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Kalman filter-based modelling and forecasting of stochastic volatility with threshold

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We propose a parametric nonlinear time-series model, namely the Autoregressive-Stochastic volatility with threshold (AR-SVT) model with mean equation for forecasting level and volatility. Methodology for estimation of parameters of this model is developed by first obtaining recursive Kalman filter time-update equation and then employing the unrestricted quasi-maximum likelihood method. Furthermore, optimal one-step and two-step-ahead out-of-sample forecasts formulae along with forecast error variances are derived analytically by recursive use of conditional expectation and variance. As an illustration, volatile all-India monthly spices export during the period January 2006 to January 2012 is considered. Entire data analysis is carried out using EViews and matrix laboratory (MATLAB) software packages. The AR-SVT model is fitted and interval forecasts for 10 hold-out data points are obtained. Superiority of this model for describing and forecasting over other competing models for volatility, namely AR-Generalized autoregressive conditional heteroscedastic, AR-Exponential GARCH, AR-Threshold GARCH, and AR-Stochastic volatility models is shown for the data under consideration. Finally, for the AR-SVT model, optimal out-of-sample forecasts along with forecasts of one-step-ahead variances are obtained.

Keywords: AR-SV model; AR-SVT model; asymmetric volatility; Kalman filter; optimal out-of-sample forecasts; UQML method

1. Introduction

Linear Gaussian models [5] are not able to capture the realistic feature of changing conditional variance due to heteroscedastic errors. To handle such a situation, the Autoregressive conditional heteroscedastic (ARCH) nonlinear time-series model was introduced [8], in which squared residual series is significantly autocorrelated. The zero conditional mean process $\{\varepsilon_t\}$ is said to follow ARCH(q), if conditional distribution of $\{\varepsilon_t\}$ given available information $\psi_{t-1} = \{\varepsilon_s,\ s \le t-1\}$ is

$$\varepsilon_t|\psi_{t-1} \sim f\,(0, h_t)\,,$$

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where $f(\cdot)$ is the probability density function of a zero-mean random variable and h_t satisfies the variance equation given by

$$h_t = a_0 + \sum_{i=1}^{q} a_i \varepsilon_{t-i}^2,$$

$a_0 > 0,\ a_i \geq 0$ for all i and $\sum_{i=1}^{q} a_i < 1$. Then, the process $\{x_t\}$ defined by

$$x_t = \varphi_0 + \varphi_1 x_{t-1} + \cdots + \varphi_p x_{t-m} + \varepsilon_t$$

is said to follow AR(m)-ARCH(q) model if the residual series $\{\varepsilon_t\}$ is ARCH(q). In [9], this model was applied to study volatility present in some onion prices data. The fitted model provided a good description of underlying mechanism in terms of significant ARCH parameters due to squared residual series and exogenous variable as well as changing forecast intervals for hold-out data. The AR-ARCH model has also been used as basic 'building blocks' for Markov switching and mixture models [14]. However, the ARCH model has the drawback that, when its order is very large, estimation of a large number of parameters is required which reduces their efficiency. Even if q is moderate, variance equation of the ARCH(q) model is only capable of capturing short-range dependence of squared residuals in the form of rapid decay of their autocorrelation functions (ACF).

To overcome above difficulties, the AR-Generalized ARCH (GARCH) model was proposed [4] in which conditional variance of the residual series is a linear function of its own lags, given by

$$\varepsilon_t = \xi_t h_t^{1/2}, \quad h_t = a_0 + \sum_{i=1}^{q} a_i \varepsilon_{t-i}^2 + \sum_{j=1}^{p} b_j h_{t-j},$$

where

$$\xi_t \sim \text{IID}\,(0,1)\,, \quad a_0 > 0, \quad a_i \geq 0,\ i = 1,2,\ldots,q. \quad b_j \geq 0,\ j = 1,2,\ldots,p.$$

It can be shown that the squared residuals of a GARCH model follow the Autoregressive moving average (ARMA) model with parameters (p, q), i.e. the ARMA (p, q) model of the form

$$\varepsilon_t^2 | \psi_{t-1} = a_0 + \sum_{i=1}^{\text{Max}(p,q)} (a_i + b_i)\, \varepsilon_{t-i}^2 + \sum_{j=i}^{p} b_j \nu_{t-j} + \nu_t,$$

where $a_i = 0$ if $i > q$, $b_j = 0$ if $j > p$ and $\nu_t = \varepsilon_t^2 | \psi_{t-1} - h_t$. Generally, in the AR($m$)-GARCH ($p$, q) model, values of p and q are considered as unity. In [1], performance of the GARCH model was evaluated for modelling daily value-at-risk (VaR) of perfectly distributed portfolios in five stock indices using a number of distributional assumptions and sample sizes. However, a serious limitation of this model is that the residual series cannot capture in an appropriate way the asymmetric effects of past on future volatility, also known as 'leverage effects' in financial literature.

