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ORIGINAL ARTICLE

MAXIMUM LIKELIHOOD AND UNIFORMLY MINIMUM VARIANCE UNBIASED ESTIMATION OF P(Y < X) FOR GOMPERTZ DISTRIBUTION

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Abstract : Maximum likelihood estimator of R = P(Y < X) is derived when all the parameters of Gompertz distribution for X and Y are unknown and unequal. Maximum likelihood estimator for estimating 'R' is also derived, when one parameter is known (and equal), but other parameter is unknown for X and Y. Uniformly minimum variance unbiased estimator (UMVUE) of P(Y < X) is also derived for the case when one parameter is known (and unequal), but other parameter is unknown for both X and Y. A new approach has been applied for deriving the UMVUE of R using estimator of power of unknown parameter. Performance of the estimators have been examined using bootstrap technique.

Key words : Maximum likelihood estimator, Uniformly minimum variance unbiased estimator, P(Y < X), Gompertz distribution.

1. Introduction

In statistical literature, the quantity P(Y< X) is typically referred to as stress strength reliability. P(Y<X) determines the reliability of a system of strength X subjected to stress Y. For several distributions estimation of P(Y < X) are discussed by Chao (1982) for the family of exponential distributions. Kotz *et al.* (2003) have reviewed work done on estimation of stress strength reliability over the last four decades [Church and Harris (1970), Downton (1973), Woodward and Kelly (1973), Tong (1974, 1975a, 1975b), Awad and Gharraf (1986)] for the family of Burr XII distributions [Constantine *et al.* (1986) and Ismail *et al.* (1986)] for the family of gamma distributions [Surles and Padgett (1998, 2001), Raqab and Kundu (2005)] for the family of Burr X distributions [Kundu and Gupta (2005)] for the family of generalized exponential distributions [Ali and Woo (2005a, 2005b)] for the family of burr III distributions. Hardly any work on estimation of P(Y< X) for Gompertz distribution is available in literature. The Gompertz distribution was formulated by Gompertz (1825) to fit mortality tables. This distribution does not seem to have received enough attention, possibly because of its complicated form. Recently, many authors have contributed to the statistical methodology and characterization of Gompertz distribution. For example [Read (1983), Downton (1973), Makany (1991), Rao and Damaraju (1992), Franses (1994) and Wu and Lee (1999)]. Garg *et al.* (1970) studied the properties of the Gompertz distribution and obtained the MLEs for the parameters. Chen (1997) developed an exact confidence interval and an exact joint confidence region for the parameters of the Gompertz distribution under type II censoring. Saraçoğlu *et al.* (2009) have discussed on comparative study on estimators for stress strength reliability in the Gompertz case, when one parameter is known.

In Section 2, we have derived maximum likelihood estimator of P(Y< X) when all parameters of Gompertz distribution for X and Y are unknown and unequal. In Section 3, we have derived maximum likelihood estimator of P(Y< X), when one parameter is known (and equal), but other parameter is unknown for X and Y. In Section 4, we have derived UMVUE of P(Y< X) when one parameter is known (and unequal), but other parameter is unknown for both X and Y. In Section 5, simulation results have been discussed.

2. Maximum Likelihood Estimation of P(Y<X), when (c_1, λ_1) and (c_2, λ_2) are Unknown and Unequal

Let X be the strength of a system and Y be the stress

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acting on it. Then X and Y will be random variables from Gompertz with parameters (c_1, λ_1) and (c_2, λ_2), respectively. That is, the probability density functions (pdf) and the cumulative distribution functions of X and Y are, respectively.

$$f(x) = \lambda_1 e^{c_1 x} e^{\left[-\lambda_1 c_1^{-1}\left\{e^{c_1 x}-1\right\}\right]};\ x>0,\ c_1>0,\ \lambda_1>0, \quad (1)$$

$$F(x) = 1 - e^{\left[-\lambda_1 c_1^{-1}\left\{e^{c_1 x}-1\right\}\right]} \quad (2)$$

and

$$f(y) = \lambda_2 e^{c_2 y} e^{\left[-\lambda_2 c_2^{-1}\left\{e^{c_2 y}-1\right\}\right]};\ y>0,\ c_2>0,\ \lambda_2>0, \quad (3)$$

$$F(y) = 1 - e^{\left[-\lambda_2 c_2^{-1}\left\{e^{c_2 y}-1\right\}\right]}. \quad (4)$$

