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Source: *Sankhyā: The Indian Journal of Statistics (2003-2007)*, Vol. 68, No. 1 (Feb., 2006), pp. 111-129
Published by: Springer on behalf of the Indian Statistical Institute
Stable URL: http://www.jstor.org/stable/25053479

Sankhyā : The Indian Journal of Statistics
2006, Volume 68, Part 1, pp. 111-129

On Mixture Nonlinear Time-series Modelling and Forecasting for ARCH Effects

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Abstract

In the class of Nonlinear time-series models, Gaussian mixture transition distribution (GMTD) and Mixture autoregressive (MAR) models may be employed to describe those data sets that depict sudden bursts, outliers and flat stretches at irregular time-epochs. In order to capture volatility explicitly, recently a new family, viz. MAR-Autoregressive conditional heteroscedastic (MAR-ARCH) has been introduced in the literature. In this paper, these three families are studied by considering weekly wholesale onion price data during April, 1998 to March, 2002. Presence of ARCH in detrended and deseasonalised series is tested by Naive-Lagrange multiplier (Naive-LM) test. Estimation of parameters is done using Expectation-Maximization (EM) algorithm and best model from each family is selected on basis of Bayesian information criterion (BIC). The salient feature of work done is that, for selected models, formulae for carrying out out-of-sample forecasting up to three-steps ahead have been obtained theoretically, perhaps for the first time, by recursive use of conditional expectation and conditional variance. In respect of out-of-sample data, results derived enable us to compute best predictor, prediction error variance, and predictive density. It is concluded that a two-component MAR-ARCH provides best description of the data for modelling as well as forecasting purposes.

AMS (2000) *subject classification.* Primary 62J02; Secondary 62P20.
Keywords and phrases. Autoregressive conditional heteroscedasticity, GMTD model, MAR model, MAR-ARCH model, EM algorithm, volatility, stochastic trend, BIC, out-of-sample forecasting

1 Introduction

Box-Jenkins Autoregressive integrated moving average (ARIMA) models have dominated analysis of time-series data since last six decades or so. However, there are many instances in which such models are not appropriate. For example, Le et al. (1996) indicated their inadequacy for hourly viscosity readings in a chemical process due to frequent flat stretches with bursts. Recently Ghosh et al. (2006) have shown that India's marine products export data contains shocks at irregular time-epochs and fitted bilinear time series model to data. Given time-series data $\{Y_1, Y_2, \ldots, Y_n\}$, autocovariance function, $\gamma_j = cov(Y_t, Y_{t+j}); j = 0, 1, 2, \ldots$ used in linear time-series are only one aspect of joint distribution of $\{Y_1, Y_2, \ldots, Y_n\}$; other aspects may contain vital information missed by the γ_j. For example, when stock market becomes volatile in certain periods, it is expected that stock prices would rise or decline sharply and hence presence of bimodal conditional distribution is observed (Wong and Li, 2000). The volatility is described by presence of conditional heteroscedastic errors (Engle, 1982) and hence study of autocorrelation of squared series is required, unlike linear ARIMA models. Main limitation of ARIMA methodology is that these yield "Linear" models. Fortunately, during last two decades or so, a number of "Nonlinear" time-series models have been developed. Many such models can be shown to have marginal and conditional distributions to be multimodal due to presence of ARCH errors. The zeroth-order self-exciting threshold autoregressive model (Tong, 1995) with all lag 1 coefficients being zero, has been shown to have a mixture of Gaussian distribution marginally and hence may exhibit multimodality. Jalali and Pemberton (1995) extended the class of zeroth-order threshold autoregressive models to a much richer class of *mixture models.* Amongst their many properties, we observe that their autocovariance structure has the same form as that of linear ARMA models although we only get a subset of possible autocovariance functions from such models.

In this paper, an attempt has been made to study the performance of mixture time-series models, viz. GMTD, MAR and MAR-ARCH models. As an illustration, these models have been applied to weekly onion price data during April, 1998 to March, 2002. Tests for presence of unit root have been made before fitting trend followed by seasonal adjustment through correlogram analysis. The detrended and deseasonalised series have shown presence of volatility due to the fact that chance of a sharp increase or decrease at some points is higher than that of a moderate change at some other point. This fact is explained through within and out-of-sample predictive density which shows multimodality of predictive distribution and nonconstancy of volatility functions. Also Naive-LM test, when the conditional mean is unspecified, is applied for testing presence of ARCH leading to volatility in data set. The best models from each of GMTD, MAR, and MAR-ARCH families are selected on the basis of BIC criterion. Subsequently, for selected models, formulae for carrying out out-of-sample forecasting up to three-steps ahead have been obtained theoretically, perhaps for the first time, by recursive use of conditional expectation and conditional variance. In respect of out-of-sample data, the results derived enable us to compute best predictor, prediction error variance, and predictive density for the data. Forecast intervals are computed and it may be noticed that these depend on changing conditional variance. Through simulation, up to three-step ahead predictive densities based on estimated conditional expectation and conditional variance have been computed for these models. This, in turn, explains appropriateness of mixture modelling for data depicting varying behaviour of densities at future time-epochs. Finally, it is concluded that, for data under consideration, a two-component MAR-ARCH provides best description of the data for modelling as well as forecasting purposes.

2 Some Preliminaries

In this section, we discuss briefly three mixture nonlinear time-series models as well their estimation procedures.

2.1. Mixture nonlinear times-series models. The GMTD model, given by Le et al. (1996), is defined as

$$F(y_t|\mathbf{y}^{t-1}) = \alpha_0\Phi\{(y_t - \sum_{j=1}^{p}\phi_{0j}y_{t-j})/\sigma_0\} + \sum_{i=1}^{p}\alpha_i\Phi\{(y_t - \phi_i y_{t-i})/\sigma_i\} \quad (2.1)$$

where $\Phi(.)$ is c.d.f. of standard Gaussian variate. This model can accommodate the situation where series is reasonably well approximated by an AR model and has occasional bursts and outliers. In this case, main AR component of series would be captured by first term of (2.1), and additional components, such as outliers and bursts, would be captured by other terms. For instance, occasional outliers may be captured by a term in the model with a large variance, σ_i^2, and a small α_i, and bursts can be accommodated with a large α_i. Flat stretches can be captured with a very small variance.