Accordingly, several extensions in the variance equations for modelling heteroscedastic residual series, such as Exponential GARCH (EGARCH) and Threshold GARCH (TGARCH), have been proposed in the literature. The variance equation of residual series of the AR(m)-EGARCH(p, q) model is given by

$$\varepsilon_t = \xi_t \sigma_t = \xi_t \exp\left(\frac{h_t}{2}\right), \quad h_t = a_0 + \sum_{i=1}^{q} \frac{a_i\left(|\varepsilon_{t-i}| + \gamma_i \varepsilon_{t-i}\right)}{\sigma_{t-i}} + \sum_{j=1}^{p} b_j h_{t-j}.$$

Note that when ε_{t-i} is positive or there is 'good news', total effect of ε_{t-i} is $(1+\gamma_i)\,|\varepsilon_{t-i}|$, whereas when ε_{t-i} is negative or there is 'bad news', total effect of ε_{t-i} is $(1-\gamma_i)\,|\varepsilon_{t-i}|$. As γ_i is expected to be negative, ‘bad news’ would have a larger impact on volatility. Another advantage of this model over the GARCH model is that the conditional variance σ_t^2 here is guaranteed to be positive regardless of the values of the coefficients because log $\left(\sigma_t^2\right)$ rather than σ_t^2 is modelled. Another variant of the GARCH model, which is capable of modelling leverage effect of heteroscedastic errors is the AR(m)-TGARCH(p, q) model, whose variance equation is given by Goudarzi and Ramanarayanan [11]:

$$\varepsilon_t = \xi_t h_t^{1/2}, \quad h_t = a_0 + \sum_{i=1}^{q} a_i \varepsilon_{t-i}^2 + \sum_{i=1}^{q} \gamma_i s_{t-i}^* \varepsilon_{t-i}^2 + \sum_{j=1}^{p} b_j h_{t-j},$$

where

$$s_t^* = \begin{cases} 1 & \text{if } \varepsilon_t < 0, \\ 0 & \text{otherwise.} \end{cases}$$

That is, depending on whether ε_{t-i} is above or below the threshold value of zero, ε_{t-i}^2 has different effects on the conditional variance h_t. Here also, γ_i is expected to be negative for ‘bad news’ in order to have larger impact. This model is also known as the GJR-GARCH model as Glosten, Jagannathan, and Runkle proposed essentially the same model. The EGARCH and TGARCH models have been applied to describe asymmetric volatility in the Indian stock market [11]. Furthermore, the GJR-GARCH model was employed for modelling asymmetry and persistence under impact of sudden changes in volatility of the Indian stock market [13].

In [18], an alternate family of models for describing heteroscedastic errors, called the stochastic volatility (SV) model, was proposed in which the underlying conditional variance is represented by an unobserved stochastic process. A heartening aspect of this model is that it is capable of modelling asymmetric volatility as it can capture one-step-ahead conditional variance based on sign of past observation. However, in the SV model, conditional mean of the series is assumed as zero, which is not realistic. So, an attempt was made in [18] by taking it to be some non-zero known constant δ. Since asymmetric effect of present level of the series on one-step-ahead volatility should be controlled by unknown threshold parameter, therefore for a more realistic regime-specific modelling of asymmetric volatility, δ is unknown and so should be estimated. Furthermore, in all the work related to SV models done so far, the series was not taken as correlated.

Accordingly, in this article, we propose a modification of the SV model by incorporating both the above aspects and the new model will be called the Autoregressive-SV with threshold (AR-SVT) model. Furthermore, methodology for estimation of parameters of the AR-SVT model would be developed by first obtaining recursive Kalman filter (KF) time-update equation, along similar lines as [12]. Subsequently, the unrestricted quasi-maximum likelihood (UQML) method in conjunction with trial and error approach would be adopted for estimation of parameters. Optimal one-step and two-step-ahead out-of-sample forecasts formulae along with forecast error variances would also be derived analytically by recursive use of conditional expectation. Finally, the methodology would be illustrated on volatile all-India monthly spices export data and its superiority over other competing models would be shown.

