hence, the estimate of the mean and its variance given in (3.1.4) and (3.1.6) respectively would be

$$\bar{x}_h = \sum_{i=1}^{h} p \sqrt{\frac{a_h}{a_i}} \quad \frac{\Delta^{*}_{hi}}{\Delta^{*}_{h}} \left(\bar{x}'_i - \bar{x}''_i \right) + \bar{x}''_h \qquad \ldots (3.1.7)$$

$$\text{and} \quad V(\bar{x}_h) = \frac{a_h}{nq} \left(1 - p \frac{\Delta^{*}_{hh}}{\Delta^{*}_{h}} \right) \qquad \ldots (3.1.8)$$

where Δ^{*}_{ji} is the cofactor of the element in the j th row and i th column of Δ^{*}_{h} .

If $\delta_{ij} = \delta$ for all i and j then the equation (3.1.8) would be

$$V(\bar{x}_h) = \frac{a_h}{nq} \left[1 - p \frac{1 + (h-2)q\delta}{\left[1+(h-1)q\delta \right] (1-q\delta)} \right] \qquad \ldots (3.1.9)$$

Particular Cases

(i) Two occasions: (h = 2)

Putting h = 2 in (3.1.4) the estimate of the population mean at the second occasion becomes,

$$\hat{\bar{x}}_2 = \sum_{i=1}^{2} p a_2 \frac{\Delta_{2i}}{\Delta_2} (\bar{x}'_i - \hat{\bar{x}}''_i) + \bar{x}''_2$$

$$= - \frac{pq\, a_2\, \beta_{12}}{a_1 a_2 - q^2 \beta_{12}^2} (\bar{x}'_1 - \bar{x}''_1)$$

$$+ \frac{p a_1 a_2}{a_1 a_2 - q^2 \beta_{12}^2} (\bar{x}'_2 - \bar{x}''_2) + \bar{x}''_2 \qquad ..(3.1.10)$$

Putting h = 2 in the equation (3.1.6) one gets

$$V(\hat{\bar{x}}_2) = \frac{a_2}{nq} \left(1 - \frac{p a_1 a_2}{a_1 a_2 - q^2 \beta_{12}^2}\right) \qquad ... \ (3.1.11)$$

This result is essentially the same as obtained by D.Singh and Kathuria (1969).

(ii) Three occasions: (h = 3)

$$\hat{\bar{x}}_3 = p a_3 \left[\frac{\Delta_{31}}{\Delta_3} (\bar{x}'_1 - \hat{\bar{x}}''_1) + \frac{\Delta_{32}}{\Delta_3} (\bar{x}'_2 - \hat{\bar{x}}''_2) \right.$$

$$\left. + \frac{\Delta_{33}}{\Delta_3} (\bar{x}'_3 - \hat{\bar{x}}''_3) \right] + \bar{x}''_3 \qquad ... \ (3.1.12)$$

where

$$\Delta_3 = \begin{vmatrix} \alpha_1 & q\beta_{12} & q\beta_{13} \\ q\beta_{12} & \alpha_2 & q\beta_{23} \\ q\beta_{13} & q\beta_{23} & \alpha_3 \end{vmatrix}$$

$$= \alpha_1\alpha_2\alpha_3 - q^2(\alpha_1\beta_{23}^2 + \alpha_2\beta_{13}^2 + \alpha_3\beta_{12}^2) + 2q^3\beta_{12}\beta_{13}\beta_{23}$$

$$\Delta_{31} = q^2\beta_{12}\beta_{23} - q\beta_{13}\alpha_2$$

$$\Delta_{32} = q^2\beta_{12}\beta_{13} - q\beta_{23}\alpha_1$$

$$\Delta_{33} = \alpha_1\alpha_2 - q^2\beta_{12}^2$$

$$V(\bar{x}_3) = \frac{\alpha_3}{nq}\left[1 - \frac{p\alpha_3(\alpha_1\alpha_2 - q^2\beta_{12}^2)}{\Delta_3}\right]$$

$$= \frac{\alpha_3}{n}\left[1 - \frac{pq(\alpha_1\beta_{23}^2 + \alpha_2\beta_{13}^2 - 2q\beta_{12}\beta_{13}\beta_{23})}{\alpha_1\alpha_2\alpha_3 - q^2(\alpha_1\beta_{23}^2 + \alpha_2\beta_{13}^2 + \alpha_3\beta_{12}^2) + 2q^3\beta_{12}\beta_{13}\beta_{23}}\right] \quad \ldots\ldots (3.1.13)$$

When $\alpha_t = \alpha$ and $\beta_{tt'} = \beta$ for all t

(3.1.13) can be written as

$$V(\bar{x}_3) = \frac{\alpha}{n}\left(1 - \frac{2pq\beta^2}{\alpha^2 + \alpha\beta q - 2q^2\beta^2}\right) \quad \ldots\ldots (3.1.14)$$

This result is the same as obtained by D. Singh (1968).

3.2. Estimate of mean at the j th occasion when data on h occasions is available (h > j)

Sometimes it is required to revise the estimate at the j th occasion when information upto h occasions is available (h > j). An unbiased estimate of the population mean at the j th occasion is,

$$ {}_h\bar{x}_j = a_1(\bar{x}'_1 - \bar{x}''_1) + a_2(\bar{x}'_2 - \bar{x}''_2) + \ldots + a_j(\bar{x}'_j - \bar{x}''_j) + \bar{x}''_j $$

$$ + \ldots\ldots + a_h(\bar{x}'_h - \bar{x}''_h) \qquad \ldots \quad (3.2.1) $$

Now, it can easily be seen that

$$ npq\, V({}_h\bar{x}_j) = \sum_{i=1}^{h} a_i^2 a_i + q \sum_{i \neq i'}^{h} a_i a_{i'} \beta_{ii'} + pa_j(1-2a_j) $$

$$ = \sum_{i=1}^{h} \sum_{i'=1}^{h} a_i a_{i'} \gamma_{ii'} + pa_j(1-2a_j) \qquad \ldots\ldots (3.2.2) $$

where $\gamma_{ii'}$ is as defined in section 3.1.

The optimum values of a_i's which will minimise the variance $V({}_h\bar{x}_j)$ may be obtained by solving the equations

$$ \frac{d}{da_i} V({}_h\bar{x}_j) = 0, \quad (i = 1, 2, \ldots\ldots h) \qquad \ldots (3.2.3) $$

On solving these equations the optimum value of a_i would be

a_i

$a_i = pa_j \frac{\Delta_{ji}}{\Delta_h}$ where Δ_{ji} as defined earlier

is the cofactor of the element common to the j th row and the i th column in Δ_h. Putting the values of a_i in (3.2.1) and (3.2.2) the estimate of the mean at the j th occasion and its variance would be

$$ {}_h\bar{x}_j = \sum_{i=1}^{h} pa_j \frac{\Delta_{ji}}{\Delta_h} (\bar{x}'_i - \bar{x}''_i) + \bar{x}''_j \qquad \ldots (3.2.4)$$

$$ = \sum_{i=1}^{h} p\sqrt{\frac{a_j}{a_i}} \frac{\Delta^*_{ji}}{\Delta^*_h} (\bar{x}'_i - \bar{x}''_i) + \bar{x}''_j \qquad \ldots (3.2.5)$$

and

$$ V({}_h\bar{x}_j) = \frac{a_j}{nq} \left(1 - pa_j \frac{\Delta_{jj}}{\Delta_h}\right) \qquad \ldots (3.2.6)$$

$$ = \frac{a_j}{nq} \left(1 - p \frac{\Delta^*_{jj}}{\Delta^*_h}\right) \qquad \ldots (3.2.7)$$

3.3. A recurrence relationship between a_h and a_{h-1}

In section 3.1. the sample mean has been obtained in the form as given in (3.1.1) which contains the coefficients 'a_i'. Earlier research workers have established some relationship between the coefficients assigned to the estimates on the two consecutive occasions. Similar relationship has also been obtained in the present investigation.

If $\delta_{ih} = \delta_{i,h-1}\,\delta_{h-1,h}$ $(i = 1, 2, \ldots, h-2)$

then $\overset{*}{\Delta}_h = \left[1 + (p-q)\,\delta^2_{h-1,h}\right] \overset{*}{\Delta}_{h-1} - p^2\,\delta^2_{h-1,h}\,\overset{*}{\Delta}_{h-2}$

... (3.3.1)

Now,

$$a_h = p\alpha_h \frac{\Delta_{hh}}{\Delta_h} = p\,\frac{\overset{*}{\Delta}_{hh}}{\overset{*}{\Delta}_h} = p\,\frac{\overset{*}{\Delta}_{h-1}}{\overset{*}{\Delta}_h} \quad \ldots (3.3.2)$$

and a_{h-1} the corresponding coefficient when there are $h-1$ occasions is

$$a_{h-1} = p\alpha_{h-1} \frac{\overset{(h-1)}{\Delta}_{h-1,h-1}}{\Delta_{h-1}} = p\,\frac{\overset{(h-1)^*}{\Delta}_{h-1,h-1}}{\overset{*}{\Delta}_{h-1}} = p\,\frac{\overset{*}{\Delta}_{h-2}}{\overset{*}{\Delta}_{h-1}}$$

... (3.3.3)

where $\overset{*}{\Delta}_{ji}$ as defined earlier in section 3.1 is the cofactor of the element in the j th row and i th column of $\overset{*}{\Delta}_h$. The superscript (h-1) denotes that the determinant pertains to h-1 occasions. From (3.3.1), (3.3.2) and (3.3.3), it follows that

$$a_h = \frac{p}{\left[1 + (p-q)\,\delta^2_{h-1,h}\right] - p\,\delta^2_{h-1,h}\,a_{h-1}}$$ where $a_1 = p$

therefore,

$$\frac{1-a_h}{a_h} = \frac{q}{p}\,(1 - \delta^2_{h-1,h}) + \delta^2_{h-1,h}(1-a_{h-1})$$

.... (3.3.4)

when $\delta_{i,i+1} = \delta$ for $i = 1, 2, \ldots, h-1$

this result is same as obtained by D. Singh and Kathuria (1969). It can further be seen that a limiting value of a_h when $a_h = a_{h-1} = a$ (say) can be given by

$$a = 1 - \frac{(1-\delta^2) \pm \sqrt{(1-\delta^2)[1-\delta^2(1-4pq)]}}{2p\,\delta^2} \qquad \ldots. \quad (3.3.5)$$

3.4. A recurrence relationship between $\bar{x}_h$ and $\bar{x}_{h-1}$

The unbiased linear estimate of the population mean at the h th occasion in a two stage design as given in (3.1.7) can be put as

$$\bar{x}_h = p \frac{\Delta^*_{hh}}{\Delta^*_h} \left[\bar{x}'_h + \frac{\sqrt{a_h}}{\Delta^*_{hh}} \sum_{i=1}^{h-1} \frac{\Delta^*_{hi}}{\sqrt{a_i}} (\bar{x}'_i - \bar{x}''_i) \right] + \left(1 - \frac{p\Delta^*_{hh}}{\Delta^*_h} \right) \bar{x}''_h \qquad \ldots. \quad (3.4.1)$$

where Δ^*_h and Δ^*_{hi} are as defined in section 3.1.

Similarly

$$\bar{x}_{h-1} = p\sqrt{a_{h-1}} \left[\sum_{i=1}^{h-1} \frac{\Delta^{(h-1)*}_{h-1,i}}{\sqrt{a_i}\,\Delta^*_{h-1}} (\bar{x}'_i - \bar{x}''_i) \right] + \bar{x}''_{h-1} \qquad \ldots. \quad (3.4.2)$$

Subtracting $\bar{x}'_{h-1}$ on both sides from (3.4.2), one gets

$$\frac{(\bar{x}_{h-1} - \bar{x}'_{h-1})}{\sqrt{a_{h-1}}} = \frac{p}{\overset{*}{\Delta}_{h-1}} \sum_{i=1}^{h-2} \frac{\Delta^{(h-1)*}_{h-1,i}}{\sqrt{a_i}} (\bar{x}'_i - \bar{x}''_i)$$

$$+ \left(\frac{p\,\Delta^{(h-1)*}_{h-1,h-1}}{\sqrt{a_{h-1}}\;\overset{*}{\Delta}_{h-1}} - \frac{1}{\sqrt{a_{h-1}}} \right) (\bar{x}'_{h-1} - \bar{x}''_{h-1})$$

$$\ldots\ldots \quad (3.4.3)$$

Now it can easily be seen that if $\delta_{ih} = \delta_{i,h-1}\, \delta_{h-1,h}$ $(i=1,2,..h-2)$

then

$$\overset{*}{\Delta}_{hi} = p\; \delta_{h-1,h}\; \Delta^{(h-1)*}_{h-1,i} \qquad \text{for } i = 1,2,\ldots\ldots,h-2$$

$$\overset{*}{\Delta}_{h,h-1} = \delta_{h-1,h} \left(p\, \overset{*}{\Delta}_{h-2} - \overset{*}{\Delta}_{h-1} \right)$$

Therefore, from (3.4.1)

$$\bar{x}_h = p\, \frac{\overset{*}{\Delta}_{hh}}{\Delta^*_h} \Bigg[\bar{x}'_h + \frac{\sqrt{a_h}}{\Delta^*_{h-1}}\, \delta_{h-1,h} \Bigg\{ \sum_{i=1}^{h-2} p\, \frac{\Delta^{(h-1)*}_{h-1,i}}{\sqrt{a_i}} (\bar{x}'_i - \bar{x}''_i)$$

$$+ \frac{(p\,\Delta^*_{h-2} - \Delta^*_{h-1})}{\sqrt{a_{h-1}}} (\bar{x}'_{h-1} - \bar{x}''_{h-1}) \Bigg\} \Bigg] + \left(1 - p\, \frac{\overset{*}{\Delta}_{h-1}}{\Delta^*_h}\right) \bar{x}''_h$$

$$= \frac{p\,\overset{*}{\Delta}_{hh}}{\Delta^*_h} \left[\bar{x}'_h + \frac{\sqrt{a_h}}{\sqrt{a_{h-1}}}\, \delta_{h-1,h} (\bar{x}_{h-1} - \bar{x}'_{h-1}) \right]$$

$$+ \left(1 - p\, \frac{\overset{*}{\Delta}_{h-1}}{\Delta^*_h}\right) \bar{x}''_h$$

$$= a_h \left[\bar{x}'_h + \frac{\sqrt{\sigma_h}}{\sqrt{\sigma_{h-1}}} \delta_{h-1,h} (\bar{x}_{h-1} - \bar{x}'_{h-1}) \right] + (1-a_h) \bar{x}''_h$$

.... (3.4.4)

This is the same expression as as obtained by Tikkiwal(1951) and Patterson (1950) under more restrictive correlation models. It may be remarked that the correlation model considered in the present investigation reduces to the product model as considered by Tikkiwal if one is interested in all the occasions.