The K-component MAR model, denoted as MAR $(K; p_1, p_2, \ldots, p_K)$, is (Wong and Li, 2000):

$$F(y_t|\mathbf{y}^{t-1}) = \sum_{k=1}^{K}\alpha_k\Phi\{(y_t - \phi_{k0} - \phi_{k1}y_{t-1} - \cdots - \phi_{kp_k}y_{t-p_k})/\sigma_k\}.$$

Its conditional distribution can be multimodal. The conditional expectation of y_t is

$$E(y_t|\mathbf{y}^{t-1}) = \sum_{k=1}^{K} \alpha_k(\phi_{k0} + \phi_{k1}y_{t-1} + \cdots + \phi_{kp_k}y_{t-p_k}) = \sum_{k=1}^{K} \alpha_k \mu_{kt} \quad (2.2)$$

where

$$\mu_{kt} = \phi_{k0} + \phi_{k1}y_{t-1} + \cdots + \phi_{kp_k}y_{t-p_k}, 1, 2, \ldots, K.$$

The conditional variance of y_t, which is dependent on conditional means of components, is given by

$$Var(y_t|\mathbf{y}^{t-1}) = \sum_{k=1}^{K} \alpha_k \sigma_k^2 + \sum_{k=1}^{K} \alpha_k \mu_{k,t}^2 - \left(\sum_{k=1}^{K} \alpha_k \mu_{k,t}\right)^2. \quad (2.3)$$

Since $\sum_{k=1}^{K} \alpha_k \mu_{k,t}^2 - \left(\sum_{k=1}^{K} \alpha_k \mu_{k,t}\right)^2$ is non-negative, conditional variance will be large when $\mu_{k,t}$'s differ greatly and smallest conditional variance is $\sum_{k=1}^{K} \alpha_k \sigma_k^2$. Fourth order moment of $\{Y_t\}$ is greater than three unless $\sigma_1 = \sigma_2 = \cdots = \sigma_K$ and $\mu_{1,t} = \mu_{2,t} = \cdots = \mu_{K,t}$. This is certainly a desirable property in modelling of economic time-series as empirical distribution of economic data usually exhibits a thicker tail than that of a Gaussian distribution. Although MAR models are able to capture conditional heteroscedasticity (Engle, 1982), recently another family which explicitly involves presence of ARCH errors in individuals mixtures, called MAR-ARCH models, has been introduced by Wong and Li (2001). The model, denoted as MAR-ARCH $(K; p_1, p_2, \ldots, p_K; q_1, q_2, \ldots, q_K)$, is defined by

$$F\left(y_t|\mathbf{y}^{t-1}\right) = \sum_{k=1}^{K} \alpha_k \Phi\left[e_{k,t}(h_{k,t})^{-1/2}\right] \quad (2.4)$$

where

$$\begin{aligned} e_{k,t} &= y_t - \phi_{k0} - \phi_{k1}y_{t-1} - \cdots - \phi_{kp_k}y_{t-p_k},\ h_{k,t} \\ &= \beta_{k0} + \beta_{k1}e_{k,t-1}^2 + \cdots + \beta_{kp_k}e_{k,t-q_k}^2. \end{aligned}$$

Here $\alpha_1 + \cdots + \alpha_K = 1, \alpha_k > 0 (k = 1, 2, \ldots, K)$. To avoid possibility of zero or negative conditional variance, following condition of β_{ki}'s must be imposed:

$$\beta_{k0} > 0, k = 1, 2, \ldots, K;\ \beta_{ki} \geq 0,\ i = 1, 2, \ldots, q_k,\ k = 1, 2, \ldots, K.$$

Shape of conditional distribution of series changes over time as conditional means and variances of components, which depend on past values of time-series in different ways, differ. Conditional variance of y_t is given by

$$\text{Var }(y_t|\mathbf{y}^{t-1}) = \sum_{k=1}^{K} \alpha_k h_{k,t} + \sum_{k=1}^{K} \alpha_k \mu_{k,t}^2 - \left(\sum_{k=1}^{K} \alpha_k \mu_{k,t}\right)^2 .$$

The first term allows modelling of dependence of conditional variance on past 'errors', while second and third terms model change of conditional variance due to difference in conditional means in the components.

Before applying GMTD, MAR and MAR-ARCH families to data, it has to be ensured that stationarity conditions given respectively in Le et al. (1996), Wong and Li (2000), and Wong and Li (2001) are satisfied.

2.2. Estimation of parameters. The EM algorithm, which is most readily available procedure for estimating mixture models, is employed for estimation of parameters. Suppose that observation $Y = (y_1, \ldots, y_n)$ is generated from MAR-ARCH model (2.4). Let $Z = (Z_1, \ldots, Z_n)$ be unobserved random variable, where Z_t is a K-dimensional vector with component k equal to one, if y_t comes from kth component of conditional distribution function, and zero otherwise. Denote kth component of Z_t as $Z_{k,t}$. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_K)$, $\theta_k = (\phi_{k0}, \phi_{k1}, \ldots, \phi_{kp_k}),'$ $\beta_k = (\beta_{k0}, \beta_{k1}, \ldots, \beta_{kq_k})'$, $k = 1, 2, \ldots, K$ and $\boldsymbol{\theta} = (\alpha', \theta_1', \beta_1', \ldots, \theta_K', \beta_K')'$ where $'$ denotes transpose of a vector or a matrix. Further, let $p = \max(p_1, p_2, \ldots, p_K)$, $q = \max(q_1, q_2, \ldots, q_K)$. The (conditional) log-likelihood is given by

$$\begin{aligned} l &= \sum_{t=p+q+1}^{n} l_t \\ &= \sum_{t=p+q+1}^{n} \left\{ \sum_{k=1}^{K} Z_{k,t} \log(\alpha_k) - \sum_{k=1}^{K} (Z_{k,t}/2) \log(h_{k,t}) - \sum_{k=1}^{K} \left(Z_{k,t} e_{k,t}^2 / 2h_{k,t}\right) \right\}, \end{aligned} \tag{2.5}$$

where $N = n - p - q$. First order derivatives of log-likelihood with respect to $\boldsymbol{\theta}$ were derived by Wong and Li (2001). Iterative EM procedure estimates parameters by maximizing log-likelihood function (2.5). It comprises an E-step and an M-step described as follows:

E-STEP. Suppose that θ is known. The missing data Z are replaced by their conditional expectation, conditional on the parameters and on observed data Y. In this case, conditional expectation of kth component of Z_t is just conditional probability that observation Y_t comes from kth component of mixture distribution, conditional on θ and Y. Let $\tau_{k,t}$ be conditional expectation of $Z_{k,t}$. Then E-step equations are:

$$\tau_{k,t} = \alpha_k (h_{k,t})^{-1/2} \varphi[e_{k,t}(h_{k,t})^{-1/2}] \left[\sum_{l=1}^{K} \alpha_l (h_{l,t})^{-1/2} \varphi[e_{l,t}(h_{l,t})^{-1/2}] \right]^{-1}$$

where φ is the p.d.f. of a standard normal variate.