2. Description of the AR-SVT model

Although the EGARCH model is capable of describing the asymmetric effects of ‘returns’ on volatility in finance market [3], yet this model has the limitation that it is not able to specify the underlying volatility, which depends on continuous information in the past driven by a process separate from ‘returns’ per se. Accordingly, there should be two error processes, which are contemporaneously correlated, but having their marginal distributions as independently and identically distributed. To this end, univariate discrete-time SV model for a zero-mean process $\{y_t\}$ is given by Taylor [18]:

$$y_t = \varepsilon_t \exp\left(\frac{h_t^*}{2}\right), \quad t = 1, \ldots, T, \tag{1}$$

$$h_t^* = \alpha + \varphi h_{t-1}^* + \eta_{t-1}, \tag{2}$$

where ε_t, η_t are symmetrically distributed white noise processes with variances σ_ε^2 and σ_η^2, respectively. The parameter φ measures persistence of shocks to volatility, where $|\varphi| < 1$. When φ is close to unity and σ_η^2 is close to 0, evolution of volatility over time is very smooth. Since ε_t and η_{t-1} are independent, conditional variance of y_t given information up to time $t-1$, namely ψ_{t-1}, depends on past observations by the contemporaneous dependence present in $(\varepsilon_{t-1},\ \eta_{t-1})'$. Using Equations (1) and (2), it may be noted that

$$\begin{aligned}\text{cov}(y_t^2, y_{t-1}) &= E\{\varepsilon_t^2 \exp(h_t^*)\varepsilon_{t-1}\exp(0.5h_{t-1}^*)\}\\ &= E[\exp\{\alpha + (\varphi + 0.5)h_{t-1}^*\}E\{\varepsilon_{t-1}\exp(\eta_{t-1})\}],\end{aligned}$$

which is not zero due to dependence between ε_{t-1} and η_{t-1}. Therefore, the SV model is also capable of modelling volatility y_t^2 with respect to 'return' y_{t-1}. It may be noted that Equation (2) may be rewritten as $\left(h_{t+1}^* - \alpha^*\right) = \varphi\left(h_t^* - \alpha^*\right) + \eta_t$, where $\alpha^* = \alpha/(1-\varphi)$, which leads to rescaled SV model given by

$$y_t = \sigma_* \exp\left(\frac{h_t}{2}\right)\varepsilon_t, \tag{3}$$

$$h_{t+1} = \varphi h_t + \eta_t, \tag{4}$$

where $\sigma_* = \exp(\alpha^*/2)$ and $h_t = h_t^* - \alpha^*$. Therefore, estimate of σ_*^2 may be used to estimate α^*, which gives estimate of α in Equation (2). A good description of SV models is given in [2,6].

It may be pointed out that the SV model has generally been applied without modelling its conditional mean equation. Furthermore, it has been highlighted in [16] that differencing does not necessarily transform non-stationary data into an independent process. Therefore, it is desirable to consider the Autoregressive-SV (AR-SV) model given by

$$\Delta x_t - \rho\Delta x_{t-1} = y_t, \tag{5}$$

where

$$y_t = \sigma_* \exp\left(\frac{h_t}{2}\right)\varepsilon_t, \tag{6}$$

$$h_{t+1} = \varphi h_t + \eta_t. \tag{7}$$

In [12], mean of y_t in Equations (3) and (4) has been taken to be a known constant, say δ. However, this is not appropriate as δ should be looked upon as a threshold value of the point of asymmetry due to volatility and therefore estimated from the data. Thus, we propose an extension of the above AR-SV model by modifying Equation (6) as

$$y_t - \delta = \sigma_* \exp\left(\frac{h_t}{2}\right)\varepsilon_t. \tag{8}$$

The model given by Equations (5), (7), and (8) will be called as the AR-SVT model.

When ε_t is symmetrically distributed around zero, ACF of squared residuals y_t^2 of the AR-SV model [10] is given by

$$\rho_\tau^{(2)} = \rho_{h,\tau} \frac{\exp\left(\sigma_h^2\right) - 1}{\kappa_2 \exp\left(\sigma_h^2\right) - 1}, \quad \tau \geq 1, \tag{9}$$

where $\sigma_h^2 = \sigma_\eta^2 / \left(1 - \varphi^2\right)$, $\rho_{h,\tau} = \varphi^\tau$, and κ_2 is kurtosis of ε_t. Evidently, Equation (9) also holds for squared residuals $(y_t - \delta)^2$ of the AR-SVT model.