Let, X_1, X_2, ..., X_n and Y_1, Y_2, ..., Y_m be the two independent random samples taken from the Gompertz distribution with parameters (c_1, λ_1) and (c_2, λ_2), respectively. Then, likelihood and log-likelihood function based on the above samples are given as follows.

$$L_x = (\lambda_1)^n e^{c_1 \sum_{i=1}^{n} x_i} e^{-\lambda_1 c_1^{-1} \sum_{i=1}^{n}\left(e^{c_1 x_i}-1\right)}$$

and (5)

$$L_y = (\lambda_2)^m e^{c_2 \sum_{j=1}^{m} y_j} e^{-\lambda_2 c_2^{-1} \sum_{j=1}^{m}\left(e^{c_2 y_j}-1\right)}$$

Now,

$$logL_x = n\log\lambda_1 + c_1 \sum_{i=1}^{n} x_i - \lambda_1 c_1^{-1} \sum_{i=1}^{n}\left(e^{c_1 x_i}-1\right)$$

and (6)

$$logL_y = m\log\lambda_2 + c_2 \sum_{j=1}^{m} y_j - \lambda_2 c_2^{-1} \sum_{j=1}^{m}\left(e^{c_2 y_j}-1\right)$$

From (5), we have

$$\frac{\partial \log \mathrm{L_x}}{\partial \lambda_1} = \frac{n}{\lambda_1} - \frac{\sum_{i=1}^{n}\left(e^{c_1 x_i}-1\right)}{c_1}$$

Hence, $\frac{\partial \log \mathrm{L_x}}{\partial \lambda_1} = 0$

$$\Rightarrow \hat{\lambda}_1 = \frac{nc_1}{\sum_{i=1}^{n}\left(e^{c_1 x_i}-1\right)} \quad (7)$$

Now,

$$\frac{\partial \log L_x}{\partial c_1} = \sum_{i=1}^{n} x_i - \frac{n\sum_{i=1}^{n} x_i e^{c_1 x_i}}{\sum_{i=1}^{n}\left(e^{c_1 x_i}-1\right)} + \frac{n}{c_1}$$

Hence, $\frac{\partial \log L_x}{\partial c_1} = 0$

$$\Rightarrow \bar{x} - \frac{\sum_{i=1}^{n} x_i e^{c_1 x_i}}{\sum_{i=1}^{n}\left(e^{c_1 x_i}-1\right)} + \frac{1}{c_1} = 0 \quad (8)$$

Similarly for Y,

$$\hat{\lambda}_2 = \frac{mc_2}{\sum_{j=1}^{m}\left(e^{c_2 y_j}-1\right)} \quad (9)$$

and

$$\Rightarrow \bar{y} - \frac{\sum_{j=1}^{m} y_i e^{c_2 y_j}}{\sum_{j=1}^{m}\left(e^{c_2 y_j}-1\right)} + \frac{1}{c_2} = 0 \quad (10)$$

Through iterative procedures, we can solve $\hat{c}_1$ and $\hat{c}_2$ from (8) and (10) and thereby, we can estimate $\hat{\lambda}_1$ and $\hat{\lambda}_2$ from (7) and (9), respectively.

From (5), we have

$$R = P(Y<X) = \int_0^\infty P(Y<X)\, f(x)\,dx$$

From (4), we have

$$R = \int_0^\infty \left[1 - e^{\left[\lambda_2 c_2^{-1}\left\{e^{c_2 x}-1\right\}\right]}\right] \lambda_1 e^{c_1 x} e^{\left[\lambda_1 c_1^{-1}\left\{e^{c_1 x}-1\right\}\right]} dx$$

$$= 1 - \int_0^\infty \lambda_1 e^{c_1 x} e^{\left[-\lambda_2 c_2^{-1}\left(e^{c_2 x}-1\right) - \lambda_1 c_1^{-1}\left(e^{c_1 x}-1\right)\right]} dx$$

Putting $\lambda_1 c_1^{-1} e^{c_1 x} = t$

$$R = 1 - e^{\lambda_1 c_1^{-1} + \lambda_2 c_2^{-1}} \int_{\lambda_1 c_1^{-1}}^{\infty} e^{-\lambda_2 c_2^{-1}\left(tc_1/\lambda_1\right)^{c_2/c_1}} e^{-t} dt \quad (11)$$