M-STEP. Suppose missing data are known. The estimates of parameters θ can be obtained by maximizing log likelihood l. This can be done on replacing $Z_{k,t}$ by $\tau_{k,t}$ in first order derivatives of log-likelihood (2.5). The parameter estimates of α are

$$\hat{\alpha} = 1/(n-p-q) \sum_{t=p+q+1}^{n} \tau_{k,t}; k = 1, \ldots, K.$$

Newton-Raphson method is used for parameter estimates of θ_k's and β_k's. Starting with initial values $\theta_k^{(0)}$ and $\beta_k^{(0)}$, values of θ_k and β_k in subsequent iterations are given by

$$\theta_k^{(i+1)} = \theta_k^{(i)} + \left\{ \left. \frac{\partial^2 l}{\partial \theta_k^2} \right|_{\theta^{(i)} \beta^{(i)}} \right\} \left. \frac{\partial l}{\partial \theta_k} \right|_{\theta^{(i)} \beta^{(i)}} \tag{2.6}$$

and

$$\beta_k^{(i+1)} = \beta_k^{(i)} + \left\{ \left. \frac{\partial^2 l}{\partial \beta_k^2} \right|_{\theta^{(i+1)} \beta^{(i)}} \right\} \left. \frac{\partial l}{\partial \beta_k} \right|_{\theta^{(i+1)} \beta^{(i)}} \tag{2.7}$$

The above particular M-step is obtained by iterating (2.6) and (2.7) until convergence is achieved. Final estimates of parameter vector $\boldsymbol{\theta}$ are obtained by iterating E-steps and M-steps until convergence is achieved. The standard errors of parameter estimates can be computed by Missing information principle (Louis, 1982). The observed information matrix, I, can be computed from complete information matrix, I_c, and missing information matrix, I_m, with the relation:

$$I = I_c - I_m = E\left(-N \frac{\partial^2 l}{\partial \theta^2} \middle| \theta, Y \right)_{\hat{\theta}} - Var\left(N \frac{\partial l}{\partial \theta} \middle| \theta, Y \right)_{\hat{\theta}}.$$

The formulae for computing I_c and I_m are given in Wong (1998). The dispersion matrix of estimates $\hat{\theta}$ is given by inverse of observed information matrix, I. It should be noted that EM estimation procedure may also be employed in a similar manner for estimating parameters for GMTD and MAR models. Since there is no software package available for execution of EM algorithm and for estimating information matrices, relevant computer programs are developed in MATLAB, Ver. 5.3 and can be obtained from first author on request.

3 An Illustration

The above methodology is applied to weekly wholesale onion price data of Nasik variety at Azadpur Mandi, New Delhi, India during the period from first week of April, 1998 to first week of November, 2001 comprising 172 observations. Weekly onion price showed marked volatility by touching value of Rs.4000 per quintal in October, 1998. It remained stable in range of Rs.450 to Rs.700 depicting flat stretches with occasional bursts of large amplitude to tune of Rs.850 to Rs.900 during October, 1999. In subsequent year, price remained on an average of Rs.350 in first half whereas it remained above Rs. 500 for second half exhibiting another phase of flat stretches.

3.1. Modelling trend and seasonal fluctuations. A formal test procedure for testing presence of stochastic trend in case of an integrated process, proposed by Dickey and Fuller (1979), is based on the model

$$y_t = \mu + \beta_t + \rho y_{t-1} + e_t^*, \tag{3.1}$$

where e_t^* is a stationary process with mean zero and variance σ^2. The null hypothesis $H_0 : \rho = 1$ in (3.1) is tested based on statistic analogous to regression statistic $\hat{\tau}_\tau$ and is given by -1.84, which is not significant at 5% level. Substituting $\rho = 1$ in (3.1), model to be considered reduces to

$$\triangle_1 \; y_t = \mu + \beta t + e_t^*.$$

Regressing $\triangle_1 \; y_t$ on linear trend, least square estimates of μ and β are computed as 6.06 and -0.04 respectively. Then estimated model is

$$\triangle_1 \; y_t = 6.06 - 0.04t + \hat{e}_t^*. \tag{3.2}$$

It may be noted that trend coefficient in above equation involving growth rate, viz. $\hat{\beta}$ is negative. Negative value of $\hat{\beta}$, in turn, ensures that deterministic part of $\triangle_1 \; y_t$ series is capable of taking both positive as well as negative values. This implies that the original series y_t may increase or decrease according as above deterministic part is positive or negative. This, as is desirable, is in consonance with onion price data which exhibits similar pattern (Fig. 1).

We now consider Estimated autocorrelation function (EACF) of residual time-series $\{\hat{e}_t^*\}$ and compare it with that of ARMA-type models for seasonal time-series. Here we considered correlation around $s, 2s, \ldots$, where s denotes number of "Seasons per year". As we are considering weekly data, "Season" represents "Week". Further, duration of onion crop is 3 months and so "Year" represents "3 Months". Since there would be 3×4, i.e. 12 observations per onion crop, therefore, in present illustration, $s = 12$. The four relevant EACF's, viz. $\hat{e}_t^*, \triangle_1 \hat{e}_t^*, (\triangle_1 \hat{e}_t^*)^c$ (when $\triangle_1\ e_t$ is corrected on 12 dummy variables corresponding to 12 seasons) and $\triangle_1 \triangle_{12} \hat{e}_t^*$ for identifying appropriate seasonal models have been computed. The significant EACF values of $\triangle_1 \triangle_{12} \hat{e}_t^*$ at lags 1,11,12 and 13 suggest parsimonious model structures, known as 'Box-Jenkins airline model', given by

$$\triangle_1 \triangle_s \hat{e}_t^* = (1 + \theta_1 L)(1 + \theta_s L^s) e_t, t = s + 2, s + 3, \ldots. \tag{3.3}$$

Estimated parameters θ_1 and θ_2 are computed as -0.17 and 0.15 respectively using SAS, version 8.1. The tests for presence of nonseasonal and seasonal stochastic trends and appropriateness of double filter $\triangle_1 \triangle_s$ described above has also been performed using OSCB auxiliary regression, proposed by Osborn et al. (1988) which confirms that presence of seasonal and nonseasonal unit root cannot be rejected.

3.2. Testing for ARCH. One method of testing for presence of ARCH in estimated residual series $\{\hat{\varepsilon}_t\}$ in (3.3) is based on the statistic TR^2, where T is total number of observations and R^2 is multiple correlation coefficient between squared residual, $\hat{\eta}_t^2$ ($\hat{\eta}_t$ obtained after fitting $\hat{\varepsilon}_t$ on its conditional mean μ_t) and p of its lags. Assuming that conditional mean is correctly specified, Engle (1982) showed that TR^2 is asymptotically equivalent to an LM test and is distributed asymptotically as a $\chi^2(p)$ random variable under H_0. For present data, TR^2 and LM values are computed as 14.87 and 26.47 respectively, which are significant at 5% level. Lumsdaine and Ng (1999) suggested 'naïve' approach of approximating unknown conditional mean by computing 'recursive residuals' $\hat{w}_t$, which contain true conditional mean not captured by usual regression function. The final model to be estimated is

$$\hat{\varepsilon}_t = \mathbf{Z}_t' \boldsymbol{\gamma} + g(\hat{w}_{t-1}) + \nu_t \tag{3.4}$$

where $\mathbf{Z}_t$ is vector of lagged values of $\hat{\varepsilon}_t$, $\boldsymbol{\gamma}$ is vector of regression coefficients, $g(\hat{w}_{t-1})$ is a (possibly nonlinear) function of the recursive residuals $\hat{w}_{t-1}$ and

ν_t is the error term. The quantity, $\hat{\nu}_t^2$, is used for testing ARCH effect by TR^2 and LM statistics and the values obtained, using (3.4), are 7.40 and 20.01 which are significant at 5% level.