3.5. Single stage design

If the design is unistage, then Sw which does not appear in the expression for variance and may be assumed to be zero. The primary stage units in that case become the ultimate sampling units and as such a_t (vide section 3.1) reduces to $Sb_t^2 = S_t^2$ (say) and $\beta_{tt'}$ reduces to $\rho_{tt'} S_t S_{t'}$ where

$$S_t^2 = \frac{1}{N-1} \sum_{i=1}^{N} (\bar{X}_{it} - \bar{X}_t)^2$$ and $\rho_{tt'}$ is the correlation

between the same units (psu's) at the t th and t' th occasions. The estimate of the mean at the h th occasion and its variance are given by

$$\bar{x}_h = \sum_{i=1}^{h} p S_h^2 \frac{\Delta_{hi}}{\Delta_h} (\bar{x}'_i - \bar{x}''_i) + \bar{x}''_h \qquad ... (3.5.1)$$

and

$$V(\bar{x}_h) = \frac{S_h^2}{nq}\left(1 - p\, S_h^2 \,\frac{\Delta_{hh}}{\Delta_h}\right) \qquad \ldots (3.5.2)$$

where

$$\Delta_h = \begin{vmatrix} S_1^2 & q\rho_{12}S_1S_2 & q\rho_{12}S_1S_3 & \cdots & q\rho_{1h}S_1S_h \\ q\rho_{12}S_1S_2 & S_2^2 & q\rho_{23}S_2S_3 & \cdots & q\rho_{2h}S_2S_h \\ q\rho_{13}S_1S_3 & q\rho_{23}S_2S_3 & S_3^2 & \cdots & q\rho_{3h}S_3S_h \\ \vdots & \vdots & \vdots & & \vdots \\ q\rho_{1h}S_1S_h & q\rho_{2h}S_2S_h & \cdots & \cdots & S_h^2 \end{vmatrix}$$

$$V(\bar{x}_h) = \left(\prod_{i=1}^{h} S_i^2\right)\Delta_h^* \quad \text{where}$$

$$\Delta_h^* = \begin{vmatrix} 1 & q\rho_{12} & q\rho_{13} & \cdots & q\rho_{1h} \\ q\rho_{12} & 1 & q\rho_{23} & \cdots & q\rho_{2h} \\ q\rho_{13} & q\rho_{23} & 1 & \cdots & q\rho_{3h} \\ \vdots & \vdots & \vdots & & \vdots \\ q\rho_{1h} & q\rho_{2h} & q\rho_{3h} & \cdots & 1 \end{vmatrix}$$

Substituting these values in (3.5.1) and (3.5.2) one gets

$$\bar{x}_h = \sum_{i=1}^{h} p\,\frac{S_h}{S_i}\,\frac{\Delta_{hi}^*}{\Delta_h^*}\left(\bar{x}_i' - \bar{x}_i''\right) + \bar{x}_h'' \qquad \ldots (3.5.3)$$

and

$$V(\bar{x}_h) = \frac{S_h^2}{nq}\left[1 - p\,\frac{\triangle^*_{hh}}{\triangle^*_h}\right] \quad \ldots (3.5.4)$$

Also from section (3.3) it follows directly that $\delta_{ij} = \rho_{ij}$ and thus

$$V(\bar{x}_h) = \frac{S_h^2}{nq}(1 - a_h) \quad \ldots (3.5.5)$$

where

$$\frac{1 - a_h}{a_h} = \frac{q}{p}\left(1 - \rho^2_{h-1,h}\right) + \rho^2_{h-1,h}(1 - a_{h-1})$$

and

$$\bar{x}_h = a_h\left[\bar{x}'_h + \frac{S_h}{S_{h-1}}\rho_{h-1,h}\left(\bar{x}_{h-1} - \bar{x}'_{h-1}\right)\right] + (1 - a_h)\bar{x}''_h$$

$$\ldots (3.5.6)$$

which is the same as obtained by Tikkiwal (1951) and Patterson (1950). It is remarkable that the variance depends, apart from replacement fraction and ρ, only upon the variance at the last occasion.

3.6. Estimate of change in the means between any two occasions

It may be of interest to estimate the change between any two occasions, not necessarily the consecutive ones and to work out the variance of these estimates. Suppose the data is available for h occasions and it is desired to find the change between the j th and j' th occasions where $h > j' > j$.

An unbiased estimate of the change between j th and j' th occasion utilising the entire information of h occasions is given by

$$C_{jj'} = \sum_{i=1}^{h} a_i (\bar{x}'_i - \bar{x}''_i) + (\bar{x}''_{j'} - \bar{x}''_j) \qquad \ldots (3.6.1)$$

and variance of this estimate is given by

$$npq\; V(C_{jj'}) = \sum_{i=1}^{h} a_i^2 \alpha_i + q \sum_{i \neq i'}^{h} a_i a_{i'} \beta_{ii'} - 2p(a_{j'} \alpha_{j'} - a_j \alpha_j)$$

$$+ \; p(\alpha_j + \alpha_{j'})$$

$$= \sum_{i=1}^{h} \sum_{i'=1}^{h} a_i a_{i'} \gamma_{ii'} - 2p\,(n\, a_{j'} \alpha_{j'} - a_j \alpha_j) + (\alpha_j + \alpha_{j'}) \quad \ldots \quad (3.6.2)$$

where $\gamma_{ii'}$ is same as defined in equation (3.1.3)

Optimum values of a_i's (i = 1, ... h) which will minimise $V(C_{jj'})$ may be obtained by solving the equations

$$\frac{d}{d_{ai}} V(C_{jj'}) = 0, \qquad (i = 1, 2, \ldots\ldots h)$$

or $\qquad PA = E \qquad \ldots . (3.6.3)$

where P and A are same as defined in section 3.1 and

$$E = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ -pa_j \\ 0 \\ \cdot \\ pa_{j'} \\ 0 \\ 0 \\ \cdot \\ 0 \end{bmatrix}$$ is a column vector of h elements.

Assuming Δ_h which is same as in section 3.1 to be non zero and solving (3.6.3) for a_i's one gets,

$$a_i = \frac{1}{\Delta_h} (p\, a_{j'}\, \Delta_{j'i} - pa_j\, \Delta_{ji})$$

Putting these values of a_i's in (3.6.1) the estimate of the change becomes

$$C_{jj'} = \sum_{i=1}^{h} \frac{1}{\Delta_h} (pa_{j'}\, \Delta_{j'i} - pa_j\, \Delta_{ji})(\bar{x}'_i - \bar{x}''_i) + (\bar{x}'_{j'} - \bar{x}''_j)$$

..... (3.6.4)

and the expression for variance given in (3.6.2) becomes

$$npq\, V(C_{jj'}) = \sum_{i=1}^{h} a_i \sum_{i'=1}^{h} \gamma_{ii'} \frac{pa_{j'}\, \Delta_{j'i'} - pa_j \Delta_{ji'}}{\Delta_h}$$

$$-2p(a_{j'}a_{j'} - a_j a_j) + p (a_j + a_{j'})$$

$$\text{Now} \quad \sum_{i'=1}^{h} v_{ii'} \Delta_{ki'} = \Delta_h \quad \text{for } i = k \qquad \text{where } i \; k \; h$$

$$= 0 \quad \text{for } i \neq k$$

$$\therefore \; nqV(C_{jj'}) = \frac{p}{\Delta_h} \left[2a_j a_{j'} \Delta_{jj'} - a_j^2 \Delta_{jj} - a_{j'}^2 \Delta_{j'j'} \right] + (a_j + a_{j'}) \quad \ldots . \; (3.6.5)$$

$$= \frac{p}{\overset{*}{\Delta}_h} \left[2\sqrt{a_j a_{j'}} \; \overset{*}{\Delta}_{j'j} - a_j \overset{*}{\Delta}_{jj} - a_{j'} \overset{*}{\Delta}_{j'j'} \right] + (a_j + a_{j'}) \quad \ldots . \; (3.6.6)$$

If $a_t = a$ for all t (3.6.4) becomes

$$C_{jj'} = \sum_{i=1}^{h} \frac{p}{\overset{*}{\Delta}_h} (\overset{*}{\Delta}_{j'i} - \overset{*}{\Delta}_{ji})(\overline{x}'_i - \overline{x}''_i) + (\overline{x}''_{j'} - \overline{x}''_j) \quad \ldots\ldots (3.6.7)$$

and (3.6.6) becomes

$$nq\,V(C_{jj'}) = \frac{pa}{\overset{*}{\Delta}_h} \left[2\overset{*}{\Delta}_{j'j} - \overset{*}{\Delta}_{jj} - \overset{*}{\Delta}_{j'j'} \right] + 2a \quad \ldots\ldots (3.6.8)$$

Suppose one is interested in finding the variance of the estimate of change between two consecutive occasions say h th and (h-1) th, then from (3.6.6),

$$nqV(C_{h-1,h}) = \frac{p}{\overset{*}{\Delta}_h} \left[2\sqrt{\alpha_h \alpha_{h-1}}\ \overset{*}{\Delta}_{h,h-1} - \alpha_{h-1}\overset{*}{\Delta}_{h-1,h-1} - \alpha_h \overset{*}{\Delta}_{hh} \right] + \alpha_h + \alpha_{h-1} \quad \ldots\ (3.6.9)$$

when $\alpha_t = \alpha$ for all t

$$nqV(C_{h-1,h}) = \frac{p\alpha}{\overset{*}{\Delta}_h} \left[2\overset{*}{\Delta}_{h,h-1} - \overset{*}{\Delta}_{h-1,h-1} - \overset{*}{\Delta}_{hh} \right] + 2\alpha \quad \ldots\ (3.6.10)$$

<u>Particular case</u>

<u>Two occasions</u> ($h = 2$)

Putting $h = 2$ in 3.6.7 and 3.6.10 the estimate and the variance become

$$C_{12} = \frac{p}{1-q\delta}\left(\bar{x}'_2 - \bar{x}'_1\right) + \frac{q(1-\delta)}{1-q\delta}\left(\bar{x}''_2 - \bar{x}''_1\right) \quad ..(3.6.11)$$

and

$$V(C_{12}) = \frac{2\alpha}{n}\,\frac{(1-\delta)}{1-q\delta} \quad \ldots\ (3.6.12)$$

It can be seen that $C_{jj'} = {}_h\bar{x}_{j'} - {}_h\bar{x}_j$ where ${}_h\bar{x}_j$ and ${}_h\bar{x}_{j'}$ are same as in section 3.2. Hence (3.6.5) can also be obtained from the following relation

$$V(C_{jj'}) = V({}_h\bar{x}_j) + V({}_h\bar{x}_{j'}) - 2Cov({}_h\bar{x}_{j'}, {}_h\bar{x}_j) \quad \ldots\ (3.6.13)$$

3.7. An overall estimate of mean

An overall unbiased linear estimate of the population mean over h occasions for the sampling pattern under investigation can be put as

$$E_h = \sum_{i=1}^{h} \omega_i \left[a_i (\bar{x}'_i - \bar{x}''_i) + \bar{x}'_i \right] \quad \ldots \quad (3.7.1)$$

where ω_i ($i = 1, 2, \ldots, h$) are some suitable weights depending upon the relative importance of the occasions. For example in a milk yield survey for estimating the total availability of milk per day in an area if $\bar{x}_t$ is the average daily milk yield per animal in milk at the 't' th occasion then ω_i's ($i = 1, 2, \ldots, h$) can be the proportions of the animals in milk estimated at the t th occasion such that $\sum_{t=1}^{h} \omega_t = 1$.

Neglecting finite population correction factors when N and M are large and covariance terms which are of order 1/N and 1/M, the variance would be

$$npq\, V(E_h) = \sum_{i=1}^{h} a_i^2 \omega_i^2 \alpha_i + q \sum_{i \neq i'}^{h} a_i a_{i'} \omega_i \omega_{i'} \beta_{ii'}$$

$$- 2p \sum_{i=1}^{h} a_i \omega_i^2 \alpha_i + p \sum_{i=1}^{h} \omega_i^2 \alpha_i$$

$$= \sum_{i=1}^{h} \sum_{i'=1}^{h} a_i a_{i'} \omega_i \omega_{i'} \gamma_{ii'} - 2p \sum_{i=1}^{h} a_i \omega_i^2 \alpha_i + p \sum_{i=1}^{h} \omega_i^2 \alpha_i \quad \ldots \quad (3.7.2)$$

where $\gamma_{ii'} = q\beta_{ii'}$ for $i \neq i'$ and equal to α_i for $i = i'$

Optimum values of a_i's ($i = 1, 2, \ldots, h$) which will minimise the variance $V(\bar{E}_h)$ are obtained from the equations

$\frac{d}{da_i} V(\bar{E}_h) = 0$ ($i = 1, 2, \ldots\ldots, h$) which can be expressed as

$$P_O \; A_O = B_O$$

where

$$P_O = \begin{bmatrix} \omega_1 Y_{11} & \omega_2 Y_{12} & \omega_3 Y_{13} & \cdots\cdots & \omega_h Y_{1h} \\ \omega_1 Y_{21} & \omega_2 Y_{22} & \omega_3 Y_{23} & \cdots\cdots & \omega_h Y_{2h} \\ \omega_1 Y_{31} & \omega_2 Y_{32} & \omega_3 Y_{33} & \cdots\cdots & \omega_h Y_{3h} \\ \vdots & \vdots & \vdots & & \vdots \\ \omega_1 Y_{h1} & \omega_2 Y_{h2} & \omega_3 Y_{h3} & \cdots\cdots & \omega_h Y_{hh} \end{bmatrix}$$

is a h x h matrix of coefficients.

$A'_O = [a_1, a_2, a_3, \ldots, a_i \ldots, a_h]$ is a row vector of unknowns and

$$B_O = \begin{bmatrix} p\omega_1 e_1 \\ p\omega_2 e_2 \\ p\omega_3 e_3 \\ \vdots \\ p\omega_i e_i \\ \vdots \\ p\omega_h e_h \end{bmatrix}$$

is a column vector of h elements

Assuming the matrix P_O to be non-singular

$$A_O = P_O^{-1} \; B_O$$

It can be easily seen that the estimates of a_i's are:

$$a_i = \frac{p}{\omega_i \Delta_h} \sum_{j=1}^{h} \Delta_{ji} \alpha_j \omega_j \qquad (i = 1, 2, \ldots, h)$$

where Δ_{ji} is the cofactor of the element in the j th row and i th column of Δ_h.