If we write the following identity

$$e^{-\lambda_2 c_2^{-1}\left(tc_1/\lambda_1\right)^{c_2/c_1}} = \sum_{k=0}^{\infty} (-1)^k \frac{\left(\lambda_2 c_2^{-1}\right)^k \left(tc_1/\lambda_1\right)^{kc_2/c_1}}{k!} \quad (12)$$

The final form of (11) is regarded as follows:

$$R = 1 - e^{\lambda_1 c_1^{-1} + \lambda_2 c_2^{-1}} \sum_{k=0}^{\infty} (-1)^k \frac{\left(\lambda_2 c_2^{-1}\right)^k \left(c_1/\lambda_1\right)^{kc_2/c_1}}{k!} \times \left[\int_0^{\infty} t^{kc_2/c_1} e^{-t} dt - \int_0^{\lambda_1 c_1^{-1}} t^{kc_2/c_1} e^{-t} dt \right]$$

$$= 1 - e^{\lambda_1 c_1^{-1} + \lambda_2 c_2^{-1}} \sum_{k=0}^{\infty} (-1)^k \frac{\left(\lambda_2 c_2^{-1}\right)^k \left(c_1/\lambda_1\right)^{kc_2/c_1}}{k!} \times \left[\Gamma\left(kc_2 c_1^{-1} + 1\right) - \int_0^{\lambda_1 c_1^{-1}} t^{kc_2/c_1} e^{-t} dt \right], \quad (13)$$

Where, Γ(.) is a gamma function.

If we write the identity

$$e^{-t} = \sum_{i=0}^{\infty} (-1)^i \frac{t^i}{i!}. \quad (14)$$

From (14), we can write (13) as

$$R = 1 - e^{\lambda_1 c_1^{-1} + \lambda_2 c_2^{-1}} \sum_{i=0}^{\infty} (-1)^k \frac{\left(\lambda_2 c_2^{-1}\right)^k \left(c_1/\lambda_1\right)^{kc_2/c_1}}{k!} \times \left[\Gamma\left(kc_1 c_1^{-1} + 1\right) - \sum_{i=0}^{\infty} (-1)^i \frac{\left(\lambda_1 c_1^{-1}\right)^{kc_2 c_1^{-1} + i + 1}}{\left(kc_2 c_1^{-1} + i + 1\right) i!} \right].$$

From one to one property of MLE, we get MLE of R as

$$\hat{R} = 1 - e^{\hat{\lambda}_1 \hat{c}_1^{-1} + \hat{\lambda}_2 \hat{c}_2^{-1}} \sum_{k=0}^{\infty} (-1)^k \frac{\left(\hat{\lambda}_2 \hat{c}_2^{-1}\right)^k \left(\hat{c}_1/\hat{\lambda}_1\right)^{k\hat{c}_2/\hat{c}_1}}{k!} \times \left[\Gamma\left(k\hat{c}_2 \hat{c}_1^{-1} + 1\right) - \sum_{i=0}^{\infty} (-1)^i \frac{\left(\hat{\lambda}_1 \hat{c}_1^{-1}\right)^{k\hat{c}_2 \hat{c}_1^{-1} + i + 1}}{\left(k\hat{c}_2 \hat{c}_1^{-1} + i + 1\right) i!} \right] \quad (15)$$

3. Maximum Likelihood Estimation of P(Y<X), when $c_1 = c_2 = c$ and are known but λ_1 and λ_2 are unknown

We have

$$P = \int_0^{\infty} \left[1 - e^{\left[\lambda_2 c^{-1}\left\{e^{cx} - 1\right\}\right]} \right] \lambda_1 e^{cx} e^{\left[\lambda_1 c^{-1}\left\{e^{cx} - 1\right\}\right]} dx$$

$$= 1 - \int_0^{\infty} \lambda_1 e^{cx} e^{\left[-\lambda_2 c^{-1}\left(e^{cx} - 1\right) - \lambda_1 c^{-1}\left(e^{cx} - 1\right)\right]} dx,$$

Putting $e^{cx} - 1 = z$, we get

$$R = 1 - \lambda_1 c^{-1} \int_0^{\infty} e^{-(\lambda_1 + \lambda_2) c^{-1} z} dx$$

$$= 1 - \frac{\lambda_1}{\lambda_1 + \lambda_2}$$

$$= \frac{\lambda_2}{\lambda_1 + \lambda_2} \quad (16)$$

From (7) and (9), we get

$$\hat{\lambda}_1 = \frac{nc}{\sum_{i=1}^{n} \left(e^{cx_i} - 1\right)} \quad \text{and} \quad \hat{\lambda}_2 = \frac{mc}{\sum_{j=1}^{m} \left(e^{cy_j} - 1\right)}$$