3.3. Fitting of models. We consider two-component and three-component GMTD models for detrended and deseasonalised weekly onion price series. The order selection criterion followed here is BIC as, unlike other criteria, viz. Akaike information criterion and Final prediction error, it leads to a consistent order selection (Fan and Yao, 2003). The best GMTD model defined by (2.1) is found to be

$$F(\hat{\varepsilon}_t|\hat{\varepsilon}_{t-1}) = 0.11\Phi\left\{(\hat{\varepsilon}_t+0.28\hat{\varepsilon}_{t-1}-0.29\hat{\varepsilon}_{t-2})/0.54\right\} + 0.58\Phi\left\{(\hat{\varepsilon}_t-0.61\hat{\varepsilon}_{t-1})/0.65\right\} + 0.31\Phi\left\{(\hat{\varepsilon}_t-0.14\hat{\varepsilon}_{t-2})/3.14\right\}. \tag{3.5}$$

with BIC value as 386.72. The standard errors for $(\hat{\alpha}_0, \hat{\alpha}_1, \hat{\alpha}_2, \hat{\sigma}_0, \hat{\sigma}_1, \hat{\sigma}_2, \hat{\phi}_{01}, \hat{\phi}_{02}, \hat{\phi}_1, \hat{\phi}_2)$ are (0.09, 1.28, 2.1, 1.24, 2.05, 0.95, 2.56, 1.65, 0.98, 2.45) respectively. It is observed that occasional outliers in time-period July, 1998 and August, 1998, October, 1999 and November, 2001 is captured by large $\hat{\sigma}_2^2$ with small $\hat{\alpha}_2$ and bursts can be accommodated with a larger $\hat{\alpha}_1$ during October, 1998. The mixture containing AR part captures flat stretches by small variance $\hat{\sigma}_0^2$. Also it represents pure replacement type outlier since coefficients are opposite in sign but same in magnitude during period of flat stretches. The three-component GMTD model has been found to be inferior as its BIC value is computed as 474.11. As $\hat{\alpha}_0$ is close to zero, it is found that best GMTD(2) model satisfies first and second order stationarity conditions.

In case of MAR model also, we consider two and three component models for detrended and deseasonalised weekly onion price series. The best two-component MAR model, defined by (2.2) with $\phi_{k0} = 0, k = 1, 2$ is found to be MAR (2; 2, 1) having a BIC value of 521.63. The model is given by

$$F(\hat{\varepsilon}_t|\hat{\varepsilon}_{t-1}) = 0.27\Phi\left\{(\hat{\varepsilon}_t - 0.434\ \hat{\varepsilon}_{t-1}-0.38\hat{\varepsilon}_{t-2})/6.34\right\} + 0.73\Phi\left\{(\hat{\varepsilon}_t + 0.27\ \hat{\varepsilon}_{t-1})/0.24\right\}.$$

Standard errors for $(\hat{\alpha}_1, \hat{\alpha}_2, \hat{\sigma}_1, \hat{\sigma}_2, \hat{\phi}_{11}, \hat{\phi}_{12}, \hat{\phi}_{21})$ are (0.05, 0.20, 1.71, 0.04, 0.20, 0.16, 0.04) respectively. The best three-component MAR model is a MAR (3; 2, 2, 1) with $\phi_{k0} = 0, k = 1, 2, 3$ and the model is given by

$$F(\hat{\varepsilon}_t|\hat{\varepsilon}_{t-1}) = 0.33\Phi\left\{(\hat{\varepsilon}_t - 0.56\ \hat{\varepsilon}_{t-1} - 0.36\ \hat{\varepsilon}_{t-2})/2.08\right\} + 0.64\Phi\left\{(\hat{\varepsilon}_t + 0.26\ \hat{\varepsilon}_{t-1} + 0.13\ \hat{\varepsilon}_{t-2})/0.19\right\} + 0.03\Phi\left\{(\hat{\varepsilon}_t + 5.72\hat{\varepsilon}_{t-1})/9.44\right\}. \tag{3.6}$$

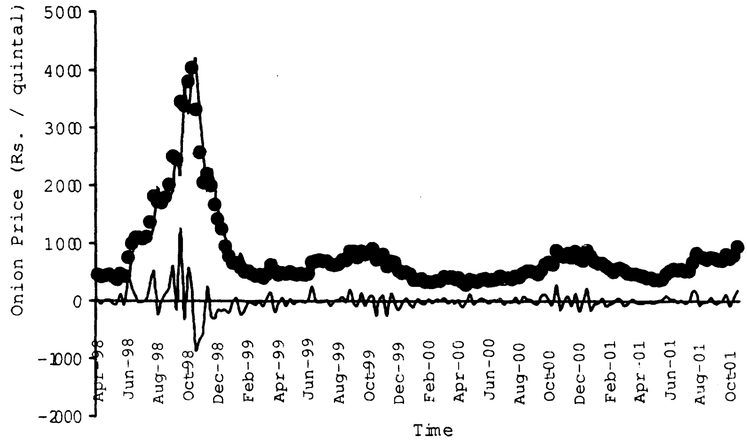


Figure 1: Fitted MAR-ARCH (2; 0, 1; 1, 1) model along with data points and error series

with BIC value as 383.84, which is less than that for the best GMTD model (386.72). The standard errors for $(\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3, \hat{\sigma}_1, \hat{\sigma}_2, \hat{\sigma}_3, \hat{\phi}_{11}, \hat{\phi}_{12}, \hat{\phi}_{21}, \hat{\phi}_{22}, \hat{\phi}_{31})$ are (0.06, 0.06, 0.11, 0.47, 0.04, 9.44, 0.11, 0.09, 0.03, 0.05, 1.63) respectively. Here, third component is non-stationary AR process, but fitted MAR (3; 2, 2,1) model satisfies first and second order stationarity conditions.