3. Estimation of parameters of the AR-SVT model

We proceed along similar lines as [12]. However, it may be pointed out that there is an error in Equation (7) of Harvey and Shephard [12]. Furthermore, there is also an error in deriving the expression for filtered log-volatility estimate $\hat{h}_{t+1|t}$ in Section 2.2.1 of Harvey and Shephard [12]. Therefore, for the benefit of readers, brief details of the methodology are given below.

A straightforward algebra using Equations (5), (7), and (8) gives the conditional linear state space form with uncorrelated measurement and transition equation errors for the AR-SVT model as

$$Z_t = \log\left(y_t - \delta\right)^2 = \omega + h_t + \xi_t, \tag{10}$$

$$h_{t+1} = \left(\varphi - \frac{\gamma^* s_t}{\sigma_\xi^2}\right) h_t + s_t \left[\mu^* + \frac{\gamma^* \left\{\log(y_t - \delta)^2 - \omega\right\}}{\sigma_\xi^2}\right] + \eta_t^+, \tag{11}$$

$$\begin{pmatrix} \xi_t \\ \eta_t^+ \end{pmatrix} | s_t \sim \text{ID}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_\xi^2 & 0 \\ 0 & \sigma_1^2 I_t + \sigma_2^2 (1 - I_t) - \gamma^{*2}/\sigma_\xi^2 \end{pmatrix}\right), \tag{12}$$

where

$$\omega = \log \sigma_*^2 + E\left\{\log\left(\varepsilon_t^2\right)\right\}, \quad \mu^* = E_+\left(\eta_t\right) = E\left(\eta_t | s_t = 1\right) = -E_-\left(\eta_t\right),$$

$$\xi_t = \log\left(\varepsilon_t^2\right) - E\left\{\log\left(\varepsilon_t^2\right)\right\}, \quad \gamma^* = \text{Cov}_+\left(\eta_t, \xi_t\right) = -\text{Cov}_-\left(\eta_t, \xi_t\right),$$

$$\sigma_\eta^2 = 0.5\left(\sigma_1^2 + \sigma_2^2\right) + \mu^{*2}, \quad s_t = \begin{cases} -1 & \text{if } y_t - \delta < 0, \\ +1 & \text{otherwise.} \end{cases}$$

Notice that Equation (12) is not the same as Equation (7) of Harvey and Shephard [12]. This is due to the fact that the authors had inadvertently considered the unconditional variance of η_t^+ instead of its conditional variance.

Referring to Equations (5)–(8) and (10)–(12), it may be pointed out that the number of parameters to be estimated in our model is 11, namely $\sigma_*^2, \varphi, \sigma_\varepsilon^2, \rho, \delta, \omega, \mu^*, \gamma^*, \sigma_\xi^2, \sigma_1^2, \sigma_2^2$. All of them, except σ_*^2 and σ_ε^2, can be estimated using the UQML method, along similar lines as [12]. To this end, recursive KF equation of an unobserved log-volatility process h_t may be obtained as follows [7]: Let $Z_t = \log\left(y_t - \delta\right)^2$ and let ψ_t^{sq} be the information set of squared observations up to time epoch t. Initialize $E\left[h_0 | \psi_{-1}^{\text{sq}}, s_0\right] = 0$ and

$$\text{Var}\left\{h_0 | \left(\psi_{-1}^{\text{sq}}, s_0\right)\right\} = \frac{\sigma_\eta^2}{1 - \varphi^2} = p_{0|-1}. \tag{13}$$

Then, using the fact that ξ_0 and s_0 are independently distributed in Equation (12), best predictor of Z_0 conditional on $\left(\psi_{-1}^{\text{sq}}, s_0\right)$, denoted by $\hat{Z}_{0|\psi_{-1}^{\text{sq}}, s_0}$, is

$$\hat{Z}_{0|\psi_{-1}^{\text{sq}}, s_0} = E\left[Z_0 | \left(\psi_{-1}^{\text{sq}}, s_0\right)\right] = E\left[\omega + h_0 + \xi_0 | \left(\psi_{-1}^{\text{sq}}, s_0\right)\right] = \omega. \tag{14}$$