From one to one property of MLE, we get MLE of R from (16) as

$$\widehat{P} = \frac{\hat{\lambda}_2}{\hat{\lambda}_1 + \hat{\lambda}_2} \quad (17)$$

4. UMVUEs of P(Y<X) when $c_1 \neq c_2$ and are known, but λ_1 and λ_2 are unknown

Throughout this section, we assume that λ_1 is unknown, but c_1 is known. Let $X_1, X_2, ..., X_n$ be a random sample of size n from (1).

Lemma 4.1 : Let, $S = c_1^{-1} \sum_{i=1}^{n} \left(e^{c_1 x_i} - 1\right)$, then S is complete and sufficient for the distribution (1). Moreover, the pdf of S is

$$\frac{\lambda_1^n}{\Gamma(n)} s^{n-1} e^{-\lambda_1 s}; \; s > 0 \, , \, \lambda_1 > 0 \quad (18)$$

Proof : From (1), the joint pdf of $X_1, X_2, ..., X_n$ is

$$f^*\left(x_1, x_2, \ldots, x_n; \lambda_1, c_1\right) = \left(\lambda_1\right)^n e^{c_1 \sum_{i=1}^{n} x_i} e^{-\lambda_1 c_1^{-1} s}. \quad (19)$$

It follows from (19) and Fisher-Nayman factorization theorem [Rohatgi (1976), pp. 346] that S is sufficient for the distribution (1). From the additive property of exponential distribution, S has Gamma distribution with parameters (n, λ_1). Since the distribution of S belongs to exponential family of distributions, it is also complete.

The following lemma provides the UMVUES of the powers (positive, as well as negative) of λ_1.

Lemma 4.2 : For q ∈ (–∞, ∞), the UMVUES of λ_1^{-q} and λ_1^{q} are given, respectively, by

$$\hat{\lambda}_1^{-q} = \left\{\frac{\Gamma(n)}{\Gamma(n+q)}\right\} S^q;\ (n+q) \ge 0 \tag{20}$$

and

$$\hat{\lambda}_1^{q} = \begin{cases} \dfrac{\Gamma(n)}{\Gamma(n-q)} S^{-q};\ (n-q) \ge 0 \\ 0, \text{ otherwise} \end{cases} \tag{21}$$

Proof : From Lemma 4.1, we have

$$E(S^q) = \frac{\lambda_1^n}{\Gamma(n)} \int_0^\infty s^{n+q-1} e^{-\lambda_1 s}\, ds$$

$$= \frac{\Gamma(n+q)}{\Gamma(n)} \lambda_1^{-q}$$

and (20) follows from Lehmann-Scheffé theorem [Rohatgi (1976), pp. 357]. Similarly, from Lemma 4.1,

$$E(S^{-q}) = \frac{\lambda_1^n}{\Gamma(n)} \int_0^\infty s^{n-q-1} \exp(-\lambda_1 s)\, ds$$

$$= \frac{\Gamma(n-q)}{\Gamma(n)} \lambda_1^{q};\ q < n$$

and (21) follows.

In the following lemma, variance of $\hat{\lambda}_1^q$ and $\hat{\lambda}_1^{-q}$ are derived.

Lemma 4.3 : For $\hat{\lambda}_1^q$ and $\hat{\lambda}_1^{-q}$ derived in Lemma 4.2.

$$\text{Var}(\hat{\lambda}_1^q) = \left\{\frac{\Gamma(n)\Gamma(n-2q)}{\Gamma^2(n-q)} - 1\right\} \lambda_1^{2q};\ 2q < n$$

and

$$\text{Var}(\hat{\lambda}_1^{-q}) = \left\{\frac{\Gamma(n)\Gamma(n+2q)}{\Gamma^2(n+q)} - 1\right\} \lambda_1^{-2q}$$

Proof : We know that

$$\text{Var}(\hat{\lambda}_1^q) = E\left[\hat{\lambda}_1^q - \lambda_1^q\right]^2$$

Which on using Lemma 4.2, gives

$$\text{Var}(\hat{\lambda}_1^q) = E\left[\hat{\lambda}_1^q - \lambda_1^q\right]^2 = \left\{\frac{\Gamma(n)}{\Gamma(n-q)}\right\}^2$$

$$\times E(S^{-2q}) + \lambda_1^{2q} - 2\lambda_1^q \frac{\Gamma(n)}{\Gamma(n-q)} E(S^{-q}).$$

Hence, the first assertion, similarly, we can prove the second assertion.