We now consider fitting of two-component and three-component MAR-ARCH models. The best two-component MAR-ARCH model, defined by (2.4) with $\phi_{k0} = 0, k = 1, 2$ is found to be MAR-ARCH (2; 0, 1; 1, 1) having BIC value of 307.98. The model is given by

$$F(\hat{\varepsilon}_t|\hat{\varepsilon}_{t-1}) = 0.75\Phi\left\{(\varepsilon'_{1,t}/\sqrt{h_{1,t}}\right\} + 0.25\Phi\left\{(\varepsilon'_{2,t}/\sqrt{h_{2,t}}\right\} \qquad (3.7)$$

where $\varepsilon'_{1,t} = \hat{\varepsilon}_t, h_{1,t} = 0.14 + 0.38\varepsilon'^2_{1,t-1}, \varepsilon'_{2,t} = \hat{\varepsilon}_t + 0.84\hat{\varepsilon}_{t-1}$ and $h_{2,t} = 1.61 + 1.54\varepsilon'^2_{2,t-1}$. The standard errors for $(\hat{\alpha}_1, \hat{\alpha}_2, \hat{\phi}_{21}, \hat{\beta}_{10}, \hat{\beta}_{11}, \hat{\beta}_{20}, \hat{\beta}_{21})$, are (0.08, 0.03, 0.29, 0.04, 0.16, 0.61, 0.83) respectively. It is observed that occasional outliers in time-period July, 1998 and August, 1998, October 1999 and November, 2001 are captured by large $h_{2,t}$ with small $\hat{\alpha}_2$ and bursts can be accommodated with a larger $\hat{\alpha}_1$ during October, 1998. Flat stretches during April, 1999 to June, 1999 are captured by low value of $Var(y_t|\mathbf{y}^{t-1})$. The best three-component model is a MAR-ARCH (3; 1, 1, 0; 2, 0, 1) with $\phi_{k0} = 0, k = 1, 2, 3$ having BIC value of 442.05. The parameter estimates and standard errors of $(\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3, \hat{\phi}_{11}, \hat{\phi}_{21}, \hat{\beta}_{10}, \hat{\beta}_{11}, \hat{\beta}_{12}, \hat{\beta}_{20}, \hat{\beta}_{30}, \hat{\beta}_{31})$, are (0.44,

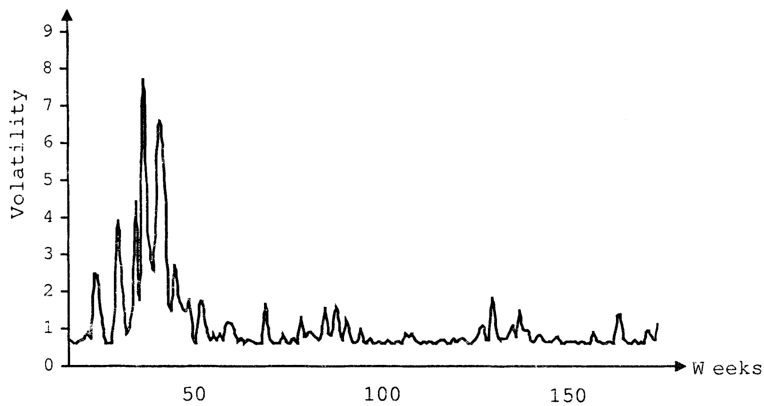


Figure 2: Volatility from fitted MAR-ARCH (2; 0, 1; 1, 1) model in seasonally adjusted onion price series

0.48, 0.09, 0.41, −0.27, 0.13, 0.37, 1.35, 0.19, 5.54, 0.42) and (0.11, 0.13, 0.20, 0.21, 0.04, 0.09, 0.26, 0.91, 0.05, 3.56, 1.13) respectively. It is noticed that best MAR-ARCH (2; 0, 1; 1, 1) model satisfies first and second order stationarity conditions. The fitted MAR-ARCH (2; 0, 1; 1, 1) model along with data points and residuals is depicted in Figure 1.

Further, one-step and two-step ahead predictive distributions from fitted three models are computed. It is observed that predictive densities exhibit unimodality or bimodality according as volatility (conditional variance) is low or high. This implies that future values have small or large probabilities of taking extreme values according as conditional variances are low or high. To save space, volatility function for fitted MAR-ARCH model only is depicted in Figure 2.

4 Best Multistep Ahead Predictors

It has been shown above that onion price data shows seasonal fluctuations with stochastic trend. Accordingly, h-step ahead forecast, $\hat{y}_{T+h}$ is given by

$$\hat{y}_{T+h} = \hat{T}_{T+h} + \hat{\hat{e}}^*_{T+h} + \hat{\varepsilon}_{T+h|T}. \tag{4.1}$$

where $\hat{T}_{T+h}$, $\hat{\hat{e}}^*_{T+h}$ and $\hat{\varepsilon}_{T+h|T}$ indicate respectively h-step ahead forecasts of trend value, seasonal value and residual series. Using (3.2) and (3.3):

$$\hat{T}_{T+h} = y_{T+h-1} + 6.06 - 0.04(T+h) \tag{4.2}$$

and

$$\begin{aligned}\hat{\hat{e}}^{*}_{T+h} = \hat{e}^{*}_{T+h-1} + \hat{e}^{*}_{T+h-12} - \hat{e}^{*}_{T+h-13} + \hat{\theta}_1 \hat{\varepsilon}_{T+h-1} \\ + \hat{\theta}_{12}\hat{\varepsilon}_{T+h-12} - \hat{\theta}_1\hat{\theta}_{12}\hat{\varepsilon}_{T+h-13},\end{aligned} \tag{4.3}$$

where $\hat{\theta}_1 = -0.17, \hat{\theta}_{12} = 0.15, T = 172$. In order to compute $\hat{\hat{e}}^{*}_{T+h}$ as a function of $\hat{\hat{e}}^{*}_{T+h-j}$, forecast values of $\hat{y}_{T+h-j}$ are $\hat{T}_{T+h-j}$ are used. The expression for $\hat{\varepsilon}_{T+h|T}$ can be obtained for fitted MAR, GMTD, and MAR-ARCH models by using (3.6), (3.5), and (3.7) respectively. It may be noted that detrended-deseasonalized onion price series $\{\hat{\varepsilon}_t\}$ exhibits presence of ARCH effects. Recursive conditional expectation is employed for obtaining best predictor, prediction error variance, and h-step ahead predictive density as discussed below. In our future discussion, we confine ourselves to computing only up to three-step ahead forecasts. However, the same procedure can easily be followed to obtain more than three-step ahead forecasts. But, from practical point of view, such forecasts are generally not very reliable due to large mean square prediction error.