Substituting the values of a_i's in (3.7.1) and (3.7.2)

$$E_h = \sum_{i=1}^{h} \omega_i \left[\left(\frac{p}{\omega_i \Delta_h} \sum_{j=1}^{h} \Delta_{ji} \alpha_j \omega_j \right) (\bar{X}'_i - \bar{X}''_i) + \bar{X}''_i \right]$$

$$\ldots\ldots \quad (3.7.3)$$

and

$$npq\, V(E_h) = \sum_{i=1}^{h} \sum_{i'=1}^{h} \left[\left[\left(\frac{p}{\omega_i \Delta_h} \sum_{j=1}^{h} \Delta_{ji} \alpha_j \omega_j \right) \left(\frac{p}{\omega_{i'} \Delta_h} \sum_{j=1}^{h} \Delta_{ji'} \alpha_j \omega_j \right) \right] \times$$

$$\omega_i \omega_{i'} V_{ii'} \Bigg] - 2p \sum_{i=1}^{h} a_i \omega_i^2 \alpha_i + p \sum_{i=1}^{h} \omega_i^2 \alpha_i$$

$$= \frac{p^2}{\Delta_h^2} \sum_{i=1}^{h} \sum_{j=1}^{h} \sum_{j'=1}^{h} \Delta_{ji} \omega_j \omega_{j'} \alpha_j \alpha_{j'} \sum_{i'=1}^{h} V_{ii'} \Delta_{j'i'}$$

$$- 2p \sum_{i=1}^{h} a_i \omega_i^2 \alpha_i + p \sum_{i=1}^{h} \omega_i^2 \alpha_i$$

$$\text{Since} \quad \sum_{i'=1}^{h} V_{ii'} \Delta_{j'i'} = \Delta_h \quad \text{if } i = j'$$

$$= 0 \quad \text{if } i \neq j'$$

Therefore,

$$npq\, V(E_h) = \frac{p^2}{\Delta_h} \sum_{i=1}^{h} \sum_{j=1}^{h} \Delta_{ji}\, \omega_j \alpha_j\, \omega_i \alpha_i - 2p \sum_{i=1}^{h} a_i \omega_i^2 \alpha_i + p \sum_{i=1}^{h} \omega_i^2 \alpha_i$$

$$= -p \sum_{i=1}^{h} a_i \omega_i^2 \alpha_i + p \sum_{i=1}^{h} \omega_i^2 \alpha_i$$

$$\therefore V(E_h) = \frac{1}{nq} \Big[p \Big(\frac{\Delta_{h\omega}}{\Delta_h} - 1 \Big) + \sum_{i=1}^{h} \omega_i^2 \alpha_i \Big] \quad \ldots (3.7.4)$$

where

$$\Delta_{h\omega} = \begin{vmatrix} \alpha_1 & q\beta_{12} & q\beta_{13} \cdots & q\beta_{1h} & \omega_1 \alpha_1 \\ q\beta_{12} & \alpha_2 & q\beta_{23} \cdots & q\beta_{2h} & \omega_2 \alpha_2 \\ q\beta_{13} & q\beta_{23} & \alpha_3 \cdots & q\beta_{3h} & \omega_3 \alpha_3 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ q\beta_{1h} & q\beta_{2h} & q\beta_{3h} \cdots & \alpha_h & \omega_h \alpha_h \\ \omega_1 \alpha_1 & \omega_2 \alpha_2 & \omega_3 \alpha_3 \cdots & \omega_h \alpha_h & 1 \end{vmatrix}$$

Particular cases

(i) Two occasions (h = 2)

An overall unbiased linear estimate of the population mean and when there are two occasions is given by

$$E_2 = \frac{p\alpha_2 (a_1\omega_1 + q\omega_2\beta_{12})(\bar{x}'_1 - \bar{x}''_1) + p\alpha_1 (a_2\omega_2 + q\omega_1\beta_{12})(\bar{x}'_2 - \bar{x}''_2)}{\alpha_1\alpha_2 - q^2\beta_{12}^2}$$

$$+ \omega_1 \bar{x}'_1 + \omega_2 \bar{x}'_2 \quad \ldots (3.7.5)$$

and its variance

$$V(E_2) = \frac{1}{nq}\left[\frac{-p\left[\alpha_1\alpha_2(\alpha_1\omega_1^2+\alpha_2\omega_2^2-2\omega_1\omega_2 q\beta_{12})\right]}{\alpha_1\alpha_2 - q^2\beta_{12}^2} + \omega_1^2\alpha_1+\omega_2^2\alpha_2\right] \quad \ldots\ (3.7.6)$$

Under the restrictions $\alpha_1 = \alpha_2 = \alpha$ and $\beta_{12} = \beta$

$$E_2 = \frac{p\alpha}{\alpha^2 - q^2\beta^2}\left[(\alpha\omega_1 - q\omega_2\beta)(\bar{x}'_1 - \bar{x}''_1) + (\alpha\omega_2 - q\omega_1\beta)(\bar{x}'_2 - \bar{x}''_2)\right]$$

$$+\ \omega_1\bar{x}'_1 + \omega_2\bar{x}'_2 \quad \ldots\ (3.7.7)$$

and

$$V(E_2) = \frac{\alpha}{n}\left[\frac{(\omega_1^2+\omega_2^2)(\alpha^2-q\beta^2)+2p\alpha\omega_1\omega_2}{\alpha^2-q^2\beta^2}\right] \quad \ldots\ (3.7.8)$$

These expressions are as obtained by D. Singh (1968).

(ii) <u>Three occasions</u> (h=3)

Putting h = 3 in (3.7.3) the overall estimate of mean is of the form

$$E_3 = \omega_1 a_1(\bar{x}'_1 - \bar{x}''_1) + \omega_2 a_2(\bar{x}'_2 - \bar{x}''_2) + \omega_3 a_3(\bar{x}'_3 - \bar{x}''_3)$$

$$+\ \omega_1\bar{x}'_1 + \omega_2\bar{x}'_2 + \omega_3\bar{x}'_3 \quad \ldots\ (3.7.9)$$

where

$$a_1 = \frac{p}{\omega_1 \Delta_3} (\alpha_1 \omega_1 \Delta_{11} + \alpha_2 \omega_2 \Delta_{21} + \alpha_3 \omega_3 \Delta_{31})$$

$$a_2 = \frac{p}{\omega_2 \Delta_3} (\alpha_1 \omega_1 \Delta_{12} + \alpha_2 \omega_2 \Delta_{22} + \alpha_3 \omega_3 \Delta_{32})$$

$$a_3 = \frac{p}{\omega_3 \Delta_3} (\alpha_1 \omega_1 \Delta_{13} + \alpha_2 \omega_2 \Delta_{23} + \alpha_3 \omega_3 \Delta_{33})$$

$$\Delta_3 = \alpha_1 \alpha_2 \alpha_3 - q^2 (\alpha_1 \beta_{23}^2 + \alpha_2 \beta_{13}^2 + \alpha_3 \beta_{12}^2) + 2q^3 \beta_{12} \beta_{13} \beta_{23}$$

$$\Delta_{11} = \alpha_2 \alpha_3 - q^2 \beta_{23}^2 \; ; \; \Delta_{12} = q^2 \beta_{13} \beta_{23} - q \alpha_3 \beta_{12}$$

$$\Delta_{13} = q^3 \beta_{12} \beta_{23} - q \alpha_2 \beta_{13} \; ; \; \Delta_{32} = q^2 \beta_{12} \beta_{13} - q \alpha_1 \beta_{23}$$

$$\Delta_{33} = \alpha_1 \alpha_2 - q^2 \beta_{12}^2 \; ; \; \Delta_{22} = \alpha_1 \alpha_3 - q^2 \beta_{13}^2$$

The variance of the estimate, putting h=3 in (3.7.4) is

$$V(E_3) = \frac{1}{nq} \left[p \left(\frac{\Delta_{3\omega}}{\Delta_3} - 1 \right) + \sum_{i=1}^{3} \omega_i^2 \alpha_i \right] \quad \ldots (3.7.10)$$

where

$$\Delta_{3\omega} = \begin{vmatrix} \alpha_1 & q\beta_{12} & q\beta_{13} & \omega_1 \alpha_1 \\ q\beta_{12} & \alpha_2 & q\beta_{23} & \omega_2 \alpha_2 \\ q\beta_{13} & q\beta_{23} & \alpha_3 & \omega_3 \alpha_3 \\ \omega_1 \alpha_1 & \omega_2 \alpha_2 & \omega_3 \alpha_3 & 1 \end{vmatrix}$$

when $\alpha_t = \alpha$

and $\beta_{tt'} = \beta$ for all t and t', ($t \neq t'$)

$$V(E_3) = \frac{a}{n}\left[\frac{a^2+a\beta q - 2q\beta^2}{a^2+a\beta q - 2q^2\beta^2}\right]\left(\sum_{i=1}^{3}\omega_i^2\right)$$

$$+ \frac{a}{n}\left[\frac{pa\beta}{a^2+a\beta q - 2q^2\beta^2}\right]\left[2\omega_1\omega_2 + 2\omega_1\omega_3 + 2\omega_2\omega_3\right] \qquad \ldots\ldots (3.7.11)$$

If $\omega_1 = \omega_2 = \omega_3 = 1/3$

$$V(E_3) = \frac{a}{3n}\left[1 + \frac{2p\beta}{a+2q\beta}\right] \qquad \ldots\ldots (3.7.12)$$

These results are the same as obtained by D. Singh (1968)

3.8. Optimum replacement fraction

Variance of the estimate of mean at the h th occasion as given in (3.1.8) ~~can also be put as~~ is

$$V(\bar{x}_h) = \frac{a_h}{nq}\left[1 - \frac{\Delta^*_{hh}}{\Delta^*_h}\right] \qquad \ldots (3.8.1)$$

Also variance of the improved estimate of the mean at the j th occasion based on the information upto h occasions ($j < h$) vide (3.2.7) is

$$V({}_h\bar{x}_j) = \frac{a_j}{nq}\left[1 - p\frac{\Delta^*_{jj}}{\Delta^*_h}\right] \qquad \ldots\ldots (3.8.2)$$

under the conditions

$$a_t = a, \quad \beta_{tt'} = \beta \quad \text{and} \quad \delta_{tt'} = \delta \qquad \text{for all } t, t' \; (t \neq t')$$

it can be seen that $\overset{*}{\Delta}_{jj} = \overset{*}{\Delta}_{hh}$ for all j.

The equation (3.8.2) reduces to

$$V({}_h\bar{x}_j) = \frac{a_j}{nq} \left[1 - p \frac{\overset{*}{\Delta}_{hh}}{\overset{*}{\Delta}_h} \right] \quad \ldots \ (3.8.3)$$

Evidently, the optimum 'q', for which $V(\bar{x}_h)$ is minimum also minimises $V({}_h\bar{x}_j)$. Therefore, the replacement fraction which is obtained by minimising the variance of the mean at the h th occasion is not only optimum for the h th occasion but is also optimum for the improved estimates on all the previous occasions.

The optimum replacement fraction is given by the equation.

$$\frac{d}{dq} V(\bar{x}_h) = 0 \quad \ldots . \ (3.8.4)$$

This equation after simplification can be expressed as

$$\delta \left[\delta(h-1) - (h-2) \right] q^2 - 2q + 1 = 0 \quad \ldots . \ (3.8.5)$$

Solving (3.8.5) the optimum q, for h occasions, is given by

$${}_hq_O = \frac{1 - \sqrt{1 - \delta\left[\delta(h-1) - (h-2)\right]}}{\delta\left[\delta(h-1) - (h-2)\right]}$$

$$= \frac{1}{1 + \sqrt{1 - \delta\left[\delta(h-1) - (h-2)\right]}}$$

$$= \frac{1}{1+\sqrt{(1-\delta)^2 + h\delta(1-\delta)}} \qquad \ldots (3.8.6)$$

Since δ is usually less than unity, it is evident from (3.8.6) that optimum q would be small, if the number of occasions is large.

If δ is unity then, the optimum replacement fraction is also unity. When $\delta = 0$, optimum q is half. Optimum values of q and the gain in precision of successive sampling with partial replacement over complete replacement for different values of $\delta \geq 0.5$ and h are given in Table 1.

It can be seen from the table that (for $\delta \geq 0.5$) as δ increases the replacement percentage and the gain in precision both increase. It means that for larger δ, a larger portion of the new units should be added to the sample on the second and the subsequent occasions. For fixed δ although the replacement fraction decreases with the increase of h but the precision increases with the increase of h.

Particular cases

(i) Two occasions (h = 2)

Putting h = 2 in equation (3.8.6) one obtains

$${}_2q_0 = \frac{1}{1+\sqrt{1-\delta^2}} \qquad \ldots\ldots (3.8.7)$$

(ii) Three occasions (h = 3)

$$ {}_3q_O = \frac{1}{1+\sqrt{1+\delta-2\delta^2}} \quad \ldots\ (3.8.8)$$

These values of optimum replacement fraction are same as obtained by D. Singh (1968). In the case of unistage δ's reduce to correlation coefficients i.e. ρ's, and optimum q's are given by

$$ {}_hq_O = \frac{1}{1+\sqrt{1-\rho\left[(h-1)\rho-(h-2)\right]}} \quad \ldots\ (3.8.9)$$

4. SAMPLING ON h OCCASIONS IN A TWO STAGE DESIGN RETAINING A FRACTION p OF THE SECONDARY STAGE UNITS (SSU'S) FROM ALL THE PSU'S

4.1. Estimate of mean at h th occasion

Let N be the number of psu's in the population, each containing M ssu's. A random sample of n psu's is drawn and from each selected psu a sample of 'm' ssu's is selected. At both the stages selection is done without replacement. All the psu's selected are retained on all occasions. On the second and the subsequent occasions a fixed fraction 'p' of the ssu's is retained and the sample is supplemented by fresh fraction 'q' of ssu's drawn by simple random sampling ($p + q = 1$).

An unbiased linear estimate of the population mean on the h th occasion for the sampling pattern given in (3.1) utilising all the information collected upto and including the h th occasion can be written as

$$\bar{x}_h = \sum_{i=1}^{h} b_i \left(\bar{x}_i' + \bar{x}''_i \right) + \bar{x}''_h \qquad \ldots\ldots (4.1.1)$$

and its variance

$$V(\bar{x}_h) = \sum_{i=1}^{h} b_i^2 \left[V(\bar{x}'_i - \bar{x}''_i) \right] + V(\bar{x}''_h) + \sum_{i<i'} b_i b_{i'} \operatorname{Cov}\left[(\bar{x}'_i - \bar{x}''_i), (\bar{x}'_{i'} - \bar{x}''_{i'}) \right] + \sum_{i=1}^{h} b_i \operatorname{Cov}\left[\bar{x}''_h, (\bar{x}_i' - \bar{x}''_i) \right]$$

$$= \frac{1}{npq}\left[\sum_{i=1}^{h}\sum_{i'=1}^{h} b_i\, b_{i'}\, Y^*_{ii'} - 2p\sigma^*_h b_h + p\sigma^*_h\right] + \frac{Sb_h^2}{n} \qquad \ldots(4.1.3)$$

where $Y^*_{ii'} = q\sigma^2_{ii'} = q\sigma\, q\rho''_{i,i'}\,\frac{S_{wi}\,Sw_{i'}}{m}$ = for $i \neq i'$

$$= \sigma^*_i = \frac{Sw_i^2}{m} \quad \text{for } i = i'$$

$$V(\bar{x}'_i) = \frac{Sb_i^2}{n} + \frac{Sw_i^2}{hmp}$$

$$V(\bar{x}''_i) = \frac{Sb_i^2}{n} + \frac{Sw_i^2}{nmq}$$

$$Cov(\bar{x}'_i, \bar{x}''_i) = \frac{S_{bi}^2}{n}$$

$$Cov(\bar{x}'_i, \bar{x}''_{i'}) = Cov(\bar{x}''_i, \bar{x}'_{i'}) = Cov(\bar{x}''_i, \bar{x}''_{i'}) = \rho'_{ii'}\frac{Sb_i\,Sb_{i'}}{n}$$

and

$$Cov(\bar{x}'_i, \bar{x}'_{i'}) = \rho'_{ii'}\frac{S_{bi}\,Sb_{i'}}{n} + \rho''_{i,i'}\frac{Sw_i\,Sw_{i'}}{npm}$$

ρ' is the correlation between psu's and ρ'' is between ssu's within psu's.