Therefore, joint distribution of $(h_0,\ Z_0)$ has mean $(0, \omega)$, where

$$q_{0(\psi_{-1}^{\text{sq}}, s_0)} = \text{Var}\left[Z_0 |\left(\psi_{-1}^{\text{sq}}, s_0\right)\right] = p_{0|-1} + \sigma_\xi^2, \tag{15}$$

$$\text{Cov}\left\{h_0,\ Z_0 |\left(\psi_{-1}^{\text{sq}}, s_0\right)\right\} = \text{Var}\left\{h_0 |\left(\psi_{-1}^{\text{sq}}, s_0\right)\right\} = p_{0|-1}. \tag{16}$$

Using Equations (13)–(16), given $\left(\psi_0^{\text{sq}}, s_0\right)$, best linear predictor of h_0 which reduces to quasi-conditional mean of h_0 is

$$\widehat{h}_{0|0} = \frac{p_{0|-1}}{p_{0|-1} + \sigma_\xi^2} (Z_0 - \omega) \tag{17}$$

and the corresponding prediction error variance of h_0 is

$$p_{0|0} = p_{0|-1} - \frac{p_{0|-1}^2}{p_{0|-1} + \sigma_\xi^2}. \tag{18}$$

Now, given $\left(\psi_0^{\text{sq}}, s_0\right)$, using Equations (17) and (18), and observing independence of η_0^+ and ξ_0 except on s_0 in Equation (12), the best linear predictor of h_1 is

$$\begin{aligned} \widehat{h}_{1|0} = E\left[h_1|\psi_0\right] &= E\left[\left(\varphi - \frac{\gamma^* s_0}{\sigma_\xi^2}\right) h_0 + s_0 \left\{\mu^* + \left(\frac{\gamma^*}{\sigma_\xi^2}\right)(Z_0 - \omega)\right\} + \eta_0^+ |\psi_0\right] \\ &= \left(\varphi - \frac{\gamma^* s_0}{\sigma_\xi^2}\right) h_{0|0} + s_0 \left\{\mu^* + \left(\frac{\gamma^*}{\sigma_\xi^2}\right)(Z_0 - \omega)\right\} \end{aligned} \tag{19}$$

and the corresponding error variance of h_1 is

$$p_{1|0} = p_{0|0}\left(\varphi - \frac{\gamma^* s_0}{\sigma_\xi^2}\right)^2 + \sigma_1^2 I_0 + \sigma_2^2 (1 - I_0) - \left(\frac{\gamma^{*2}}{\sigma_\xi^2}\right). \tag{20}$$

Using Equations (19) and (20), the best predictor of Z_1 conditional on $\left(\psi_0^{\text{sq}},\ s_1\right)$, denoted by $\hat{Z}_{0|(\psi_0^{\text{sq}},\ s_1)}$, is

$$\hat{Z}_{1|\psi_0^{\text{sq}},\ s_1} = \omega + \hat{h}_{1|0}, \tag{21}$$

where the error variance of Z_1 is given by

$$q_{1|(\psi_0^{\text{sq}},\ s_1)} = p_{1|0} + \sigma_\xi^2. \tag{22}$$

Now, given $\left(\psi_1^{\text{sq}}, s_1\right)$, to obtain measurement equation $\widehat{h}_{1|1}$ and $p_{1|1}$ of KF, note that covariance between h_1 and Z_1 is obtained as $p_{1|0}$. Therefore, from Equations (19)–(22), the best linear predictor of h_1 which reduces to quasi-conditional mean of h_1 is

$$\widehat{h}_{1|1} = \hat{h}_{1|0} + \frac{p_{1|0}}{p_{1|0} + \sigma_\xi^2}\left\{Z_1 - \omega - \hat{h}_{1|0}\right\}, \tag{23}$$

$$p_{1|1} = p_{1|0} - \frac{p_{1|0}^2}{p_{1|0} + \sigma_\xi^2}. \tag{24}$$

Using Equations (23) and (24), given $\left(\psi_1^{\text{sq}}, s_1\right)$, a straightforward but lengthy algebra yields the best linear predictor of h_2 given $\psi_1$as

$$\hat{h}_{2|1} = E\left[\left(\varphi - \frac{\gamma^* s_1}{\sigma_\xi^2}\right) h_1 + s_1 \left\{\mu^* + \frac{\gamma^*}{\sigma_\xi^2 \left(Z_1 - \omega\right)}\right\} + \eta_1^+ | \psi_1^{\text{sq}}\right]$$
$$= \left(\varphi - \frac{\gamma^* s_1}{\sigma_\xi^2}\right)\left(\frac{\sigma_\xi^2}{p_{1|0} + \sigma_\xi^2}\right)\hat{h}_{1|0} + \frac{Z_1 - \omega}{p_{1|0} + \sigma_\xi^2}\left\{\gamma^* s_1 + \varphi p_{1|0}\right\} + s_1 \mu^*.$$