We now derive best predictors for mixture nonlinear time-series models. To this end, first consider MAR model. Introduce an additional parameter ϕ_{32} in this model which would enable us to obtain, as particular cases, relevant results for both MAR and GMTD models. One-step ahead best predictor of error series is $\hat{\varepsilon}_{T+1|T}$, where

$$\hat{\varepsilon}_{T+1|T} = E\left[\hat{\varepsilon}_{T+1}|\hat{\varepsilon}_T, \hat{\varepsilon}_{T-1}, \ldots\right]. \tag{4.4}$$

Using (2.2), we get

$$\begin{aligned}\hat{\varepsilon}_{T+1|T} =& \alpha_1\left(\phi_{11}\hat{\varepsilon}_T + \phi_{12}\hat{\varepsilon}_{T-1}\right) + \alpha_2\left(\phi_{21}\hat{\varepsilon}_T + \phi_{22}\hat{\varepsilon}_{T-1}\right) \\ &+ \alpha_3\left(\phi_{31}\hat{\varepsilon}_T + \phi_{32}\hat{\varepsilon}_{T-1}\right).\end{aligned} \tag{4.5}$$

Further, two-step ahead best predictor of error series, $\hat{\varepsilon}_{T+2|T+1}$, is given by

$$\hat{\varepsilon}_{T+2|T} = E\left[E\left\{\hat{\varepsilon}_{T+2}|\hat{\varepsilon}_{T+1}, \hat{\varepsilon}_T, \cdots\right\}|\hat{\varepsilon}_T, \hat{\varepsilon}_{T-1}, \ldots\right].$$

This, on using (4.4) and simplifying, yields

$$\begin{aligned}\hat{\varepsilon}_{T+2|T} =& \left(\alpha_1\phi_{11} + \alpha_2\ \phi_{21} + \alpha_3\ \phi_{31}\right)\hat{\varepsilon}_{T+1|T} \\ &+ \left(\alpha_1\phi_{12} + \alpha_2\phi_{22} + \alpha_3\ \phi_{32}\right)\hat{\varepsilon}_T.\end{aligned} \tag{4.6}$$

Proceeding along similar lines, three-step best predictor of error series can easily be seen to be

$$\hat{\varepsilon}_{T+3|T} = (\alpha_1\phi_{11} + \alpha_2\ \phi_{21} + \alpha_3\ \phi_{31})\ \hat{\varepsilon}_{T+2|T} + (\alpha_1\phi_{12} + \alpha_2\ \phi_{22} + \alpha_3\ \phi_{32})\,\hat{\varepsilon}_{T+1|T}. \tag{4.7}$$

Forecasts for best MAR (3; 2, 2, 1) and GMTD(2) models can be obtained from (4.5)–(4.7) on putting $\phi_{32} = 0$ and $\phi_{22} = \phi_{31} = 0$ respectively.

For best fitted MAR-ARCH model, viz. MAR-ARCH (2; 0, 1; 1, 1), the first three-steps ahead best predictors of error series are:

$$\hat{\varepsilon}_{T+1|T} = E\left[\hat{\varepsilon}_{T+1}|\hat{\varepsilon}_T, \hat{\varepsilon}_{T-1}, \ldots\right] = \alpha_2\phi_{21}\hat{\varepsilon}_T, \tag{4.8}$$

$$\hat{\varepsilon}_{T+2|T} = E\left[\hat{\varepsilon}_{T+2}|\hat{\varepsilon}_T, \hat{\varepsilon}_{T-1}, \ldots\right] = \alpha_2\phi_{21}\hat{\varepsilon}_{T+1|T} \tag{4.9}$$

and $$\hat{\varepsilon}_{T+3|T} = E\left[\hat{\varepsilon}_{T+3}|\hat{\varepsilon}_T, \hat{\varepsilon}_{T-1}, \ldots\right] = \alpha_2\phi_{21}\hat{\varepsilon}_{T+2|T}. \tag{4.10}$$

5 Evaluation of Forecasting Performance

We now study forecasting performance of above fitted mixture nonlinear time-series models for hold-out weekly onion price data from second week of November, 2001 to fourth week of March, 2002 comprising nineteen observations. To this end, one-step and two-step forecasts for various models are first computed using (4.1), (4.2), (4.3), (4.5), (4.6), (4.8) and (4.9). Then, two statistics, viz. Root mean square errors and Mean absolute errors are evaluated and the same are reported in Table 1. A perusal indicates that, for hold-out data, performance of MAR-ARCH model is best, followed by MAR, and then followed by GMTD models in respect of one-step as well as two-step ahead forecasting. In other words, capability of MAR-ARCH model to describe and forecast onion price data is clearly demonstrated.

TABLE 1. FORECASTING PERFORMANCE FOR HOLD-OUT WEEKLY ONION PRICE DATA FROM SECOND WEEK OF NOVEMBER, 2001 TO FOURTH WEEK OF MARCH, 2002 (RS. PER QUINTAL).

	Root mean square errors of various models		
Number of steps	GMTD	MAR	MAR-ARCH
One-step	116.17 (81.95)	113.47 (77.40)	97.65 (66.91)
Two-step	148.76 (111.62)	143.71 (106.36)	123.23 (90.48)

Note: Figures with brackets ( ) indicate corresponding mean absolute errors of forecasts.

We shall now confine our attention to studying two aspects concerning out-of-sample forecasting, viz. variances of forecast errors, and predictive density. This would enable us to forecast a few steps ahead unobserved data points.

5.1. Variances of forecast errors. It may be mentioned that, unlike well-known Box-Jenkins Autoregressive integrated moving average methodology, variances of forecast errors for mixture nonlinear time-series models depend on information up to present time-epochs. For MAR model with an additional parameter ϕ_{32}, one-step ahead forecast error variance, σ^2_{T+1}, is given by

$$\sigma^2_{T+1} = V\left[\hat{\varepsilon}_{T+1}|\hat{\varepsilon}_T, \hat{\varepsilon}_{T-1}, \ldots\right].$$

This, on using (2.3), can be written as

$$\begin{aligned}\sigma^2_{T+1} =& \alpha_1\sigma_1^2 + \alpha_2\sigma_2^2 + \alpha_3\sigma_3^2 + \alpha_1\mu^2_{1,T+1} + \alpha_2\mu^2_{2,T+1} + \alpha_3\mu^2_{3,T+1} \\ &- (\alpha_1\mu_{1,T+1} + \alpha_2\mu_{2,T+1} + \alpha_3\mu_{3,T+1})^2 .\end{aligned} \tag{5.1}$$

where

$$\mu_{k,T+1} = \phi_{k1}\hat{\varepsilon}_{T+i-1} + \phi_{k2}\hat{\varepsilon}_{T+i-2}, k = 1,2,3;\ i = 1,2,\ldots$$

Now, two-step ahead forecast error variance, σ^2_{T+2}, is given by

$$\sigma^2_{T+2} = V\left[\hat{\varepsilon}_{T+2}|\hat{\varepsilon}_T, \hat{\varepsilon}_{T-1}, \ldots\right].$$

which can be expressed as

$$\begin{aligned}\sigma^2_{T+2} =& V\left[E\left\{\hat{\varepsilon}_{T+2}|\hat{\varepsilon}_{T+1}, \hat{\varepsilon}_T, \ldots,\right\}|\hat{\varepsilon}_T, \hat{\varepsilon}_{T-1}, \ldots\right] \\ &+ E\left[V\left\{\hat{\varepsilon}_{T+2}|\hat{\varepsilon}_{T+1}, \hat{\varepsilon}_T, \ldots,\right\}|\hat{\varepsilon}_T, \hat{\varepsilon}_{T-1}, \ldots\right].\end{aligned}$$