Optimum values of b_i's $(i = 1, 2, \ldots, h)$ which will minimise the variance may be obtained by solving the equations

$$\frac{d}{d_{bi}} V(\bar{x}_h) = 0, \quad (i = 1, 2, \ldots, h) \qquad \ldots (4.1.4)$$

These equations are of the same form as equation given in (1) vide section 3.1. Solving these equations for h unknowns, b_i (i = 1, 2, ..., h) is obtained as

$$b_i = p\, a_h^* \frac{D_{hi}}{D_h}$$

Substituting the value of b_i equation (4.1.1) takes the form

$$\bar{x}_h = \sum_{i=1}^{h} p\, a_h^* \frac{D_{hi}}{D_h} \left(\bar{x}'_i - \bar{x}''_i \right) + \bar{x}''_h \qquad \ldots. \; (4.1.5)$$

and equation (4.1.3) becomes

$$V(\bar{x}_h) = \frac{a_h^*}{nq} \left[1 - p a_h^* \frac{D_{hh}}{D_h} \right] + \frac{Sb_h^2}{n}$$

$$= \frac{a_h^*}{nq} \left[1 - b_h \right] + \frac{Sb_h^2}{n} \qquad \ldots. \; (4.1.6)$$

D_{hi} being the cofactor of the element common to the h th row and i th column in D_h where

$$D_h = \begin{vmatrix} a_1^* & q\beta_{12}^* & q\beta_{13}^* & \ldots\ldots & q\beta_{1h}^* \\ q\beta_{12}^* & a_2^* & q\beta_{23}^* & \ldots\ldots & q\beta_{2h}^* \\ q\beta_{13}^* & q\beta_{23}^* & a_3^* & \ldots\ldots & q\beta_{3h}^* \\ \ldots & \ldots & \ldots & \ldots\ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots\ldots & \ldots \\ q\beta_{1h}^* & q\beta_{2h}^* & q\beta_{3h}^* & \ldots\ldots & a_h^* \end{vmatrix}$$

$$= \left(\prod_{i=1}^{h} \frac{S^2 w_i}{m} \right) D^*_h$$

Where

$$D^*_h = \begin{vmatrix} 1 & q\rho''_{12} & q\rho''_{13} & \cdots & q\rho''_{1h} \\ q\rho''_{12} & 1 & q\rho''_{23} & \cdots & q\rho''_{2h} \\ q\rho''_{13} & q\rho''_{23} & 1 & \cdots & q\rho''_{3h} \\ \vdots & \vdots & \vdots & & \vdots \\ q\rho''_{1h} & q\rho''_{2h} & q\rho''_{3h} & \cdots & 1 \end{vmatrix}$$

Similarly $D_{hh} = \left[\prod_{i=1}^{h-1} \frac{Sw_i^2}{m} \right] D^*_{hh}$

Now

$$\frac{a^*_h \, D_{hh}}{D_h} = \frac{D^*_{hh}}{D^*_h}$$

Hence the estimate of the mean and its variance take the form

$$\bar{x}_h = \sum_{i=1}^{h} p \frac{Sw_h \, D^*_{hi}}{Sw_i \, D^*_h} \left(\bar{x}'_i - \bar{x}''_i \right) + \bar{x}''_h \quad \ldots \quad (4.1.7)$$

and

$$V(\bar{x}_h) = \frac{Sw_h^2}{amq} \left[1 - \frac{p \, D^*_{hh}}{D^*_h} \right] + \frac{S^2_{bh}}{n} \quad \ldots \quad (4.1.8)$$

It is interesting note that the variance depends upon the components of variation only at the last occasion. As such the assumptions of equality of the components for all occasions is not needed. Also it can be seen that variance is independent of p'.

Particular case

Two occasions $(h = 2)$

$$\bar{x}_2 = -pq \frac{Sw_2 (\bar{x}'_1 - \bar{x}''_1)}{Sw_1 (1 - q^2 \rho''^2_{12})} + \frac{p(\bar{x}'_2 - \bar{x}''_2)}{(1 - q^2 \rho''^2_{12})} + \bar{x}''_2 \quad \ldots (4.1.9)$$

and

$$V(\bar{x}_2) = \frac{Sw_2^2}{nmq} \left[1 - \frac{p}{(1 - q^2 \rho''^2_{12})} \right] + \frac{Sb_2^2}{n} \quad \ldots (4.1.10)$$

These results have also been obtained by D. Singh and Kathuria (1969) under the restrictions

$$Sw_1^2 = Sw_2^2 = Sw^2 \quad \text{and}$$
$$Sb_1^2 = Sb_2^2 = Sb^2$$

4.2. Estimate of the mean at the j th occasion when data on h occasions is available $(h > j)$

An unbiased linear estimate of the population mean at the j th occasion utilising the information collected upto and including the h th occasion is given by

$$ {}_h\bar{x}_j = \sum_{i=1}^{h} b_i(\bar{x}'_i - \bar{x}''_i) + \bar{x}''_j \qquad \ldots (4.2.1) $$

and its variance,

$$ V({}_h\bar{x}_j) = \frac{1}{nqp} \left[\sum_{i=1}^{h} \sum_{i'=1}^{h} b_i b_{i'} V^*_{ii'} + pa^*_h(1-2b_h) \right] + \frac{Sb_j^2}{n} \qquad \ldots (4.2.2) $$

where $V^*_{ii'}$, a^*_h and Sb_j^2 are as defined in section 4.1.

The optimum values of b_i's which will minimize the variance $V({}_h\bar{x}_j)$ may be obtained by solving the equation $\frac{d}{d_{bi}} V({}_h\bar{x}_j) = 0 \quad (i = 1,2,\ldots,h) \quad \ldots (4.2.3)$

From this the optimum value of b_i is obtained as

$$ b_i = pa^*_j \frac{D_{ji}}{D_h} $$ where as defined earlier in section 4.1 D_{ji} is the cofactor of the element common to the j th row and the i th column in D_h. Substituting these values of b_i's in (4.2.1) and (4.2.2) and defining D^*_h as in section 4.1 the estimate of mean and its variance is obtained as

$$ {}_h\bar{x}_j = \sum_{i=1}^{h} pa^*_j \frac{D_{ji}}{D_h} (\bar{x}'_i - \bar{x}''_i) + \bar{x}''_j \qquad \ldots (4.2.4) $$

and

$$V({}_h\bar{x}_j) = \frac{\sigma_j^*}{nq}\left[1 - b_j\right] + \frac{Sb_j^2}{n}$$

$$= \frac{Sw_j^2}{nmq}\left[1 - p\,\frac{D^*_{jj}}{D^*_h}\right] + \frac{Sb_j^2}{n} \quad \ldots (4.2.5)$$

4.3. A recurrence relationship between b_h and b_{h-1}

To establish a recurrence relationship between the coefficients in h th occasion (b_h) and that in (h-1) th occasion (b_{h-1}), the procedure adopted is the same as in Section 3.3. From section 4.1

$$b_h = p\,a_h^*\,\frac{D_{hh}}{D_h} = p\,\frac{D^*_{hh}}{D^*_h} = p\,\frac{D^*_{h-1}}{D^*_h}$$

and

$$b_{h-1} = p\,a^*_{h-1}\,\frac{D^{(h-1)}_{(h-1,\,h-1)}}{D_{h-1}} = p\,\frac{D^{(h-1)*}_{h-1,\,h-1}}{D^*_{h-1}} = p\,\frac{D^*_{h-2}}{D^*_{h-1}}$$

$$\ldots (4.3.1)$$

where D^*_{ji} is the cofactor of the element in the j th row and i th column of D^*_h. The superscript (h-1) denotes that the determinant pertains to (h-1) occasions

If $\rho''_{ih} = \rho''_{i,h-1}\,\rho''_{h-1,h}$ $\quad (i = 1, 2, \ldots, h-2)$

then following the method adopted in section 3.3 it is seen that

$$b_h = \frac{p}{\left[1 + (p-q)\,\rho''^2_{h-1,h}\right] - p\rho''^2_{h-1,h}\, b_{h-1}} \quad \ldots (4.3.2)$$

Therefore

$$\frac{1 - b_h}{b_h} = \frac{q}{p}\left(1 - \rho''^2_{h-1,h}\right) + \rho''^2_{h-1,h}\left(1 - b_{h-1}\right) \quad \ldots\ldots (4.3.3)$$

when $\rho''_{i,i+1} = \rho''$ for $i = 1, 2, \ldots, h-1$, this result is the same as obtained by D. Singh and Kathuria (1969). The limiting value of b_h when sampling is carried over a sufficient number of occasions is obtained by writing $b_h = b_{h-1} = b$ in (4.3.2) and then solving for b which is given by

$$b = 1 - \frac{-\left(1 - \rho''^2_{h-1,h}\right) \pm \sqrt{\left(1 - \rho''^2_{h-1,h}\right)\left[1 - \rho''^2_{h-1,h}(1-4pq)\right]}}{2p\,\rho''^2_{h-1,h}} \quad \ldots (4.3.4)$$

4.4. <u>A recurrence relationship between the estimates of the of the means in two consecutive occasions</u>

Under the assumptions given in section 4.3 and following the procedure adopted in section 3.4, the estimate given in (4.1.5) can be put in the form

$$\bar{x}_h = p\,\frac{D^*_{hh}}{D^*_h}\left[\bar{x}'_h + \frac{Sw_h}{Sw_{h-1}}\,\rho''_{h-1,h}\,(\bar{x}_{h-1} - \bar{x}'_{h-1})\right]$$

$$+ \left(1 - p\,\frac{D^*_{hh}}{D^*_h}\right)\bar{x}''_h \qquad \ldots\ldots \quad (4.4.1)$$

If $Sw_h = Sw_{h-1}$ the expression (4.4.1) reduces to

$$\bar{x}_h = p\,\frac{D^*_{hh}}{D^*_h}\left[\bar{x}'_h + \rho''_{h-1,h}(\bar{x}_{h-1} - \bar{x}'_{h-1})\right]$$

$$+ \left(1 - p\,\frac{D^*_{hh}}{D^*_h}\right)\bar{x}''_h$$

$$= b_h\left[\bar{x}'_h + \rho''_{h-1,h}(\bar{x}_{h-1} - \bar{x}'_{h-1})\right] + (1-b_h)\,\bar{x}''_h$$

$$\ldots \quad (4.4.2)$$

4.5. Estimate of the change in the means between any two occasions

One may be interested to find the estimate of the change between the estimates of the population mean on the j th and j' th occasions and the variance of this change when data is available for h occasions where $h \geq j' > j$.

The best linear unbiased estimate of the change between j th and j' th occasions ($j' > j$) is given by

$$C^*_{j'j} = \sum_{i=1}^{h} b_i (\bar{x}'_i - \bar{x}''_i) + (\bar{x}''_{j'} - \bar{x}''_j) \quad \ldots (4.5.1)$$

where $b_i = \frac{1}{D_h} \left[p\, a^*_{j'} \, D_{j'i} - p a^*_j \, D_{ji} \right]$

and the variance of this change

$$V(C^*_{j'j}) = \frac{1}{npq} \left[\sum_{i=1}^{h} \sum_{i'=1}^{h} b_i b_{i'} v^*_{ii'} - 2p(b_{j'} a^*_{j'} - b_j a^*_j) \right.$$

$$\left. + p(a^*_j + a^*_{j'}) \right] + \frac{1}{n} \left[Sb_j^2 + Sb_{j'}^2 - 2\rho'_{jj'} Sb_j Sb_{j'} \right]$$

where $v^*_{ii'} = q\rho''_{ii'} \frac{Sw_i \; Sw_{i'}}{m}$, for $i \neq i'$

$= \frac{Sw_i^2}{m}$ for $i = i'$

(the notation being defined in section 4.1)

Now

~~$V(C^*_{j'j})$~~ $V(C^*_{j'j}) = \frac{p}{nq} \left[\frac{2a^*_j \, a^*_{j'} D_{jj'} - a^{*2}_j D_{jj} - a^{*2}_{j'} D_{j'j'}}{D_h} \right]$

$$+ \frac{1}{nq} (a^*_j + a^*_{j'}) + \frac{1}{n} \left[Sb_j^2 + Sb_{j'}^2 - 2\rho'_{jj'} Sb_j Sb_{j'} \right]$$

$\ldots (4.5.2)$

$$= \frac{p}{nq} \left[\frac{2 \left(a_j^* \, a_{j'}^* \, D_{jj'}^* - a_j^* \, D_{jj}^* - a_{j'}^* \, D_{j'j'}^* \right)}{D_h^*} \right] + \frac{1}{nq} (a_j^* + a_{j'}^*)$$

$$+ \frac{1}{n} \left[Sb_j^2 + Sb_{j'}^2 - 2\rho_{jj'}^2 \, Sb_j \, Sb_{j'} \right] \quad \ldots \ (4.5.3)$$

with the assumptions

$Sb_j = Sb_{j'} = Sb$ for all j and j'

$Sw_t = Sw_{t'} = Sw$ for all t and t'

and

$a_t^* = a^*$ for all t

with the equations (4.5.1) and (4.5.3) reduce to

$$C_{j'j}^* = \sum_{i=1}^{h} b_i \left(\bar{x}'_i - \bar{x}''_i \right) + \bar{x}''_{j'} - \bar{x}''_j$$

$$\text{with } b_i = \frac{p}{D_h^*} \left(D_{j'i}^* - D_{ji}^* \right) \quad \ldots \ (4.5.4)$$

and

$$V(C_{j'j}^*) = p \, \frac{Sw^2}{mnq} \left[\frac{2D_{j'j}^* - D_{jj}^* - D_{j'j'}^*}{D_h^*} \right] + \frac{2\, Sw^2}{nmq}$$

$$+ \frac{2Sb^2}{n} \left(1 - \rho'_{j'j} \right) \quad \ldots \ (4.5.5)$$

Particular Cases

(i) Two occasions (h = 2)

Putting j' = 2 and j = 1 in the equations (4.5.4) and (4.5.5) the estimate of the change and its variance would be as follows:

$$C^*_{21} = \frac{p}{1-q\rho''_{12}}(\bar{x}'_2 - \bar{x}'_1) + \frac{q(1-\rho''_{12})}{1-q\rho''_{12}}(\bar{x}''_2 - \bar{x}''_1) \quad \ldots (4.5.6)$$

and

$$V(C^*_{21}) = 2\left[\frac{1-\rho''_{12}}{1-q\rho''_{12}}\right]\frac{Sw^2}{mn} + 2(1-\rho'_{12})\frac{Sb^2}{n} \quad \ldots (4.5.7)$$

(ii) Three occasions (h = 3)

The estimate of the change between the third and the first occasion is given by

$$C^*_{31} = \frac{p}{D^*_3}\Big[(D^*_{31} - D^*_{11})(\bar{x}'_1 - \bar{x}''_1) + (D^*_{32} - D^*_{12})(\bar{x}'_2 - \bar{x}''_2)$$

$$+ (D^*_{33} - D^*_{13})(\bar{x}'_3 - \bar{x}''_3)\Big] + \bar{x}''_3 - \bar{x}''_1 \quad \ldots (4.5.8)$$

where

$$D^*_3 = 1 - q^2(\rho''^2_{12} + \rho''^2_{13} + \rho''^2_{23}) + 2q^3\rho''_{12}\rho''_{13}\rho''_{23}$$

$$D^*_{13} = D^*_{31} = q^2 \rho''_{12} \rho''_{23} - q\rho''_{13}$$

$$D^*_{12} = D^*_{21} = -q\rho''_{12} + q^2 \rho''_{13} \rho''_{23}$$

$$D^*_{32} = D^*_{23} = -q\rho''_{23} + q^2 \rho''_{12} \rho''_{13}$$

$$D^*_{11} = 1 - q^2 \rho''^2_{23} \qquad D^*_{22} = 1 - q^2 \rho''^2_{13}$$

$$D^*_{33} = 1 - q^2 \rho''^2_{12}$$

and

$$V(C^*_{31}) = \frac{\left[2(1 - p\rho''_{13}) + pq(\rho''_{12} + \rho''_{23})^2 - 2q(\rho''^2_{12} + \rho''^2_{13} + \rho''^2_{23}) + 4q^2 \rho''_{12} \rho''_{13} \rho''_{23} \right]}{1 - q^2(\rho''^2_{12} + \rho''^2_{13} + \rho''^2_{23}) + 2q^3 \rho''_{12} \rho''_{13} \rho''_{23}}$$

$$\frac{Sw^2}{mn} + 2(1 - \rho'_{13}) \frac{S_b^2}{n} \qquad \ldots (4.5.9)$$

4.6. An overall estimate of mean

An overall unbiased linear estimate of the population mean over h occasions for the sampling pattern under investigation can be put following the notations explained in section 4.1 as

$$E_h = \sum_{i=1}^{h} \omega_i \left[b_i (\bar{x}'_i - \bar{x}''_i) + \bar{x}''_i \right] \qquad \ldots (4.6.1)$$

where $\sum_{t=1}^{h} \omega_t = 1$

and

$$V(E_h) = \frac{1}{npq} \left[\sum_{i=1}^{h} \sum_{i'=1}^{h} b_i b_{i'} \omega_i \omega_{i'} V_{ii'}^{*} - 2p \sum_{i=1}^{h} b_i \omega_i^2 a_i^{*} + p \sum_{i=1}^{h} \omega_i^2 a_i^{*} \right] + \frac{1}{n} \left[\sum_{i=1}^{h} \omega_i^2 Sb_i^2 \right]$$

$$\pm \frac{1}{n} \left[\sum_{i \neq i'}^{h} \omega_i \omega_{i'} \rho'_{ii'} Sb_i Sb_{i'} \right] \quad \ldots (4.6.2)$$

where $V_{ii'}^{*} = q\beta_{ii'}^{*} = q\rho''_{ii'} Sw_i Sw_{i'}/m$ for $i \neq i'$