In the following lemma, we provide the UMVUE of the sampled pdf (1) at a specified point 'x'.

Lemma 4.4 : The UMVUE of $f(x;\, c_1,\, \lambda_1)$ at a specified point 'x' is

$$\hat{f}(x; c_1, \lambda_1) = \begin{cases} \dfrac{e^{c_1 x}}{S\, B((n-1),1)} \left[1 - \dfrac{e^{c_1 x} - 1}{c_1 S}\right]^{n-2} ;\ c_1^{-1}\{e^{c_1 x} - 1\} < S \\ 0, \textit{otherwise} \end{cases}$$

Proof : Since S is complete and sufficient for the distribution $f(x;\, c_1,\, \lambda_1)$, any function H(S) of S satisfying $E[H(S)] = f(x;\, c_1,\, \lambda_1)$ will be the UMVUE of $f(x;c_1,\lambda_1)$. To this end, from (1) and Lemma 4.1.

$$\frac{\lambda_1^n}{\Gamma(n)} \int_0^\infty H(s)\, s^{n-1} exp(-\lambda_1 s)\, ds = \lambda_1 e^{c_1 x} e^{-\left[\lambda_1 c_1^{-1}\{e^{c_1 x} - 1\}\right]}$$

or

$$\frac{\lambda_1^{n-1}}{\Gamma(n)} \int_0^\infty H(s)\, s^{n-1} exp\left[-\lambda_1 s + \lambda_1 c_1^{-1}\{e^{c_1 x} - 1\}\right] ds = e^{c_1 s}$$

or

$$\frac{\lambda_1^{n-1}}{\Gamma(n)} \int_{-c_1^2\{e^{c_1 x} - 1\}}^{\infty} H\left(u + c_1^{-1}\{e^{c_1 x} - 1\}\right)\left(u + c_1^{-1}\{e^{c_1 x} - 1\}\right)^{n-1} \times exp[-\lambda_1 u]\, du = e^{c_1 x} \tag{22}$$

(22) is satisfied and the lemma will hold, if we choose $H\left(u + c_1^{-1}\{e^{c_1 x} - 1\}\right)$ accordingly.

Remark 4.1 : We can write (1) as

$$f(x; c_1, \lambda_1) = exp(c_1 x) \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \left\{\frac{e^{c_1 x} - 1}{c_1}\right\}^i \lambda_1^{i+1}.$$

Using Lemma 2.1 of Chaturvedi and Tomer (2002) and lemma 4.2, the UMVUE of $f(x;\, c_1,\, \lambda_1)$ at a specified point 'x' is

$$\hat{f}(x; c_1, \lambda_1) = \frac{e^{c_1 x}}{S\, B((n-1), 1)} \sum_{i=0}^{n-2} (-1)^i \binom{n-2}{i} \left\{\frac{e^{c_1 x} - 1}{c_1 S}\right\}^i$$

$$= \begin{cases} \dfrac{e^{c_1 x}}{S\,B((n-1),1)}\left[1-\dfrac{e^{c_1 x}-1}{c_1 S}\right]^{n-2} & ; c_1^{-1}\left\{e^{c_1 x}-1\right\} < S \\ 0, \textit{otherwise} \end{cases}$$

Hence, the Lemma 4.4 follows.

Theorem 4.1 : The UMVUE of R is given by

$$\hat{P} = \frac{1}{(n-1)B((n-1),1)\,B((m-1),1)} \sum_{i=0}^{n-1}\left(\frac{1}{c_1 S}\right)^i \binom{n-1}{i} \sum_{j=0}^{i}(-1)^j$$

$$\times \binom{i}{j} \sum_{k=0}^{jc_1/c_2} \binom{jc_1/c_2}{k} (c_2 T)^k \cdot \int_0^{\left((c_1 S+1)^{c_2/c_1}-1\right)/c_2 T} u^k [1-u]^{m-2}\, du$$