In general, filtered estimate of log-volatility, $\hat{h}_{t+1|t}$, is given by the time-update equation as

$$\widehat{h}_{t+1|t} = \left(p_{t|t-1} + \sigma_\xi^2\right)^{-1}\left[\varphi\{\sigma_\xi^2 \hat{h}_{t|t-1} + p_{t|t-1}\left(Z_t - \omega\right)\} + \left(Z_t - \omega - \hat{h}_{t|t-1}\right)\gamma^* s_t\right] + s_t \mu^*, \quad (25)$$

where

$$p_{t|t-1} = p_{t-1|t-1}\left(\varphi - \frac{\gamma^* s_0}{\sigma_\xi^2}\right)^2 + \sigma_1^2 I_{t-1} + \sigma_2^2\left(1 - I_{t-1}\right) \; - \left(\frac{\gamma^{*2}}{\sigma_\xi^2}\right),$$
$$p_{t|t} = p_{t|t-1} - \frac{p_{t|t-1}^2}{\left(p_{t|t-1} + \sigma_\xi^2\right)}.$$

It may be noted that the filtered estimate in Equation (25) behaves similarly to that of the EGARCH model. If γ^* is negative, then larger negative value of $y_t - \delta$ (less bad news) imputes smaller sensitivity of $\hat{h}_{t+1|t}$ in the same direction as compared with smaller negative value (more bad news), which will cause larger sensitivity of $\hat{h}_{t+1|t}$ in the opposite direction. Similarly, smaller positive value of $y_t - \delta$ (less good news) imputes smaller sensitivity of $\hat{h}_{t+1|t}$ in the same direction as compared with larger positive value (more good news), which may cause larger sensitivity of $\hat{h}_{t+1|t}$ in the opposite direction.

In order to apply the UQML method for estimation of parameters of AR-SV and AR-SVT models, time-update equation of state given in Equation (25) is used to construct the likelihood function of Z_t. To this end, using Equations (21) and (22), it may be noted that the mean and variance of Z_t are, respectively, $\hat{Z}_{t|\left(\psi_{t-1}^{\text{sq}}, s_t\right)} = \omega + \hat{h}_{t|t-1}$ and $q_{t|\left(\psi_{t-1}^{\text{sq}}, s_t\right)} = p_{t|t-1} + \sigma_\xi^2$, where conditioning set is taken to be $\left(\psi_{t-1}^{\text{sq}}, s_t\right)$ to make inference on $\{y_t\}$. It may be pointed out that all the 11 parameters of our model, except σ_*^2 and σ_ε^2, could be estimated using the UQML method on Equation (25). The parameters σ_*^2 and σ_ε^2 cannot be estimated as these are subsumed in Equation (10). Thus, α also cannot be estimated directly as $\alpha = (1 - \varphi) \ \log\left(\sigma_*^2\right)$. One way out is to employ a trial-and-error approach, where various choices of σ_*^2 in Equation (8) are used to compute the unobserved series ε_t. An estimate of $E\left\{\log\left(\varepsilon_t^2\right)\right\}$ is the observed mean of $\log\left(\varepsilon_t^2\right)$. Then, estimate of σ_*^2 would be that value which matches $\log\left(\sigma_*^2\right) + E\left\{\log\left(\varepsilon_t^2\right)\right\}$ and ω. Note that estimate of α is $\hat{\alpha} = \left(1 - \hat{\varphi}\right)\log\left(\hat{\sigma}_*^2\right)$. Thereafter, using estimate of h_t from Equation (25) and estimate of α^*, i.e. $\hat{\alpha}^* = \log\left(\hat{\sigma}_*^2\right)$, estimate of h_t^* is obtained. This gives the estimated unobserved series ε_t which is used to obtain the kernel estimate of its distribution and its variance σ_ε^2. The goodness of fit of AR-SV and AR-SVT models is examined by computing the Akaike information criterion (AIC) and Bayesian information criterion (BIC) values, where