$= a_i^{*} = Sw_i^2/m$ for $i = i'$

Optimum values of b_i's ($i = 1, 2, \ldots, h$) which will minimise $V(E_h)$ may be obtained by solving the equations,

$$\frac{d}{d_{bi}} V(E_h) = 0 \quad (i = 1, 2, \ldots, h) \quad \ldots (4.6.3)$$

These h equations can solved following the same procedure as adopted in section 3.7. The solution obtained is

$$b_i = \frac{p}{D_h \omega_i} \sum_{j=1}^{h} D_{ji} a_j^{*} \omega_j , \quad (i = 1, 2, \ldots, h)$$

Substituting the values of b_i's , equations (4.6.1) and (4.6.2) would be

$$E_h = \sum_{i=1}^{h} \left[\left(\frac{p}{D_h} \sum_{j=1}^{h} D_{ji} a_j^{*} \omega_j \right) (\bar{x}'_i - \bar{x}''_i) \right] + \sum_{i=1}^{h} \omega_i \bar{x}'_i$$

$$\ldots (4.6.4)$$

and

$$V(E_h) = \frac{1}{nq} \left[p \left(\frac{D_{h\omega}}{D_h} - 1 \right) + \sum_{i=1}^{h} \omega_i^2 a_i^* \right]$$

$$+ \frac{1}{n} \left[\sum_{i=1}^{h} \omega_i^2 Sb_i^2 + \sum_{i \neq i'}^{h} \omega_i \omega_{i'} \rho'_{ii'} Sb_i Sb_{i'} \right]$$

$$\ldots\ldots (4.6.5)$$

where

$$D_{h\omega} = \begin{vmatrix} a_1^* & q\beta_{12}^* & q\beta_{13}^* & \ldots\ldots & q\beta_{1h}^* & \omega_1 a_1^* \\ q\beta_{12}^* & a_2^* & q\beta_{23}^* & \ldots\ldots & q\beta_{2h}^* & \omega_2 a_2^* \\ q\beta_{13}^* & q\beta_{23}^* & a_3^* & \ldots\ldots & q\beta_{3h}^* & \omega_3 a_3^* \\ \ldots & \ldots & \ldots & \ldots\ldots & \ldots & \ldots \\ q\beta_{1h}^* & q\beta_{2h}^* & q\beta_{3h}^* & \ldots\ldots & a_h^* & \omega_h a_h^* \\ \omega_1 a_1^* & \omega_2 a_2^* & \omega_3 a_3^* & \ldots\ldots & \omega_h a_h^* & 1 \end{vmatrix}$$

Assuming $Sb_t = Sb_{t'} = S_b$ and $Sw_t = Sw_{t'} = Sw$ for all t and t'

$$a_t^* = a^* = \frac{Sw^2}{m}$$

and

$$\beta_{tt'} = \rho''_{tt'} \frac{Sw^2}{m} = \rho''_{tt'} a^*$$

Then

$$E_h = \sum_{i=1}^{h} \left(\frac{pa^*}{D_h^*} \sum_{j=1}^{h} D_{ji}^* \omega_j \right) (\bar{x}'_i - \bar{x}''_i) + \sum_{i=1}^{h} \omega_i \bar{x}'_i$$

$$\ldots(4.6.6)$$

and

$$V(E_h) = \frac{Sw^2}{mnq} \left[p \left(\frac{D_{h\omega}^*}{D_h^*} - 1 \right) + \sum_{i=1}^{h} \omega_i^2 \right]$$

$$+ \frac{Sb^2}{n} \left[\sum_{i=1}^{h} \omega_i^2 + \sum_{i \neq i'}^{h} \omega_i \omega_{i'} \rho'_{ii'} \right] \quad \ldots (4.6.7)$$

where

$$D^*_{hw} = \begin{vmatrix} 1 & q\rho''_{12} & q\rho''_{13} & \cdots & q\rho''_{1h} & \omega_1 \\ q\rho''_{12} & 1 & q\rho''_{23} & \cdots & q\rho''_{2h} & \omega_2 \\ q\rho''_{13} & q\rho''_{23} & 1 & \cdots & q\rho''_{3h} & \omega_3 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ q\rho''_{1h} & q\rho''_{2h} & q\rho''_{3h} & \cdots & 1 & \omega_h \\ \omega_1 & \omega_2 & \omega_3 & \cdots & \omega_h & 1 \end{vmatrix}$$

Particular case

Two occasions (h = 2)

Putting h = 2 in (4.6.6) and (4.6. 7) the following results are obtained

$$E_2 = \frac{p(\omega_1 - q\rho''_{12}\omega_2)(\bar{x}'_1 - \bar{x}''_1) + \omega_1 \bar{x}''_1}{1 - q^2\rho''^2_{12}}$$

$$+ \frac{p(\omega_2 - q\rho''_{12}\omega_1)(\bar{x}'_2 - \bar{x}''_2) + \omega_2 \bar{x}''_2}{1 - q^2 \rho''^2_{12}} \quad \ldots (4.6.8)$$

$$V(E_2) = \frac{Sw^2}{mn} \left[\frac{(\omega_1^2 + \omega_2^2)(1 - q\rho''^2_{12}) + 2\omega_1\omega_2 p\rho''_{12}}{1 - q^2\rho''^2_{12}} \right]$$

$$+ Sb^2/n \left[\omega_1^2 + \omega_2^2 + 2\omega_1\omega_2 \rho'_{12} \right] \quad \ldots (4.6.9)$$

If $\omega_1 = \omega_2 = \frac{1}{2}$

Then

$$V(E_2) = \frac{Sw^2}{mn} \left[\frac{1+\rho''_{12}}{1+q\,\rho''_{12}} \right] + \frac{S_b^2}{2n} \left(1+\rho'_{12} \right) \quad ..(4.6.10)$$

4.7 Optimum replacement fraction

Sometimes, one may be confronted with situations where the correlation (ρ'') between second-stage units on any two occasions does not differ much. In such cases, $\rho''_{tt'}$ may be taken to be constant for all t and t'. Under the assumptions $\rho''_{tt'}$ = constant for all t and t' ($t \neq t'$) it can be seen, as in section 3.8, that optimum q obtained by minimising $V(\bar{x}_h)$ [vide (4.1.8)] also minimises $V({}_h\bar{x}_j)$ [vide (4.2.5)] for all j. Thus q obtained in the manner explained above is not only optimum for the mean on h th occasion but also optimum for all the improved estimates on all previous occasions.

Optimum q in this case is given by

$$_hq_o = \frac{1}{1+\sqrt{(1-\rho'')^2 + h\rho''(1-\rho'')}} \quad .... (4.7.1)$$

5. COMPARISON OF ESTIMATES

It would be of interest to know the efficiency of the estimate of the mean $\bar{x}_h$ discussed in section 3 over the estimate discussed in section 4. Moreover the study of the comparison of the estimates discussed in sections 3 and 4 with the estimates obtained in simple random sampling will be useful.

The efficiencies which have been considered in this section are as follows:

(i) Efficiency of the estimate $\bar{x}_h$ given in section 3.1 over complete replacement.

(ii) Efficiency of the estimate $\bar{x}_h$ given in section 4.1 over complete replacement.

(iii) Efficiency of the estimate $\bar{x}_h$ given in section 3.1 over $\bar{x}_h$ given in section 4.1.

Comparisons of the estimates for general pattern of correlations were, however, not been possible. As such two correlations models conveniently considered are (i) $\delta_{ij} = \delta$ and (ii) $\delta_{ij} = {}^{|i-j|}$ for all i and j, ($i \neq j$).

5.1. Efficiency of the estimate of mean $\bar{x}_h$ given in section 3.1 over complete replacement.

Table 3 gives the percentage gain in efficiency (say) G) of the estimate of mean $\bar{x}_h$ given in section 3.1 over complete replacement for different values of h, q, ρ', P ρ'' and ϕ/m under the assumptions $\delta_{ij} = \delta$ for all i and j ($i \neq j$). The conclusions drawn from this table are as follows:

(i) The efficiency G increases monotonically with h for all values of q, ρ', ρ'' and ϕ/m.

(2) In most of the cases G is maximum for $q = 0.5$ approximately. Even in the case of few exceptions (when ρ' and ρ'' are very large and h is small say 2 or 3) when the maximum efficiency is attained for $q = 0.75$ approximately, the gain in efficiency as compared to the case when $q = 0.5$ is not much.

(3) For given h and q

(a) G increases as ϕ/m increases when $\rho' < \rho''$

(b) G remains constant as ϕ/m increases when $\rho' = \rho''$

(c) G decreases as ϕ/m increases when $\rho' > \rho''$

It may be remarked that ϕ/m is the ratio of variation between secondaries to the variation between primaries. Thus an increase in ϕ/m indicates an increase in the variation between secondaries and correspondingly a decrease in the between primary variations. It is perhaps, due to the reason that for an increasing between secondary variations, G increases only when correlations between secondaries is more than the correlations between primaries on different occasions. Similar arguments hold for cases (b) and (c) given above.

(4) For fixed h, q, ρ' and ϕ/m G increases as ρ'' increases.

(5) For fixed h, q, ρ'' and ϕ/m, G increases as ρ' increases.

The rates of increase in G in cases (4) and (5) above are affected by ϕ/m in the same way as in case (3) i.e. G increases rapidly with increasing ρ'' for higher values of ϕ/m and vice versa. Also G increases slowly with increasing ρ' for higher values of ϕ/m.

The percentage of efficiency of the $\bar{x}_h$ as worked out under geometric model is given in Table 6. It can be seen from the table that the incase of geometric model of correlation ($\delta_{ij} = \delta^{|i-j|}$ for all i and j, ($i \neq j$), the trend of efficiency remains same as in the model discussed earlier but the numerical values as expected are less consistently. Also for higher values of h (≥ 3), G in this model increases very slowly.

5.2. Efficiency of the estimate of mean $\bar{x}_h$ given in Section (§ 4.1) over complete replacement

The variance of the estimate of mean $\bar{x}_h$ as given in (4.1.8) is independent of ρ'. The percentage gain in efficiency of the estimate of mean over complete replacement for different values of h, q, ρ'' and ϕ/m is given in Table 4. for the model $\rho''_{ij} = \rho''$ for all i and j, ($i \neq j$) and in table 7 for the model $\rho''_{ij} = \rho''^{|i-j|}$ for all i and j. It can be seen from the table 4 that

(i) G increases as h increases for all values of q, ρ'' and ϕ/m.

(ii) G is maximum, mostly for q = 1/2 except for a few cases where ρ'' is very high and h is small (≤ 3),

(iii) For fixed ϕ/m and q , G increases as ρ'' increases.

The table overall, indicates that mostly the gain in efficiency is not much for smaller values of ρ''.

Table 7 shows that inthe case of geometric model of correlation the trend of efficiency remains same as in the previous model but the numerical values of efficiency are less consistently. As expected G practically remains constant as h increases ($h \geq 3$) for all sets of values of q, ρ'' and ϕ/m.

5.3. Efficiency of the estimate of mean $\bar{x}_h$ given in Section (3.1) over $\bar{\bar{x}}_h$ given in section (4.1)

Table 5 gives the percentage efficiency of the estimate of mean given in Section (3.1) over that given in Section (4.1) for the model $\rho''_{ij} = \rho''$ and $\delta_{ij} = \delta$ for all i and j, for different values of h, q, ρ', ρ'' and ϕ/m. The conclusions drawn from the table are as follows:

(i) G increases as ϕ/m decreases for all sets of values of h, q, ρ' and ρ''.

(ii) G increases as ρ' increases for fixed h, q, ϕ/m and ρ''.

(iii) The behaviour of G with ρ'' for a fixed values of h, q and ϕ/m is not systematic. However, for $\rho' = 0.9$, G increases as ρ'' increases for all sets of h, q and ϕ/m.

(iv) Sampling pattern discussed in section 3 is more efficient than the one discussed in section 4, for the estimate of mean for all values of h, q, ϕ/m, ρ' and ρ'' with a few exceptions. For example when q = 0.75, $\rho'' = 0.9$, $\rho' = 0.50$ and ϕ/m 0.5 for all h ($h \leq 5$) G is negative.

Percentage gain in efficiency of the estimate of mean given in section 3.1 over that given in section 4.1 for the geometric model $\delta_{ij} = \delta^{|i-j|}$ is given in table 8. It can be seen that $\delta_{ij} = \delta^{|i-j|}$ holds if $\rho'_{ij} = \rho'^{|i-j|}$, $\rho''_{ij} = \rho''^{|i-j|}$ and

$$\frac{\rho'_{ij}\, Sb_i}{Sb_j} = \frac{\rho''_{ij}\, Sw_i}{Sw_j}.$$

Therefore the estimates given in 3.1 and 4.1 are comparable under the model $\delta_{ij} = \delta^{|i-j|}$

The broad conclusions are the same as found in the case of previous model but the numerical value of efficiency is reduced consistently . G is negative for a particular combination of $\rho' = 0.5$, $\rho'' = 0.9$, $\phi/m \geq 0.5$ for all h and q considered in the table.

6. AN ILLUSTRATION

A large-scale sample survey was taken up by the Institute of Agricultural Research Statistics in Krishna delta area of Andhra Pradesh during 1967-69 to estimate the availability of milk and its disposal in different seasons in the area and the cost of production of milk.

The sampling design was one of stratified multistage random sampling with villages as primary stage units (psu's) and households in the village as second stage units (ssu's). The entire area to be surveyed was divided into eight sectors on the basis of a number of milch animals in the population. In each sector 12 villages (4 groups of three villages each) were selectdd at random. Out of these four groups of villages in each sector , two groups nearer to each other were allotted for the cost of production inquiry and the remaining two groups for studies on availability of milk . The 48 villages selected for cost study remained fixed throughout the period of enquiry (two years) , while the 48 villages selected for availability study which was continued for a period of one year were selected afresh during each season. Thus three such sets of villages were selected for availability study. Each season consisted of three to four rounds , a round being approximately of a month's duration. During the first round in a village 8 producer households were selected at random for recording the data. Out of these eight households two commercial producer households were fixed for all the rounds in a season but the remaining 6 producer households were selected afresh in the second and the subsequent rounds without replacement. The pattern of selection is the same as discussed in section 4. The interval of recording the data by trained enumerators was one month.