Using (2.2) and (2.3), we get

$$\begin{aligned}\sigma^2_{T+2} =& \left(\sum_{k=1}^{3}\alpha_k\phi_{k1}\right)^2 \sigma^2_{T+1} + \sum_{k=1}^{3}\alpha_k\sigma_k^2 \\ &+ \sum_{k=1}^{3}\alpha_k E\left[(\phi_{k1}\ \hat{\varepsilon}_{T+1} + \phi_{k2}\hat{\varepsilon}_T)^2\,|\hat{\varepsilon}_T, \hat{\varepsilon}_{T-1}, \ldots\right] \\ &-E\left[\left\{\left(\sum_{k=1}^{3}\alpha_k\phi_{k1}\right)\hat{\varepsilon}_{T+1} + \left(\sum_{k=1}^{3}\alpha_k\phi_{k2}\right)\hat{\varepsilon}_t\right\}^2 \,\middle|\, \hat{\varepsilon}_T, \hat{\varepsilon}_{T-1}, \ldots\right].\end{aligned}$$

Expressing conditional expectation of $\hat{\varepsilon}^2_{T+1}$ in terms of conditional expectation and variance of $\hat{\varepsilon}_{T+1}$, a straightforward algebra yields

$$\begin{aligned}
\sigma^2_{T+2} = &(\alpha_1\phi_{11} + \alpha_2\phi_{21} + \alpha_3\phi_{31})^2 \sigma^2_{T+1} \\
&+ \alpha_1\sigma_1^2 + \alpha_2\sigma_2^2 + \alpha_3\sigma_3^2 \\
&+ \left\{ \left(\alpha_1\phi_{11}^2 + \alpha_2\phi_{21}^2 + \alpha_3\phi_{31}^2\right) - (\alpha_1\phi_{11} + \alpha_2\phi_{21} + \alpha_3\phi_{31})^2 \right\} \hat{\varepsilon}_{T+1|T} \\
&+ 2\{(\alpha_1\phi_{11}\phi_{12} + \alpha_2\phi_{21}\phi_{22} + \alpha_3\phi_{31}\phi_{32}) \\
&- (\alpha_1\phi_{11} + \alpha_2\phi_{21} + \alpha_3\phi_{31})(\alpha_1\phi_{12} + \alpha_2\phi_{22} + \alpha_3\phi_{32})\} \hat{\varepsilon}_{T+1|T}\ \hat{\varepsilon}_T \\
&+ \left\{ \left(\alpha_1\phi_{12}^2 + \alpha_2\phi_{22}^2 + \alpha_3\phi_{32}^2\right) - (\alpha_1\phi_{12} + \alpha_2\phi_{22} + \alpha_3\phi_{32})^2 \right\} \hat{\varepsilon}^2_T. \qquad (5.2)
\end{aligned}$$

Expression for three-step prediction error variance σ^2_{T+3} is more involved due to the presence of conditional covariance between $\hat{\varepsilon}_{T+1}$ and $\hat{\varepsilon}_{T+2}$. This has been taken care of by systematic conditioning principle. After lengthy algebra, the following final expression is obtained:

$$\begin{aligned}
\sigma^2_{T+3} = &\sum_{k=1}^{3} \alpha_k\sigma_k^2 + \sum_{k=1}^{3} \alpha_k \Bigg[\phi_{k1}^2 \left(\sigma^2_{T+2} + \hat{\varepsilon}^2_{T+2|T}\right) \\
&+ \phi_{k2}^2 \left(\sigma^2_{T+1} + \hat{\varepsilon}^2_{T+1|T}\right) \\
&+ 2\phi_{k1}\phi_{k2} \left\{ \left(\sum_{k=1}^{3} \alpha_k\phi_{k1}\right) \left(\sigma^2_{T+1} + \hat{\varepsilon}^2_{T+1|T}\right) \right. \\
&\left. + \left(\sum_{k=1}^{3} \alpha_k\phi_{k2}\right) \hat{\varepsilon}_{T+1|T}\ \hat{\varepsilon}_T \right\} \Bigg] \\
&- \left(\sum_{k=1}^{3} \alpha_k\phi_{k1}\right)^2 \hat{\varepsilon}^2_{T+2|T} - \left(\sum_{k=1}^{3} \alpha_k\phi_{k2}\right)^2 \hat{\varepsilon}^2_{T+1|T} \\
&- 2\left(\sum_{k=1}^{3} \alpha_k\phi_{k1}\right) \left(\sum_{k=1}^{3} \alpha_k\phi_{k2}\right) \hat{\varepsilon}_{T+1|T}\ \hat{\varepsilon}_{T+2|T}. \qquad (5.3)
\end{aligned}$$

Prediction error variances for best MAR (3; 2, 2, 1) and GMTD(2) models can be obtained from (5.1), (5.2), and (5.3) on putting $\phi_{32} = 0$ and $\phi_{22} = \phi_{31} = 0$ respectively.

For best fitted MAR-ARCH model, viz. MAR-ARCH (2; 0, 1; 1, 1), the first three steps ahead forecast error variances are:

$$\sigma_{T+1}^2 = \alpha_1 \left(\beta_{10}+\beta_{11}\hat{\varepsilon}_T^2\right) +\alpha_2 \left\{\beta_{20}+\beta_{21} \left(\hat{\varepsilon}_T-\phi_{21}\hat{\varepsilon}_{T-1}\right)^2\right\} +\alpha_2(1-\alpha_2)\phi_{21}^2\hat{\varepsilon}_T^2$$

$$\begin{aligned}\sigma_{T+2}^2 &= \alpha_2^2\phi_{21}^2\sigma_{T+1}^2 + \alpha_1 \left\{\beta_{10} + \beta_{11} \left(\sigma_{T+1}^2 + \hat{\varepsilon}_{T+1|T}^2\right)\right\} \\ &+ \alpha_2 \left[\beta_{20}+\beta_{21} \left\{ \left(\sigma_{T+1}^2+\hat{\varepsilon}_{T+1|T}^2\right) +\phi_{21}^2\hat{\varepsilon}_T^2-2\phi_{21}\hat{\varepsilon}_{T+1|T}\ \hat{\varepsilon}_T\right\}\right] \\ &+ \alpha_2\phi_{21}^2 \left(1-\alpha_2\right) \left(\sigma_{T+1}^2 + \hat{\varepsilon}_{T+1|T}^2\right)\end{aligned}$$

$$\begin{aligned}\sigma_{T+3}^2 &= \alpha_2^2\phi_{21}^2\sigma_{T+2}^2 + \alpha_1 \left\{\beta_{10} + \beta_{11} \left(\sigma_{T+2}^2 + \hat{\varepsilon}_{T+2|T}^2\right)\right\} \\ &+ \alpha_2 \left[\beta_{20}+\beta_{21} \left\{ \left(\sigma_{T+2}^2+\hat{\varepsilon}_{T+2|T}^2\right) +\phi_{21}^2 \left(\sigma_{T+1}^2 + \hat{\varepsilon}_{T+1|T}^2\right)\right.\right. \\ &\left.\left. -2\alpha_2\phi_{21}^2\hat{\varepsilon}_{T+1|T}^2\right\}\right] + \alpha_2\phi_{21}^2 \left(1-\alpha_2\right) \left(\sigma_{T+2}^2 + \hat{\varepsilon}_{T+2|T}^2\right).\end{aligned}$$

For out-of-sample forecasting of weekly onion price data, parameters of all the three mixture nonlinear time-series models are taken as fixed. Computed values of best predictor and prediction error variance for 2nd, 3rd and 4th weeks of November, 2001 based on GMTD, MAR and MAR-ARCH models along with actual values are reported in Table 2. A perusal indicates that, in case of interval forecasts, for both MAR and MAR-ARCH models, all the forecast values lie within one standard error of forecasts. However, forecast values for MAR-ARCH model are much closer to true values than those for MAR model. Further, performance of GMTD model in out-of-sample forecasting is found to be inferior to other two models.