Proof : It follows from Lemma 4.4 that the UMVUEs of $f_1(x;\, c_1,\, \lambda_1)$ and $f_2(y;\, c_2,\, \lambda_2)$ at specified points 'x' and 'y', respectively, are

$$\hat{f}_1(x;c_1,\lambda_1) = \begin{cases} \dfrac{e^{c_1 x}}{S\,B((n-1),1)}\left[1-\dfrac{e^{c_1 x}-1}{c_1 S}\right]^{n-2}; \\ \qquad c_1^{-1}\left\{e^{c_1 x}-1\right\} < S \\ 0, \text{ otherwise} \end{cases} \tag{23}$$

and

$$\hat{f}_2(y;c_2,\lambda_2) = \begin{cases} \dfrac{e^{c_2 y}}{T\,B((m-1),1)}\left[1-\dfrac{e^{c_2 y}-1}{c_2 T}\right]^{m-2}; \\ \qquad c_2^{-1}\left\{e^{c_2 y}-1\right\} < S \\ 0, \text{ otherwise} \end{cases} \tag{24}$$

It can be shown that the UMVUE of R is given by

$$\hat{R} = \int_{y=0}^{\infty}\int_{x=y}^{\infty} \hat{f}_1(x;c_1,\lambda_1)\,\hat{f}_2(x;c_2,\lambda_2)\,dx\,dy,$$

which on using (23) and (24) gives that

$$\hat{R} = \frac{1}{S\;B((n-1),1)\;T\;B((m-1),1)} \int_{y=0}^{c_2^{-1}\log(c_2 T+1)} \int_{x=y}^{c_1^{-1}\log(c_1 S+1)}$$

$$\cdot e^{c_1 x}\left[1-\frac{e^{c_1 x}-1}{c_1 S}\right]^{n-2} e^{c_2 y}\left[1-\frac{e^{c_2 y}-1}{c_2 T}\right]^{m-2} dx\,dy$$

Putting, $z = 1-\dfrac{e^{c_1 x}-1}{c_1 S}$, we get

$$\hat{R} = \frac{1}{(n-1)B((n-1),\,1)\;B((m-1),\,1)T} \int_{y=0}^{\min\left\{c_2^{-1}\log(c_2 T+1),\,c_1^{-1}\log(c_1 S+1)\right\}}$$

$$\cdot\left[1-\frac{e^{c_1 y}-1}{c_1 S}\right]^{n-1} e^{c_2 y}\left[1-\frac{e^{c_2 y}-1}{c_2 T}\right]^{m-2} dy \tag{25}$$

Let us first consider the case, when $c_2^{-1}\log(c_2 T+1) < c_1^{-1}\log(c_1 S+1)$. In this case, from (25)

$$\hat{R} = \frac{1}{(n-1)B((n-1),1)\;B((m-1),1)T} \int_{y=0}^{c_2^{-1}\log(c_2 T+1)}$$

$$\cdot\left[1-\frac{e^{c_1 y}-1}{c_1 S}\right]^{n-1} e^{c_2 y}\left[1-\frac{e^{c_2 y}-1}{c_2 T}\right]^{m-2} dy$$

Putting $\dfrac{e^{c_2 y}-1}{c_2 T} = u,$

$$\frac{e^{c_2 y}-1}{T}\,dy = du,$$

$$e^{c_2 y} = 1 + u c_2 T$$

and $y = \dfrac{\log(1+c_2 Tu)}{c_2}$

$$\hat{R} = \frac{1}{(n-1)B((n-1),\,1)B((m-1),\,1)T} \int_0^1 \left[1-\frac{e^{\frac{c_1}{c_2}\log(1+c_2 Tu)}-1}{c_1 S}\right]^{n-1} \cdot T[1-u]^{m-2}\,du$$

$$= \frac{1}{(n-1)B((n-1),\,1)B((m-1),\,1)} \int_0^1 \left[1-\frac{(1+c_2 Tu)^{\frac{c_1}{c_2}}-1}{c_1 S}\right]^{n-1} \cdot [1-u]^{m-2}\,du$$

$$= \frac{1}{(n-1)B((n-1),1)\;B((m-1),1)} \int_0^1 \sum_{i=0}^{n-1}(-1)^i \binom{n-1}{i} \cdot \left\{\frac{(1+c_2 Tu)^{\frac{c_1}{c_2}}-1}{c_1 S}\right\}^i [1-u]^{m-2}\,du$$

$$= \frac{1}{(n-1)B\left((n-1),1\right)\,B\left((m-1),1\right)} \int_0^1 \sum_{i=0}^{n-1} \binom{n-1}{i} \cdot \left\{ \frac{1-(1+c_2Tu)^{\frac{c_1}{c_2}}}{c_1 S} \right\}^{i} [1-u]^{m-2}\, du$$