The items of information collected were particulars regarding individual animals in the selected households, production and utilisation of milk, quantity and composition of feed consumed by animals and procurement of cattle feeds etc. The data on milk yield of individual animals and quantity of feeds and fodders actually fed to them on the day of enumerators' visit were collected by actual weighment and other information through direct observation and careful inquiry.

The villages selected for cost study as mentioned earlier were kept fixed throughout the period of inquiry. In each village selected for cost study four commercial producer households were selected and were visited continuously for a period of two years by trained enumerators once in each fortnight. Main items of data collected were the same as in the case of availability study. Additional information collected in this study pertained to quantum and type of labour and wage rates.

Data collected during rainy season and pertaining to availability study has been considered here. In this example, sampling design has been considered as two-stage design spread over three the occasions. Since in an actual survey S^2b_t, Sw^2_t, $\rho'_{tt'}$ and $\rho''_{tt'}$ are not known, they are estimated from the sample estimates. Unbiased estimates of these (assuming N and M to be large) are given by

$$Est(Sb^2_t) = sb^2_t - \frac{sw^2_t}{m}$$

$$Est(Sw^2_t) = sw^2_t$$

$$Est.(\rho'_{tt'} Sb_t Sb_{t'}) = r'_{tt'} sb_t sb_{t'} - r''_{tt'} \frac{sw_t \; sw_{t'}}{m}$$

$$Est.(\rho''_{tt'} Sw_t Sw_{t'}) = r''_{tt'} sw_t sw_{t'}$$

where $sb^2_t = \frac{1}{n-1} \sum_{k=1}^{n} (\bar{y}_{tk.} - \bar{y}_{t..})^2$

$$sw^2_t = \frac{1}{n(mp-1)} \sum_{k=1}^{n} \sum_{l=1}^{mp} (y_{tkl} - \bar{y}_{tk.})^2$$

$$sb_{tt'} = \frac{1}{n-1} \sum_{k=1}^{n} (\bar{y}_{tk.} - \bar{y}_{t..})(\bar{y}_{t'k.} - \bar{y}_{t'..})$$

$$sw_{tt'} = \frac{1}{n(mp-1)} \sum_{k=1}^{n} \sum_{l=1}^{mp} (y_{tkl} - \bar{y}_{tk.})(y_{t'kl} - \bar{y}_{t'k.})$$

$$r'_{tt'} = \frac{sb_{tt'}}{sb_t \; sb_{t'}} \quad \text{and} \quad r''_{tt'} = \frac{sw_{tt'}}{sw_t \; sw_{t'}}$$

Considering round as an occasion, the current estimate as well as the improved estimates of the average daily milk yield (kg) per buffalo in milk, in each round have been obtained from (4.1.7) along with variances utilising (4.1.8). Before obtaining these estimates it would be desirable to know the estimates of the population variances and correlation coefficients.

Estimates of true variance		1st round	2nd round	3rd round
Between villages	Est Sb^2	0.6259	0.7339	0.6550
Between households within villages	Est. Sw^2	0.6975	0.5861	0.9236

Estimates of correlation coefficients between the three rounds

		1st and 2nd round	2nd and third round	1st and third round
Between villages	ρ'	0.69	0.61	0.56
Between households within villages	ρ''	0.58	0.55	0.57

It can be seen that the values of $\phi/m = (Sw_h^2)/(mSb_h^2)$ for the three occasions range from 0.10 to 0.18.

The estimates of average milk yield as mentioned earlier for are given in table 2. It is observed from the table that the percentage gain of successive sampling with partial replacement over simple random sampling (i.e. complete replacement) in this case is not substantial because of poor correlations between ssu's on successive occasions. These results are in agreement with the results presented in table 4.

The optimum replacement fraction for estimating the population mean at the third occasion has been obtained by minimising the variance given in (4.1.8) with respect to q i.e. from the equation $\frac{d}{dq} V(\bar{x}_h) = 0$. This optimum freplacement fraction worked out to be 0.506, i.e. slightly higher than half.

SUMMARY

It is well known that in sample surveys auxiliary information can be used to improve upon the estimates. In the case of successive sampling the same variate is kept under observation on different occasions. The observations on the earlier occasions are used as ancillary information to improve the estimate of the population character under study at the subsequent occasions. In this investigation an attempt has been made to obtain the minimum variance linear unbiased estimates of

1. the population mean on the most recent occasion ;
2. the change in the population mean from one occasion to another ; and
3. an overall estimate of the population mean over all occasions, for a dynamic population using a two stage design.

The study is confined to two cases viz.,

(a) Partially replacing primary stage units (psu's) and keeping second stage units (ssu's) fixed ; and

(b) Keeping psu's fixed and partially replacing ssu's, for a fixed sample size 'n' and under the retention pattern in which 'np' units are retained over all occasions and 'nq' units are selected afresh at each occasion (p + q = 1).

The entire investigation has been made under a general correlation pattern. The results obtained by some of the other research workers in the field follow particular cases of this investigation. A comparison of the efficiency of the two sampling patterns has been discussed. A suitable example is also given to illustrate the application of the estimates obtained.

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APPENDIX

(Tables)

Table 1

Optimum replacement fractions with corresponding gain in efficiencies of $\bar{x}_h$ given in (3.1) over complete replacement.

h → / δ ↓	2		3		4		5	
δ	Optimum % replaced	% gain in precision	Optimum % replaced	% gain in precision	Optimum % replaced	% gain in precision	Optimum % replaced	% gain in precision
0.5	53.5	7.2	50.0	12.5	47.2	16.7	44.9	20.2
0.6	55.5	11.0	51.5	19.2	48.5	25.5	46.1	30.7
0.7	58.3	16.7	54.0	28.7	50.9	38.1	48.3	45.8
0.8	62.5	25.0	58.1	43.2	54.8	57.7	52.1	69.7
0.9	69.6	39.3	65.3	69.3	62.1	93.9	59.5	115.0
0.95	76.2	52.4	72.4	94.7	69.5	130.8	67.1	162.6

Table 2

Estimates of the average daily milk yield per animal in milk

Occasion	Current estimate (kg)			Improved estimate (kg)		
	Estimate	S.E.	Percentage gain over SRS	Estimate	S.E.	Percentage gain over SRS
1	1.582	0.1258	-	1.553	0.1249	1.48
2	1.615	0.1335	0.72	1.631	0.1332	1.05
3	1.721	0.1299	1.06	1.721	0.1299	1.06

Table 3

Percentage efficiency of the estimate of the mean $\bar{x}_h$ given in Section 3.1 over complete replacement under the assumption $\delta_{ij} = \delta$ for all i and j (i ≠ j).

h	ρ'	$\frac{\phi}{m}$	$q=0.25$			$q=0.50$			$q=0.75$		
		ρ''	0.5	0.7	0.9	0.5	0.7	0.9	0.5	0.7	0.9
2	0.5	.025	4.99	5.10	5.21	7.14	7.30	7.46	5.76	5.90	6.05
		.050	4.99	5.20	5.41	7.14	7.45	7.78	5.76	6.04	6.33
		.100	4.99	5.39	5.81	7.14	7.75	8.40	5.76	6.30	6.87
		.250	4.99	5.89	6.88	7.14	8.53	10.11	5.76	6.99	8.43
		.500	4.99	6.54	8.35	7.14	9.56	12.54	5.76	7.93	10.75
		1.000	4.99	7.41	10.47	7.14	10.97	16.22	5.76	9.24	14.52
		2.000	4.99	8.35	12.91	7.14	12.54	20.81	5.76	10.75	19.70
	0.7	.025	10.30	10.47	10.63	15.92	16.22	16.52	14.20	14.52	14.84
		.050	10.14	10.47	10.79	15.64	16.22	16.81	13.91	14.52	15.16
		.100	9.86	10.47	11.10	15.14	16.22	17.37	13.38	14.52	15.77
		.250	9.16	10.47	11.89	13.92	16.22	18.85	12.13	14.52	17.42
		.500	8.35	10.47	12.91	12.54	16.22	20.81	10.75	14.52	19.70
		1.000	7.41	10.47	14.28	10.97	16.22	23.52	9.24	14.52	23.07
		2.000	6.54	10.47	15.75	9.56	16.22	26.59	7.93	14.52	27.17
	0.9	.025	18.53	18.78	19.04	32.81	33.41	34.03	36.63	37.64	38.69
		.050	18.05	18.54	19.04	31.70	32.84	34.03	34.81	36.68	38.69
		.100	17.19	18.09	19.04	29.73	31.80	34.03	31.74	34.98	38.69
		.250	15.15	17.01	19.04	25.32	29.33	34.03	25.43	31.14	38.69
		.500	12.91	15.75	19.04	20.81	26.59	34.03	19.70	27.17	38.69
		1.000	10.47	14.28	19.04	16.22	23.52	34.03	14.52	23.07	38.69
		2.000	8.35	12.91	19.04	12.54	20.81	34.03	10.75	19.70	38.69

Table 3 (Contd)

h	p'	φ/m	q=0.25, p'' 0.5	q=0.25, 0.7	q=0.25, 0.9	q=0.50, 0.5	q=0.50, 0.7	q=0.50, 0.9	q=0.75, 0.5	q=0.75, 0.7	q=0.75, 0.9
3	0.5	.025	9.37	9.57	9.76	12.49	12.77	13.05	9.37	9.59	9.81
		.050	9.37	9.75	10.15	12.49	13.04	13.60	9.37	9.80	10.25
		.100	9.37	10.11	10.89	12.49	13.55	14.67	9.37	10.21	11.11
		.250	9.37	11.05	12.91	12.49	14.90	17.63	9.37	11.30	13.55
		.500	9.37	12.27	15.70	12.49	16.68	21.90	9.37	12.76	17.22
		1.000	9.37	13.91	19.75	12.49	19.14	28.48	9.37	14.83	23.25
		2.000	9.37	15.70	24.55	12.49	21.90	36.94	9.37	17.22	31.79
	0.7	.025	19.43	19.75	20.08	27.95	28.48	29.03	22.74	23.25	23.78
		.050	19.13	19.75	20.39	27.44	28.48	29.56	22.27	23.25	24.29
		.100	18.58	19.75	20.98	26.53	28.48	30.58	21.41	23.25	25.28
		.250	17.24	19.75	22.53	24.35	28.48	33.29	19.40	23.25	27.99
		.500	15.70	19.75	24.55	21.90	28.48	36.94	17.22	23.25	31.79
		1.000	13.91	19.75	27.27	19.14	28.48	42.10	14.83	23.25	37.49
		2.000	12.27	19.75	30.24	16.68	28.48	48.07	12.76	23.25	44.64
	0.9	.025	35.96	36.50	37.04	60.72	61.98	63.28	62.06	64.00	66.03
		.050	34.97	35.99	37.04	58.40	60.78	63.28	58.60	62.15	66.03
		.100	33.18	35.06	37.04	54.36	58.61	63.28	52.88	58.91	66.03
		.250	29.02	32.81	37.04	45.58	53.56	63.28	41.58	51.78	66.03
		.500	24.55	30.24	37.04	36.94	48.07	63.28	31.79	44.64	66.03
		1.000	19.75	27.27	37.04	28.48	42.10	63.28	23.25	37.49	66.03
		2.000	15.70	24.55	37.04	21.90	36.94	63.28	17.22	31.79	66.03

Table 3 (contd)

h	ρ'	$\frac{\varphi}{m}$	$q=0.25$			$q=0.50$			$q=0.75$		
			ρ'' 0.5	0.7	0.9	0.5	0.7	0.9	0.5	0.7	0.9
4	0.5	.025	13.23	13.51	13.78	16.66	17.03	17.40	11.84	12.11	12.38
		.050	13.23	13.77	14.33	16.66	17.38	18.12	11.84	12.37	12.92
		.100	13.23	14.27	15.87	16.66	18.05	19.52	11.84	12.87	13.98
		.250	13.23	15.60	18.23	16.66	19.83	23.46	11.84	14.21	16.99
		.500	13.23	17.32	22.21	16.66	22.19	29.15	11.84	16.01	21.53
		1.000	13.23	19.66	28.05	16.66	25.47	38.08	11.84	18.57	29.08
		2.000	13.23	22.21	35.07	16.66	29.15	49.81	11.84	21.53	39.95
	0.7	.025	27.58	28.05	28.52	37.34	38.08	38.83	28.44	29.08	29.75
		.050	27.15	28.05	28.97	36.66	38.08	39.56	27.84	29.08	30.39
		.100	26.35	28.05	29.84	35.41	38.08	40.95	26.77	29.08	31.64
		.250	24.42	28.05	32.10	32.45	38.08	44.70	24.26	29.08	35.08
		.500	22.21	28.05	35.07	29.15	38.08	49.81	21.53	29.08	39.95
		1.000	19.66	28.05	39.13	25.47	38.08	57.14	18.57	29.08	47.36
		2.000	17.32	28.05	43.60	22.19	38.08	65.78	16.01	29.08	56.81
	0.9	.025	52.40	53.23	54.08	84.73	86.68	88.68	80.73	83.48	86.37
		.050	50.85	52.44	54.08	81.19	84.83	88.68	75.88	80.86	86.37
		.100	48.09	50.99	54.08	75.10	81.51	88.68	67.97	76.31	86.37
		.250	41.76	47.52	54.08	62.15	73.90	88.68	52.74	66.46	86.37
		.500	35.07	43.60	54.08	49.81	65.78	88.68	39.95	56.81	86.37
		1.000	28.05	39.13	54.08	38.08	57.14	88.68	29.08	47.36	86.37
		2.000	22.21	35.07	54.08	29.15	49.81	88.68	21.53	39.95	86.37

Table 3 (Contd)

h	p'	q/m	v = 0.25			v = 0.50			v = 0.75		
		p''	0.5	0.7	0.9	0.5	0.7	0.9	0.5	0.7	0.9
5	0.5	.025	16.66	17.01	17.36	19.99	20.43	20.87	13.63	13.94	14.25
		.050	16.66	17.34	18.04	19.99	20.85	21.73	13.63	14.23	14.86
		.100	16.66	17.97	19.35	19.99	21.64	23.40	13.63	14.80	16.05
		.250	16.66	19.64	22.96	19.99	23.76	28.09	13.63	16.31	19.47
		.500	16.66	21.81	28.01	19.99	26.58	34.94	13.63	18.36	24.62
		1.000	16.66	24.77	35.50	19.99	30.50	45.79	13.63	21.25	33.25
		2.000	16.66	28.01	44.65	19.99	34.94	60.31	13.63	24.62	45.84
	0.7	.025	34.90	35.50	36.11	44.89	45.79	46.71	32.51	33.25	34.01
		.050	34.34	35.50	36.70	44.05	45.79	47.60	31.83	33.25	34.76
		.100	33.31	35.50	37.81	42.63	45.29	49.32	30.59	33.25	36.20
		.250	30.83	35.50	40.76	38.93	45.79	53.96	27.72	33.25	40.17
		.500	28.01	35.50	44.65	34.94	45.79	60.31	24.62	33.25	45.84
		1.000	24.77	35.50	49.99	30.50	45.79	69.56	21.25	33.25	54.54
		2.000	21.81	35.50	55.97	26.58	45.79	80.64	18.36	33.25	65.78
	0.9	.025	67.91	69.06	70.23	105.62	108.24	110.95	95.03	98.46	102.10
		.050	65.79	67.97	70.23	100.88	105.75	110.95	89.01	95.19	102.10
		.100	62.02	65.99	70.23	92.80	101.31	110.95	79.28	89.54	102.10
		.250	53.50	61.26	70.23	75.96	91.21	110.95	60.92	77.45	102.10
		.500	44.65	55.97	70.23	60.31	80.64	110.95	45.84	65.78	102.10
		1.000	35.50	49.99	70.23	45.79	69.56	110.95	33.25	54.54	102.10
		2.000	28.01	44.65	70.23	34.94	60.31	110.95	24.62	45.84	102.10