TABLE 2. OUT-OF SAMPLE FORECASTING OF WEEKLY ONION PRICE DATA (RS. PER QUINTAL)

Week/Month/Year	Actual	Forecast value by		
	Value	GMTD	MAR	MAR-ARCH
2nd/November/2001	1050	1137.69 (55.90)	1039.80 (120.21)	1051.71 (108.27)
3rd/November/2001	1180	1052.09 (87.14)	1117.93 (125.70)	1147.78 (115.30)
4th/November/2001	825	1193.33 (98.35)	905.19 (103.59)	870.01 94.82)

Note: Figures with brackets ( ) indicate corresponding standard errors of forecasts.

5.2. Predictive density. Here, predictive density for the three mixture nonlinear time-series models, viz. GMTD, MAR, and, MAR-ARCH are studied. This gives a much more comprehensive description of the underlying phenomenon as compared to mean and variance of future observations discussed earlier. However, to save space, we give below details only about MAR-ARCH model.

One-step ahead predictive density of $\hat{\varepsilon}_{T+1}$ given $\hat{\varepsilon}_T, \hat{\varepsilon}_{T-1}, \ldots$ is a mixture of two Gaussian distributions with mixture coefficients 0.75 and 0.25. Means and variances of individual components are $(0, \beta_{10}+\beta_{11}\hat{\varepsilon}_T^2)$ and $(\phi_{21}\hat{\varepsilon}_T, \beta_{20}+\beta_{21}(\hat{\varepsilon}_T-\hat{\phi}_{21}\hat{\varepsilon}_{T-1})^2)$ respectively. Further, to study two-step ahead predictive density of $\hat{\varepsilon}_{T+2}$ given $\hat{\varepsilon}_T, \hat{\varepsilon}_{T-1}, \ldots$, we follow the naive approach given in Wong and Li (2001) for computing within sample predictive density. Since means and variances of mixture components are functions of unobserved $\hat{\varepsilon}_{T+1}$, these are estimated by their conditional expectations given $\hat{\varepsilon}_T, \hat{\varepsilon}_{T-1}, \ldots$ Finally, means and variances of individual components are obtained as:

$$\left(0, \beta_{10} + \beta_{11}\left(\sigma_{T+1}^2 + \hat{\varepsilon}_{T+1|T}^2\right)\right)$$

and

$$\left(\phi_{21}\hat{\varepsilon}_{T+1|T}, \beta_{20} + \beta_{21}\left\{\left(\sigma_{T+1}^2 + \hat{\varepsilon}_{T+1|T}^2\right) + \phi_{21}^2\hat{\varepsilon}_T^2 - 2\phi_{21}\hat{\varepsilon}_T\hat{\varepsilon}_{T+1|T}\right\}\right).$$

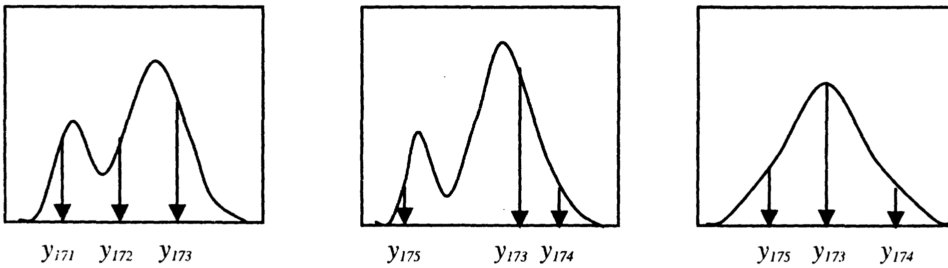
Proceeding along similar lines, means and variances of individual components of three-step ahead predictive density of $\hat{\varepsilon}_{T+3}$ given $\hat{\varepsilon}_T, \hat{\varepsilon}_{T-1}, \ldots$, are

$$\left(0, \beta_{10} + \beta_{11}\left(\sigma_{T+2}^2 + \hat{\varepsilon}_{T+2|T}^2\right)\right)$$

and

$$\left(\phi_{21}\hat{\varepsilon}_{T+2|T}, \beta_{20} + \beta_{21}\left\{\left(\sigma_{T+2}^2 + \hat{\varepsilon}_{T+2|T}^2\right) + \phi_{21}^2\left(\sigma_{T+1}^2 + \hat{\varepsilon}_{T+1|T}^2\right)\left(1 - 2\alpha_2\right)\right\}\right).$$

Predictive densities are computed through simulation based on 200 realizations and using standard uniform and standard normal variates for selecting mixture and drawing actual observations from corresponding mixture distribution and the same are exhibited in Figs. 3(a)-3(c). A perusal indicates presence of high volatility at future time-epoches 173 and 174 leading to bimodality in one- and two-step ahead predictive densities. High volatility at these time-epoches is also reflected by large one and two-step ahead prediction error variances given in last column of Table 2. Further, from Fig. 3(c), volatility for three-step ahead predictive density is not very prominent



(a) (b) (c)

Figure 3: One, two and three-step ahead predictive densities at time-epoch 172

as it is unimodal, which is also reflected by a corresponding smaller value of prediction error variance, as mentioned in last column of Table 2.

To sum up, two-component MAR-ARCH model is found to be best for modelling and forecasting of weekly wholesale onion price data.

6 Conclusion

In this paper, we consider mixture nonlinear time-series models for analyzing onion price data. Generally, in studying price phenomenon over time, it is required to standardize prices prevailing during various time-epochs with respect to some standard price index. However, in present instance, discussions with subject matter specialists suggested that there was no need to do this. The models studied in this paper are of particular importance in those situations in which the data depicts sudden bursts, flat stretches and outliers. Work is in progress to investigate some other possible extensions of ARCH modelling, like GARCH, EGARCH and TARCH, in the framework of mixture distribution and shall be reported separately in due course of time.

Acknowledgement. Authors are grateful to the referee for valuable comments.

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Paper received July 2004; revised January 2006.