$$= \frac{1}{(n-1)B\left((n-1),1\right)\,B\left((m-1),1\right)} \int_0^1 \sum_{i=0}^{n-1} \left(\frac{1}{c_1 S}\right)^{i} \binom{n-1}{i} \cdot \sum_{j=0}^{i} (-1)^{j} \binom{i}{j} (1+c_2Tu)^{j\frac{c_1}{c_2}} [1-u]^{m-2}\, du$$

$$= \frac{1}{(n-1)B\left((n-1),1\right)\,B\left((m-1),1\right)} \int_0^1 \sum_{i=0}^{n-1} \left(\frac{1}{c_1 S}\right)^{i} \binom{n-1}{i} \cdot \sum_{j=0}^{i} (-1)^{j} \binom{i}{j} \sum_{k=0}^{jc_1/c_2} \binom{jc_1/c_2}{k} (c_2Tu)^{k} [1-u]^{m-2}\, du$$

$$= \frac{1}{(n-1)B\left((n-1),1\right)\,B\left((m-1),1\right)} \sum_{i=0}^{n-1} \left(\frac{1}{c_1 S}\right)^{i} \binom{n-1}{i} \sum_{j=0}^{i} (-1)^{j} \binom{i}{j} \cdot \sum_{k=0}^{jc_1/c_2} \binom{jc_1/c_2}{k} (c_2T)^{k} \int_0^1 u^{k} [1-u]^{m-2}\, du$$

$$= \frac{1}{(n-1)B\left((n-1),1\right)\,B\left((m-1),1\right)} \sum_{i=0}^{n-1} \left(\frac{1}{c_1 S}\right)^{i} \binom{n-1}{i} \sum_{j=0}^{i} (-1)^{j} \binom{i}{j} \cdot \sum_{k=0}^{jc_1/c_2} \binom{jc_1/c_2}{k} (c_2T)^{k} B(k+1,\, m-1) \qquad (26)$$

Now, we consider the case, when $c_1^{-1}\log(c_1S+1) < c_2^{-1}\log(c_2T+1)$.

In this case, from (25)

$$\hat{R} = \frac{1}{(n-1)B\left((n-1),1\right)\,B\left((m-1),1\right)T} \int_{y=0}^{c_1^{-1}\log(c_1S+1)} \times \left[1-\frac{e^{c_1y}-1}{c_1S}\right]^{n-1} e^{c_2y} \left[1-\frac{e^{c_2y}-1}{c_2T}\right]^{m-2} dy$$

Putting $\frac{e^{c_2y}-1}{c_2T} = u$,

$$\frac{e^{c_2y}}{T}\,dy = du,$$

$$e^{c_2y} = 1 + uc_2T$$

and $y = \frac{\log(1+uc_2T)}{c_2}$

Table 1 : Simulation results.

n,m \ (λ_1, λ_2, c)	(1,2.5,0.8)	(1,1.5,0.8)	(1,7,0.8)	(5,5,0.8)
10,10	0.0453	0.05	0.0341	0.0118
	0.7593	0.65	0.9091	0.5118
	0.0094	0.0115	0.0018	0.0127
	0.3382	0.3813	0.139	0.4548
	94.695	94.39	92.7822	95.5923
10,20	0.0507	-0.0169	0.0272	-0.0125
	0.7647	0.5831	0.9022	0.4875
	0.0081	0.0122	0.0022	0.0099
	0.2887	0.4414	0.1429	0.3768
	92.4941	95.2823	91.7897	94.201
10,30	-0.0456	-0.0162	0.0162	-0.0057
	0.731	0.5838	0.8912	0.4943
	0.009	0.01	0.0016	0.0088
	0.3626	0.3818	0.1391	0.3602
	92.8182	93.2301	91.9317	93.7076
15,15	-0.0072	-0.0059	0.0115	-0.0182
	0.7068	0.5941	0.8865	0.4818
	0.0052	0.0069	0.0016	0.0084
	0.2765	0.3137	0.1434	0.3412
	94.1822	94.0804	92.9904	94.1586
15,25	0.0525	0.0021	0.0073	-0.0326
	0.7665	0.6021	0.8823	0.4674
	0.007	0.0067	0.0011	0.0057
	0.1896	0.2938	0.1211	0.2785
	92.4059	92.4626	93.2789	95.9025
25,25	0.043	0.0419	0.0254	0.0014
	0.757	0.6419	0.9004	0.5014
	0.004	0.0072	0.0013	0.0055
	0.1731	0.2848	0.1033	0.2902
	93.6482	94.5628	93.7842	94.9233
40,40	0.0372	0.0397	0.0101	0.0227
	0.7512	0.6397	0.8851	0.5227
	0.0037	0.0047	6e-04	0.0037
	0.17 93.0661	0.2125 94.1963	0.0955 95.309	0.219 94.7133