Table 4

Percentage efficiency of the estimate of mean $\bar{x}_h$ given in section 4.1 over complete replacement under the assumption $\rho''_{ij} = \rho''$ for all i and j (i ≠ j)

ρ''	$\frac{\phi}{m}$	q=0.25				q=0.50				q=0.75			
	h	2	3	4	5	2	3	4	5	2	3	4	5
0.5	.025	.11	.20	.28	.34	.16	.27	.34	.40	.13	.20	.25	.29
	.050	.22	.40	.55	.68	.31	.53	.68	.80	.26	.40	.50	.57
	.100	.43	.78	1.07	1.31	.60	1.02	1.31	1.53	.49	.78	.97	1.10
	.250	.96	1.74	2.39	2.94	1.35	2.27	2.94	3.44	1.10	1.74	2.16	2.45
	.500	1.61	2.94	4.05	4.99	2.27	3.84	4.99	5.88	1.85	2.94	3.65	4.16
	1.000	2.43	4.47	6.20	7.69	3.44	5.88	7.69	9.09	2.80	4.47	5.59	6.38
	2.000	3.27	6.06	8.45	10.52	4.65	7.99	10.52	12.49	3.77	6.06	7.59	8.69
0.7	.025	.23	.40	.53	.64	.34	.54	.67	.77	.31	.46	.55	.61
	.050	.45	.79	1.05	1.26	.66	1.06	1.33	1.51	.60	.90	1.08	1.20
	.100	.86	1.52	2.03	2.44	1.28	2.05	2.57	2.93	1.16	1.74	2.09	2.32
	.250	1.93	3.41	4.58	5.53	2.87	4.64	5.83	6.70	2.60	3.92	4.71	5.25
	.500	3.26	5.81	7.87	9.57	4.88	7.98	10.12	11.69	4.41	6.71	8.12	9.07
	1.000	4.97	8.99	12.30	15.07	7.50	12.46	15.99	18.63	6.77	10.41	12.69	14.25
	2.000	6.74	12.35	17.10	21.16	10.26	17.34	22.52	26.48	9.23	14.39	17.67	19.95
0.9	.025	.39	.66	.86	1.01	.62	.95	1.15	1.29	.68	.97	1.14	1.24
	.050	.76	1.30	1.69	2.00	1.22	1.88	2.28	2.56	1.34	1.93	2.25	2.46
	.100	1.47	2.51	3.29	3.89	2.36	3.65	4.46	5.02	2.60	3.75	4.39	4.81
	.250	3.30	5.71	7.54	8.99	5.35	8.40	10.37	11.75	5.90	8.64	10.21	11.23
	.500	5.63	9.90	13.24	15.94	9.24	14.83	18.57	21.25	10.25	15.28	18.27	20.25
	1.000	8.69	15.62	21.28	25.98	14.54	24.03	30.72	35.68	16.21	24.82	30.16	33.79
	2.000	11.93	21.98	30.54	37.93	20.34	34.83	45.63	53.99	22.84	36.08	44.71	50.78

Table 5

Percentage efficiency of the estimate of mean $\bar{X}_h$ given in section 3.1 over $\bar{X}_h$ given in section 4.1 under the assumption $\delta_{ij}=\delta$ for all i and j, ($i \neq j$).

k	ρ'	φ/m	q=0.25, ρ'' 0.5	q=0.25, 0.7	q=0.25, 0.9	q=0.50, 0.5	q=0.50, 0.7	q=0.50, 0.9	q=0.75, 0.5	q=0.75, 0.7	q=0.75, 0.9
2	0.5	.025	4.87	4.86	4.80	6.96	6.93	6.80	5.62	5.58	5.32
		.050	4.76	4.73	4.61	6.80	6.74	6.48	5.49	5.40	4.01
		.100	4.50	4.48	4.27	6.49	6.38	5.89	5.24	5.07	4.16
		.250	3.99	3.89	3.46	5.71	5.50	4.51	4.61	4.28	2.38
		.500	3.33	3.18	2.58	4.76	4.46	3.01	3.84	3.36	.45
		1.000	2.49	2.32	1.63	3.57	3.22	1.46	2.88	2.31	-1.45
		2.000	1.66	1.51	.87	2.38	2.06	.35	1.92	1.39	-2.55
	0.7	.025	10.17	10.21	10.20	15.73	15.82	15.80	14.05	14.17	14.06
		.050	9.89	9.97	9.95	15.28	15.45	15.40	13.61	13.83	13.63
		.100	9.38	9.51	9.48	14.44	14.75	14.66	12.82	13.20	12.83
		.250	8.12	8.37	8.31	12.40	12.98	12.81	10.90	11.62	10.87
		.500	6.63	6.98	6.89	10.04	10.81	10.58	8.74	9.68	8.57
		1.000	4.86	5.23	5.14	7.27	8.11	7.84	6.26	7.26	5.90
		2.000	3.16	3.49	3.41	4.69	5.40	5.16	4.00	4.84	3.52
	0.9	.025	18.39	18.51	18.57	32.60	32.96	33.20	36.45	37.22	37.75
		.050	17.78	18.00	18.13	31.28	31.96	32.41	34.46	35.85	36.85
		.100	16.68	17.08	17.31	28.95	30.13	30.93	31.08	33.42	35.17
		.250	14.05	14.79	15.23	23.65	25.72	27.22	24.06	27.81	30.95
		.500	11.12	12.09	12.69	18.12	20.70	22.68	17.53	21.79	25.79
		1.000	7.83	8.86	9.52	12.35	14.90	17.01	11.40	15.27	19.34
		2.000	4.91	5.78	6.34	7.54	9.56	11.34	6.72	9.58	12.89

Table 9 (Contd.)

k	ρ'	Φ/m	q=0.25, ρ'' 0.5	q=0.25, 0.7	q=0.25, 0.9	q=0.50, 0.5	q=0.50, 0.7	q=0.50, 0.9	q=0.75, 0.5	q=0.75, 0.7	q=0.75, 0.9
3	0.5	.025	9.14	9.12	9.04	12.10	12.16	11.98	9.14	9.08	8.75
		.050	8.92	8.89	8.73	11.00	11.85	11.50	8.92	8.82	8.16
		.100	8.52	8.46	8.16	11.36	11.26	10.63	8.52	8.32	7.09
		.250	7.49	7.38	6.81	9.99	6.80	8.52	7.50	7.10	4.52
		.500	6.24	6.09	5.27	8.33	8.06	6.15	6.25	5.67	1.68
		1.000	4.68	4.52	3.57	6.24	5.94	3.58	4.68	4.00	-1.25
		2.000	3.12	2.97	2.10	4.16	3.88	1.55	3.12	2.47	-3.15
	0.7	.025	19.18	19.27	19.29	27.60	27.79	27.81	22.49	22.69	22.58
		.050	18.65	18.81	18.84	26.77	27.13	27.17	21.77	22.15	21.94
		.100	17.66	17.96	18.01	25.25	25.85	25.98	20.46	21.14	20.75
		.250	15.23	15.80	15.91	21.58	22.79	22.96	17.36	18.60	17.81
		.500	12.39	13.17	13.32	17.39	18.99	19.25	13.87	15.50	14.31
		1.000	9.03	9.87	10.07	12.52	14.24	14.56	9.91	11.62	10.15
		2.000	5.85	6.58	6.77	8.04	9.49	9.81	6.32	7.75	6.29
	0.9	.025	35.68	35.95	36.13	60.28	61.10	61.73	61.72	63.24	64.42
		.050	34.42	34.92	35.27	57.56	59.08	60.26	57.95	60.69	62.88
		.100	32.14	33.03	33.67	52.80	55.41	57.52	51.68	56.18	60.02
		.250	26.81	28.43	29.63	42.34	46.75	50.62	39.15	46.05	52.82
		.500	20.99	23.07	24.69	31.86	37.13	42.18	28.02	35.54	44.02
		1.000	14.62	16.77	18.52	21.35	26.35	31.64	17.97	24.52	33.01
		2.000	9.09	10.85	12.34	12.87	16.69	21.09	10.52	15.21	22.01

Table 5 (Contd.)

h	ρ'	φ/m	q=0.25, ρ'' 0.5	0.7	0.9	q=0.50, 0.5	0.7	0.9	q=0.75, 0.5	0.7	0.9
4	0.5	.025	12.91	12.90	12.81	16.26	16.24	16.05	11.55	11.49	11.11
		.050	12.60	12.58	12.41	15.87	15.84	15.47	11.27	11.16	10.43
		.100	12.03	12.00	11.69	15.15	15.09	14.42	10.76	10.56	9.18
		.250	10.58	10.53	9.93	13.38	13.22	11.85	9.47	9.06	6.15
		.500	8.82	8.75	7.91	11.11	10.96	8.92	7.89	7.30	2.76
		1.000	6.61	6.55	5.58	8.33	8.16	5.63	5.92	5.21	-.82
		2.000	4.41	4.36	3.47	5.55	5.41	2.86	3.94	3.28	-3.28
	0.7	.025	27.22	27.36	27.42	36.86	37.15	37.24	28.11	28.38	28.28
		.050	26.44	26.71	26.82	35.73	36.26	36.43	27.20	27.70	27.51
		.100	25.01	25.50	25.69	33.65	34.62	34.93	25.55	26.44	26.10
		.250	21.51	22.44	22.83	28.67	30.46	31.10	21.63	23.27	22.56
		.500	17.44	18.70	19.27	23.00	25.38	26.34	17.24	19.39	18.33
		1.000	12.66	14.02	14.71	16.50	19.04	20.21	12.30	14.54	13.21
		2.000	8.18	9.35	10.00	10.55	12.69	13.83	7.83	9.69	8.36
	0.9	.025	51.96	52.41	52.76	84.09	85.42	86.52	80.27	82.47	84.26
		.050	50.01	50.85	51.50	79.96	82.40	84.46	75.00	78.92	82.26
		.100	46.51	47.98	49.16	72.82	76.96	80.62	66.35	72.70	78.52
		.250	38.44	41.06	43.26	57.51	64.30	70.94	49.50	58.96	69.09
		.500	29.81	33.11	36.05	42.67	50.54	59.12	35.01	45.03	57.58
		1.000	20.56	23.89	27.04	28.21	35.47	44.34	22.25	30.76	43.18
		2.000	12.68	15.35	18.02	16.85	22.26	29.56	12.96	18.92	28.79

Table 5 (Contd.)

h	ρ′	φ/m	θ=0.25			θ=0.50			θ=0.75		
			ρ″ 0.5	0.7	0.9	0.5	0.7	0.9	0.5	0.7	0.9
5	0.5	.025	16.26	16.26	16.18	19.51	19.51	19.32	13.30	13.24	12.84
		.050	15.87	15.88	15.72	19.04	19.04	18.68	12.98	12.87	12.09
		.100	15.15	15.16	14.88	18.18	18.17	17.50	12.39	12.19	10.72
		.250	13.33	13.37	12.81	15.99	15.99	14.62	10.90	10.51	7.39
		.500	11.11	11.17	10.40	13.33	13.33	11.28	9.09	8.51	3.63
		1.000	8.33	8.42	7.55	9.99	10.01	7.45	6.81	6.12	-.40
		2.000	5.55	5.65	4.86	6.66	6.68	4.10	4.54	3.88	-3.27
	0.7	.025	34.43	34.64	34.74	44.30	44.67	44.83	32.13	32.44	32.36
		.050	33.43	33.81	34.01	42.91	43.61	43.91	31.07	31.67	31.51
		.100	31.58	32.27	32.65	40.37	41.63	42.18	29.17	30.23	29.94
		.250	27.09	28.40	29.15	34.30	36.63	37.76	24.66	26.60	26.01
		.500	21.91	23.67	24.76	27.44	30.52	32.21	19.63	22.17	21.28
		1.000	15.85	17.75	19.05	19.63	22.89	24.97	13.98	16.62	15.50
		2.000	10.21	11.83	13.07	12.52	15.26	17.30	8.89	11.08	9.95
	0.9	.025	67.33	67.98	68.51	104.79	106.65	108.25	94.46	97.25	99.61
		.050	64.66	65.87	66.88	99.28	102.67	105.67	87.93	92.87	97.23
		.100	59.92	62.03	63.84	89.88	95.56	100.87	77.32	85.24	92.81
		.250	49.11	52.81	56.18	70.09	79.20	88.76	57.05	68.59	81.68
		.500	37.76	42.34	46.82	51.41	61.73	73.97	40.00	51.99	68.06
		1.000	25.83	30.34	35.11	33.64	42.93	55.47	25.26	35.26	51.05
		2.000	15.82	19.38	23.41	19.95	26.74	36.98	14.65	21.57	34.03

Table 6

Percentage efficiency of the estimate of mean $\bar{x}_h$ given in section 3.1 over complete replacement under the assumption $\delta_{ij} = \delta^{|i-j|}$ for all i and j (i ≠ j)

h	ρ'	φ/m	q=0.25			q=0.50			q=0.75		
		ρ''	0.5	0.7	0.9	0.5	0.7	0.9	0.5	0.7	0.9
3	0.5	.025	5.76	5.90	6.04	7.69	7.87	8.06	5.89	6.04	6.19
		.050	5.76	6.04	6.32	7.69	8.05	8.44	5.89	6.18	6.48
		.100	5.76	6.29	6.85	7.69	8.40	9.17	5.89	6.46	7.06
		.250	5.76	6.97	8.36	7.69	9.33	11.24	5.89	7.19	8.73
		.500	5.76	7.87	10.56	7.69	10.57	14.34	5.89	8.18	11.26
		1.000	5.76	9.14	14.00	7.69	12.32	19.36	5.89	9.60	15.52
		2.000	5.76	10.56	18.44	7.69	14.34	26.27	5.89	11.26	21.73
	0.7	.025	13.72	14.00	14.29	18.94	19.36	19.79	15.15	15.52	15.89
		.050	13.45	14.00	14.56	18.55	19.36	20.22	14.81	15.52	16.26
		.100	12.97	14.00	15.10	17.84	19.36	21.02	14.20	15.52	16.97
		.250	11.83	14.00	16.52	16.17	19.36	23.23	12.79	15.52	18.93
		.500	10.56	14.00	18.44	14.34	19.36	26.27	11.26	15.52	21.73
		1.000	9.14	14.00	21.15	12.32	19.36	30.76	9.60	15.52	26.02
		2.000	7.87	14.00	24.27	10.57	19.36	36.23	8.18	15.52	31.55
	0.9	.025	30.79	31.43	32.09	48.84	50.19	51.59	45.92	47.61	49.40
		.050	29.60	30.82	32.09	46.42	48.91	51.59	42.96	46.00	49.40
		.100	27.53	29.71	32.09	42.32	46.64	51.59	38.18	43.22	49.40
		.250	22.97	27.11	32.09	33.91	41.52	51.59	29.15	37.28	49.40
		.500	18.44	24.27	32.09	26.27	36.23	51.59	21.73	31.55	49.40
		1.000	14.00	21.15	32.09	19.36	30.76	51.59	15.52	26.02	49.40
		2.000	10.56	18.44	32.09	14.34	26.27	51.59	11.26	21.73	49.40