$$\text{AIC} = L\left(\hat{\theta}\right) + 2\left(\text{Number of parameter} + 1\right), \quad (26)$$

$$\text{BIC} = L\left(\hat{\theta}\right) + \left(\text{Number of parameter} + 1\right)\log\left(T + 1\right), \quad (27)$$

where

$$L\left(\hat{\theta}\right) = 2\sum_{t}\left[\log\left\{\hat{\sigma}_{*}\hat{\sigma}_{\varepsilon}\exp\left(\frac{\hat{h}_{t|t-1}}{2}\right)\right\} + \frac{1}{2}\left(\frac{y_t - \delta}{\hat{\sigma}_{*}\hat{\sigma}_{\varepsilon}\exp\left(\hat{h}_{t|t-1}/2\right)}\right)^2\right].$$

The performance of a fitted model is compared on the basis of one-step-ahead mean square prediction error (MSPE), mean absolute prediction error (MAPE), and relative mean absolute prediction error (RMAPE) criteria, where

$$\text{MSPE} = \frac{1}{N}\sum_{i=0}^{N-1}\left\{Y_{T+i+1} - \hat{Y}_{T+i+1}\right\}^2, \tag{28}$$

$$\text{MAPE} = \frac{1}{N}\sum_{i=0}^{N-1}\left\{\left|Y_{T+i+1} - \hat{Y}_{T+i+1}\right|\right\}, \tag{29}$$

$$\text{RMAPE} = \frac{1}{N}\sum_{i=0}^{N-1}\left\{\frac{\left|Y_{T+i+1} - \hat{Y}_{T+i+1}\right|}{Y_{T+i+1}}\right\} \times 100. \tag{30}$$

4. Development of out-of-sample forecast formulae

We now develop optimal k-step-ahead forecasts of x_t along with their forecast error variances for the AR-SVT model. To this end, optimal one-step-ahead forecast of x_{T+1}, denoted by $\hat{x}_{T+1|T}$, is obtained by taking conditional expectation of x_{T+1} given ψ_T. Note that

$$x_{T+1} = x_T + \Delta x_{T+1}, \tag{31}$$

which implies that

$$\hat{x}_{T+1|T} = x_T + \widehat{\Delta x}_{T+1},$$

where $\widehat{\Delta x}_{T+1}$ is the conditional expectation of Δx_{T+1} given ψ_T. For AR-SVT model of x_t written in terms of y_t in Equation (5), note from Equation (8) that conditional mean of $y_{T+1} - \delta$ is independent of ψ_T, which leads to $\widehat{\Delta x}_{T+1} = \delta + \rho\Delta x_T$. Therefore, using Equation (31), optimal one-step-ahead forecast of x_{T+1} reduces to

$$\widehat{x}_{T+1|T} = \delta + x_T + \rho\Delta x_T. \tag{32}$$

One-step-ahead forecast error variance of x_{T+1} given ψ_T, denoted by $\sigma^2_{T+1|T}$, is its conditional variance. This is obtained by taking repeated expectation on $(y_{T+1} - \delta)^2$ leading to $\sigma_*^2\sigma_\varepsilon^2\ E\left\{\exp\left(h_{T+1}\right)|\psi_T\right\}$, which can be approximated by

$$\sigma^2_{T+1|T} = \sigma_*^2\sigma_\varepsilon^2\exp\left(\hat{h}_{T+1|T}\right). \tag{33}$$

For obtaining optimal two-step-ahead forecast of x_{T+2} given ψ_T, denoted by $\hat{x}_{T+2|T}$, note that

$$\hat{x}_{T+2|T} = E\left\{\left(Ex_{T+2}|\psi_{T+1}\right)|\psi_T\right\}, \tag{34}$$

where $E\left(x_{T+2}|\psi_{T+1}\right)$ may be written in terms of x_{T+1} and Δx_{T+1}. In evaluation of the second stage of conditional expectation in Equation (34), optimal two-step-ahead forecast of x_{T+2}, using

Equations (8) and (32), reduces to

$$\hat{x}_{T+2|T} = \hat{x}_{T+1|T} + (1+\rho)\,\delta + \rho^2 \Delta x_T. \tag{35}$$