Estimates in order are bias, bootstrap estimate, MSE, confidence length and coverage percentage.

$$\hat{R} = \frac{1}{(n-1)B((n-1),1)\,B((m-1),1)T}\int_0^{\left((c_1S+1)^{c_2 c_1}-1\right)/c_2T}$$

$$\times\left[1-\frac{e^{\frac{c_1}{c_2}\log(1+c_2Tu)}-1}{c_1S}\right]^{n-1} T[1-u]^{m-2}\,du$$

$$=\frac{1}{(n-1)B((n-1),1)\,B((m-1),1)}\int_0^{\left((c_1S+1)^{c_1/c_2}-1\right)/c_2T}\sum_{i=0}^{n-1}(-1)^i$$

$$\times\binom{n-1}{i}\left\{\frac{(1+c_2Tu)^{\frac{c_1}{c_2}}-1}{c_1S}\right\}^i[1-u]^{m-2}\,du$$

$$=\frac{1}{(n-1)B((n-1),1)\,B((m-1),1)}\int_0^{\left((c_1S+1)^{c_1/c_2}-1\right)/c_2T}\sum_{i=0}^{n-1}\binom{n-1}{i}$$

$$\times\left\{\frac{1-(1+c_2Tu)^{\frac{c_1}{c_2}}}{c_1S}\right\}^i[1-u]^{m-2}\,du$$

$$=\frac{1}{(n-1)B((n-1),1)\,B((m-1),1)}\int_0^{\left((c_1S+1)^{c_1/c_2}-1\right)/c_2T}\sum_{i=0}^{n-1}\left(\frac{1}{c_1S}\right)^i\binom{n-1}{i}$$

$$\times\sum_{j=0}^{i}(-1)^j\binom{i}{j}\sum_{k=0}^{jc_1/c_2}\binom{jc_1/c_2}{k}(c_2Tu)^k[1-u]^{m-2}\,du$$

$$=\frac{1}{(n-1)B((n-1),1)\,B((m-1),1)}\sum_{i=0}^{n-1}\left(\frac{1}{c_1S}\right)^i\binom{n-1}{i}\sum_{j=0}^{i}(-1)^j\binom{i}{j}$$

$$\times\sum_{k=0}^{jc_1/c_2}\binom{jc_1/c_2}{k}(c_2T)^k\int_0^{\left((c_1S+1)^{c_2/c_1}-1\right)/c_2T}u^k[1-u]^{m-2}\,du \quad (27)$$

5. Simulation Study

To study the performance of the estimator developed when all the parameters are unknown, we simulated from the Gompertz distribution with the values of the parameters $(\lambda_1, c_1, \lambda_2, c_2) = (2, 0.5, 3,1)$ for sample size (n, m) = (15,15). The simulated data are X : 0.3488, 0.0444, 0.5277, 0.0554, 0.4027, 0.2053, 0.0701, 0.2139, 0.3790, 0.1909, 1.0587, 0.2164, 0.7535, 0.0306, 0.0231. Y: 0.2371, 0.1836, 0.0050, 0.1107, 0.0033, 0.0853, 0.4173, 0.0378, 0.6579, 0.2541, 0.0790, 0.4696, 0.1402, 0.0513, 0.3414.

R (Actual) = 0.6206.

$\hat{R} = 0.6425$

To study the performance of $\hat{R}$ developed, when $c_1 = c_2 = c$ and are known, 500 bootstrap samples are simulated from the Gompertz distribution with the values of parameters $(\lambda_1, \lambda_2, c)$ as (1,2.5,0.8), (1,1.5,0.8), (1,7,0.8) and (5,5,0.8) over different samples of sizes (n, m) = (10,10), (10,20), (10,30), (15,15), (15,25), (25,25) and (40,40). Bootstrap estimates of bias, mean, MSE, confidence length and coverage percentage are given in Table 1.

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