Table 6 (contd)

k	ρ'	$\frac{\phi}{m}$	$q=0.25$			$q=0.50$			$q=0.75$		
			ρ'' 0.5	0.7	0.9	0.5	0.7	0.9	0.5	0.7	0.9
4	0.5	.025	5.88	6.02	6.17	7.73	7.92	8.11	5.90	6.04	6.19
		.050	5.88	6.16	6.46	7.73	8.10	8.49	5.90	6.19	6.49
		.100	5.88	6.43	7.03	7.73	8.45	9.23	5.90	6.46	7.07
		.250	5.88	7.16	8.66	7.73	9.39	11.36	5.90	7.19	8.74
		.500	5.88	8.13	11.11	7.73	10.67	14.57	5.90	8.19	11.28
		1.000	5.88	9.51	15.10	7.73	12.48	19.90	5.90	9.62	15.58
		2.000	5.88	11.11	20.61	7.73	14.57	27.52	5.90	11.28	21.91
	0.7	.025	14.76	15.10	15.44	19.45	19.90	20.37	15.21	15.58	15.96
		.050	14.45	15.10	15.78	19.03	19.90	20.82	14.86	15.58	16.33
		.100	13.89	15.10	16.42	18.27	19.90	21.70	14.25	15.58	17.05
		.250	12.56	15.10	18.18	16.50	19.90	24.11	12.82	15.58	19.05
		.500	11.11	15.10	20.61	14.59	19.90	27.52	11.28	15.58	21.91
		1.000	9.51	15.10	24.18	12.48	19.90	32.68	9.62	15.58	26.33
		2.000	8.13	15.10	28.49	10.67	19.90	39.21	8.19	15.58	32.13
	0.9	.025	38.30	39.33	40.40	55.55	57.41	59.36	47.83	49.74	51.79
		.050	36.43	38.35	40.40	52.26	55.64	59.36	44.51	47.02	51.79
		.100	33.26	36.60	40.40	46.86	52.55	59.36	39.25	44.80	51.79
		.250	26.66	32.63	40.40	36.40	45.84	59.36	29.61	38.27	51.79
		.500	20.61	28.49	40.40	27.52	39.21	59.36	21.91	32.13	51.79
		1.000	15.10	24.18	40.40	19.90	32.68	59.36	15.58	26.33	51.79
		2.000	11.11	20.61	40.40	14.57	27.52	59.36	11.28	21.91	51.79

Table 6 (Contd)

h	ρ'	φ/m	q=0.25			q=0.50			q=0.75		
		ρ''	0.5	0.7	0.9	0.5	0.7	0.9	0.5	0.7	0.9
5	0.5	.025	5.89	6.04	6.19	7.73	7.92	8.11	5.90	6.04	6.19
		.050	5.89	6.18	6.48	7.73	8.10	8.49	5.90	6.19	6.49
		.100	5.89	6.46	7.06	7.73	8.46	9.24	5.90	6.46	7.07
		.250	5.89	7.19	8.72	7.73	9.40	11.37	5.90	7.19	8.74
		.500	5.89	8.18	11.24	7.73	10.68	14.60	5.90	8.19	11.28
		1.000	5.89	9.59	15.44	7.73	12.49	19.99	5.90	9.62	15.58
		2.000	5.89	11.24	21.43	7.73	14.60	27.79	5.90	11.28	21.92
	0.7	.025	15.08	15.44	15.80	19.53	19.99	20.46	15.21	15.58	15.96
		.050	14.75	15.44	16.16	19.10	19.99	20.93	14.87	15.58	16.33
		.100	14.15	15.44	16.85	18.33	19.99	21.82	14.25	15.58	17.06
		.250	12.75	15.44	18.76	16.54	19.99	24.29	12.82	15.58	19.06
		.500	11.24	15.44	21.43	14.60	19.99	27.79	11.28	15.58	21.92
		1.000	9.59	15.44	25.46	12.49	19.99	33.17	9.62	15.58	26.37
		2.000	8.18	15.44	30.50	10.68	19.99	40.09	8.19	15.58	32.21
	0.9	.025	42.69	44.03	45.44	58.17	60.31	62.58	48.20	50.18	52.30
		.050	40.29	42.75	45.44	54.43	58.27	62.58	44.79	48.29	52.30
		.100	36.29	40.50	45.44	48.41	54.76	62.58	39.42	45.09	52.30
		.250	28.34	35.52	45.44	37.09	47.28	62.58	29.66	38.43	52.30
		.500	21.43	30.50	45.44	27.79	40.09	62.58	21.92	32.21	52.30
		1.000	15.44	25.46	45.44	19.99	33.17	62.58	15.58	26.37	52.30
		2.000	11.24	21.43	45.44	14.60	27.79	62.58	11.28	21.92	52.30

Table 7

Percentage efficiency of the estimate of mean $\bar{x}_h$ given in section 4.1 over complete replacement under the assumption $\rho''_{ij} = \rho''^{|i-j|}$ for all i and j) (i ≠ j)

ρ''	$\frac{\varphi}{m}$	q=0.25				q=0.50				q=0.75			
		k 2	3	4	5	2	3	4	5	2	3	4	5
0.5	.025	.11	.13	.13	.12	.16	.17	.17	.17	.12	.13	.12	.12
	.050	.22	.26	.26	.26	.[illegible]1	.24	.24	.24	.24	.24	.24	.26
	.100	.43	.40	.50	.50	.60	.65	.65	.65	.4[illegible]	.50	.[illegible]0	.50
	.250	.96	1.10	1.12	1.12	1.35	1.44	1.45	1.45	1.1[illegible]	1.12	1.12	1.12
	.500	1.61	1.85	1.88	1.90	2.27	2.42	2.45	2.45	1.[illegible]5	1.90	1.90	1.90
	1.000	2.43	2.90	2.95	2.96	2.44	2.70	2.72	2.72	2.[illegible]0	2.96	2.96	2.96
	2.000	3.27	3.77	3.94	3.95	4.65	4.90	[illegible].02	5.02	3.77	3.85	3.85	3.95
0.7	.025	.2[illegible]	.30	.32	.32	.34	.30	.40	.46	.31	.32	.32	.32
	.050	.45	.58	.62	.64	.66	.77	.70	.70	.60	.64	.64	.64
	.100	.86	1.12	1.20	1.23	1.29	1.40	1.53	1.53	1.16	1.23	1.24	1.24
	.250	1.93	2.51	2.60	2.74	2.[illegible]7	2.35	2.42	2.44	2.60	2.74	2.77	2.77
	.500	3.26	4.26	4.57	4.66	4.92	5.71	5.95	[illegible].98	4.61	4.68	4.70	4.70
	1.000	4.97	6.54	7.02	7.16	7.50	9.92	[illegible].05	[illegible].08	6.77	7.20	7.22	7.22
	2.000	6.74	9.01	9.58	9.78	10.26	12.12	12.44	12.40	9.22	9.83	9.97	9.97
0.9	.025	.30	.50	.70	.76	.62	.93	.91	.94	.69	.91	.83	.94
	.050	.76	1.1[illegible]	1.30	1.51	1.22	1.64	1.80	1.96	1.24	1.50	1.65	1.66
	.100	1.47	2.25	2.69	2.92	2.36	3.10	3.30	3.62	2.60	3.00	3.20	3.22
	.250	2.30	5.10	6.10	6.66	5.35	7.[illegible]2	8.[illegible]5	[illegible].34	5.00	7.08	7.32	7.37
	.500	5.63	9.91	10.60	11.62	9.24	12.70	14.17	14.71	10.25	12.38	12.83	12.92
	1.000	9.69	13.92	16.90	1[illegible].51	14.54	20.50	22.89	23.93	16.21	19.90	20.57	20.73
	2.000	11.93	19.22	22.73	26.31	20.27	29.34	33.04	34.51	22.64	29.27	29.44	29.69

Table 8

Percentage efficiency of the estimate of mean $\bar{x}_h$ given in section 3.1 over $\bar{x}_h$ given in section 4.1 under the assumption $\delta_{ij} = \delta^{|i-j|}$ for all i and j (i ≠ j)

k	ρ′	φ/m	q=0.25, ρ″ 0.5	q=0.25, 0.7	q=0.25, 0.9	q=0.50, 0.5	q=0.50, 0.7	q=0.50, 0.9	q=0.75, 0.5	q=0.75, 0.7	q=0.75, 0.9
3	0.5	.025	5.62	5.58	5.41	7.50	7.45	7.17	5.75	5.70	5.33
		.050	5.48	5.42	5.09	7.32	7.22	6.69	5.61	5.50	4.81
		.100	5.24	5.11	4.49	6.99	6.80	5.79	5.36	5.16	3.85
		.250	4.60	4.35	3.10	6.16	5.78	3.67	4.71	4.31	1.54
		.500	3.83	3.46	1.60	5.13	4.59	1.37	3.92	3.34	-.99
		1.000	2.87	2.44	.15	3.84	3.21	-.94	2.94	2.23	-3.57
		2.000	1.91	1.51	-.78	2.57	1.98	-2.37	1.96	1.30	-5.09
	0.7	.025	13.57	13.65	13.61	18.73	18.89	18.80	15.00	15.15	14.95
		.050	13.15	13.34	13.23	18.14	18.44	18.28	14.51	14.78	14.44
		.100	12.41	12.73	12.56	17.07	17.60	17.27	13.63	14.11	13.46
		.250	10.61	11.20	10.86	14.52	15.49	14.84	11.54	12.41	11.06
		.500	8.55	9.34	8.85	11.62	12.91	11.95	9.19	10.35	8.31
		1.000	6.16	7.00	6.43	8.31	9.68	8.51	6.55	7.76	5.19
		2.000	3.95	4.67	4.14	5.31	6.45	5.32	4.16	5.18	2.55
	0.9	.025	30.62	31.03	31.31	48.58	49.60	50.34	45.73	47.13	48.19
		.050	29.26	30.06	30.56	45.92	47.77	49.14	42.58	45.07	47.06
		.100	26.90	28.27	29.18	41.40	44.48	46.90	37.49	41.47	44.92
		.250	21.63	23.99	25.68	32.00	36.93	41.27	27.71	33.59	39.52
		.500	16.28	19.19	21.39	23.27	28.87	34.40	19.47	25.66	32.94
		1.000	10.89	13.71	16.05	15.10	20.16	25.80	12.30	17.55	24.70
		2.000	6.54	8.75	10.70	8.90	12.62	17.20	7.13	10.83	16.47

Table 8 (Contd.)

h	ρ'	φ/m	q=0.25			q=0.50			q=0.75		
		ρ''	0.5	0.7	0.9	0.5	0.7	0.9	0.5	0.7	0.9
4	0.5	.025	5.74	5.68	5.43	7.54	7.49	7.13	5.76	5.70	5.31
		.050	5.60	4.46	5.01	7.36	7.25	6.57	5.62	5.51	4.76
		.100	5.35	5.16	4.23	7.03	6.81	5.53	5.37	5.15	3.75
		.250	4.70	4.35	2.41	6.19	5.76	3.06	4.72	4.30	1.32
		.500	3.92	3.40	.46	5.15	4.55	.35	3.93	3.33	-1.37
		1.000	2.94	2.32	-1.45	3.86	3.14	-2.43	2.95	2.23	-4.13
		2.000	1.96	1.39	-2.52	2.58	1.89	-4.14	1.97	1.28	-5.81
	0.7	.025	14.61	14.73	14.63	19.24	19.42	19.28	15.06	15.21	15.00
		.050	14.15	13.26	14.20	18.62	18.96	18.68	14.56	14.84	14.44
		.100	13.32	13.73	13.38	17.50	18.09	17.58	13.68	14.16	13.42
		.250	11.31	12.08	11.38	14.83	15.92	14.86	11.57	12.46	10.92
		.500	9.05	10.06	9.05	11.84	13.27	11.69	9.21	10.39	8.04
		1.000	6.47	7.54	6.31	8.44	9.94	7.96	6.57	7.79	4.77
		2.000	4.13	5.03	3.84	5.37	6.63	4.63	4.17	5.19	2.07
	0.9	.025	38.12	38.88	39.42	55.28	56.78	57.92	47.63	49.26	50.54
		.050	36.07	36.14	38.48	51.74	54.42	56.54	44.13	46.97	49.32
		.100	32.59	34.98	36.73	45.91	50.25	53.97	38.55	43.02	47.08
		.250	25.25	29.15	32.32	34.45	41.00	47.48	28.17	34.54	41.43
		.500	18.38	22.87	26.94	24.47	31.51	39.58	19.64	26.19	34.52
		1.000	11.91	16.03	20.20	15.59	21.66	29.67	12.36	17.82	25.89
		2.000	7.00	10.06	13.47	9.09	13.41	19.78	7.15	10.95	17.26

Table 8 (contd)

h	ρ'	$\frac{\phi}{m}$	$q = 0.25$			$q = 0.50$			$q = 0.75$		
		ρ''	0.50	0.7	0.9	0.5	0.7	0.9	0.5	0.7	0.9
5	0.5	.025	5.75	5.70	5.38	7.54	7.49	7.10	5.76	5.70	5.30
		.050	5.61	5.50	4.89	7.36	7.25	6.50	5.62	5.51	4.75
		.100	5.36	5.16	4.02	7.03	6.82	5.42	5.37	5.15	3.72
		.250	4.71	4.33	1.93	6.19	5.76	2.79	4.72	4.30	1.27
		.500	3.92	3.36	-.34	5.15	4.53	-.09	3.93	3.33	-1.45
		1.000	2.94	2.26	-2.59	3.86	3.12	-3.10	2.95	2.23	-4.26
		2.000	1.96	1.32	-3.86	2.58	1.87	-4.99	1.97	1.28	-5.99
	0.7	.025	14.93	15.07	14.92	19.32	19.51	19.33	15.06	15.21	14.99
		.050	14.45	14.70	14.43	18.69	19.04	18.72	14.57	14.84	14.43
		.100	13.58	14.03	13.53	17.56	18.18	17.56	13.68	14.16	13.40
		.250	11.50	12.36	11.34	14.87	15.99	14.72	11.57	12.46	10.88
		.500	9.17	10.30	8.78	11.85	13.32	11.40	9.21	10.39	7.97
		1.000	6.54	7.72	5.86	8.45	10.00	7.54	6.57	7.79	4.67
		2.000	4.16	5.15	3.31	5.38	6.66	4.14	4.17	5.19	1.94
	0.9	.025	42.50	43.57	44.34	57.90	59.67	61.06	48.00	49.70	51.03
		.050	39.92	41.84	43.27	53.90	57.02	59.61	44.41	47.34	49.81
		.100	35.61	38.79	41.31	47.45	52.42	56.90	38.72	43.31	47.54
		.250	26.91	31.90	36.35	35.13	42.38	50.06	28.22	34.69	41.84
		.500	19.17	24.68	30.29	24.73	32.31	41.73	19.65	26.27	34.87
		1.000	12.23	17.07	22.72	15.68	22.08	31.29	12.36	17.89	26.14
		2.000	7.11	10.61	15.14	9.12	13.60	20.86	7.15	10.96	17.43