Two-step-ahead forecast error variance of x_{T+2} given ψ_T, denoted by $\sigma^2_{T+2|T}$, is its conditional variance, which is derived by repeated expectation and variance approach. Thus

$$\sigma^2_{T+2|T} = \sigma^2_* \sigma^2_\varepsilon E\left\{\exp\left(\hat{h}_{T+2|T+1}\right)|\psi_T\right\} + (1+\rho)^2 \sigma^2_* \sigma^2_\varepsilon \exp\left(\hat{h}_{T+1|T}\right). \tag{36}$$

Now, put $t = T+1$ in Equation (25) and note that $Z_{T+1} - \omega - \hat{h}_{T+1|T} \approx \xi_t$ is independent of s_{T+1}. Also note that the conditional distribution of $(\xi_{T+1}, s_{T+1})'$ is independent of ψ_T due to the fact that distribution of ε_t is symmetric and independently and identically distributed. Furthermore, estimating $Z_{T+1} - \omega$ by $\hat{h}_{T+1|T}$, Equation (36) reduces to

$$\begin{aligned}
\sigma^2_{T+2|T} &= \sigma^2_* \sigma^2_\varepsilon \left[\exp\left\{E\left(\hat{h}_{T+2|T+1}\right)|\psi_T\right\} + (1+\rho)^2 \sigma^2_* \sigma^2_\varepsilon \exp\left(\hat{h}_{t+1|t}\right)\right] \\
&= \sigma^2_* \sigma^2_\varepsilon \left[\exp\left\{\frac{1}{p_{T+1|T} + \sigma^2_\xi}[\varphi\{\sigma^2_\xi \hat{h}_{T+1|T} + p_{T+1|t}\left(\hat{h}_{T+1|T}\right)\}]\right\}\right. \\
&\quad \left. + (1+\rho)^2 \sigma^2_* \sigma^2_\varepsilon \exp\left(\hat{h}_{T+1|T}\right)\right].
\end{aligned} \tag{37}$$

Finally, the model describing log-volatility process $\{h_t\}$ may be used to generate k-step-ahead forecasts of one-step-ahead log-volatility h^*_t in Equations (1) and (2). Note that, here, log-volatility process of the AR-SVT model is taken as logarithm of squared value of $\sigma_* \exp(h_t/2)$, that is, h^*_t, which unlike h_t in Equation (8), follows the non-zero mean process. Therefore, from the autoregressive equation followed by h_t in Equation (7) and the fact that $h_t = h^*_t - \alpha^*$, optimal k-step-ahead forecast of one-step-ahead log-volatility h^*_{T+k} is obtained as

$$\hat{h}^*_{T+k} = \alpha^* + \varphi^k \hat{h}_{T|T-1}. \tag{38}$$

5. An illustration

Spices have great medicinal value and India ranks first in the world in their production, consumption, and export. In [15], the GARCH model was applied for investigating volatility in prices of spices. This model was also applied in [17] for modelling and forecasting of India's volatile spices export data. In the present illustration, all-India data of monthly export of spices during the period January 2006–2012, obtained from Indiastat (www.indiastat.com), are considered. Out of total 73 data points, first 63 data points corresponding to the period January 2006–March 2011 are used for model building and remaining 10 data points, that is, from April 2011 to January 2012, are used for validation purpose. Perusal of the data indicates high volatility in March 2007 when export suddenly jumped almost 140% to the level of Rs. 402 crores (Rs. 1 crore = Rs. 10 million) and then abruptly dipped in the very next month to Rs. 301 crores. Volatility can also be seen in many other time points, such as August 2007, March 2008, October 2009, March 2010, and December 2010.

As the data $\{x_t\}$ indicate the presence of trend, first-order differenced series $\{\Delta x_t\}$ is considered to detrend it and the resultant series is seen to be stationary. In our subsequent data analysis, various models would be fitted to the series $\{\Delta x_t\}$. The EViews, Ver. 5 software package is employed for fitting the ARIMA model as well as the GARCH model and its variants. The only significant autocorrelation is observed at lag one. On the basis of minimum AIC and BIC criteria,

Table 1. Estimates of parameters along with their standard errors for the fitted ARIMA (1,1,0) model to $\{x_t\}$.

Parameter	Estimate	Standard error
Intercept	11.56	6.51
AR1	– 0.25	0.12

Table 2. Specification test of mean and variance equations for the fitted AR-SVT model.

	Standardized residuals			Squared standardized residuals		
Lags	ACF	*Q*-Statistics	Probability	ACF	*Q*-Statistics	Probability
1	0.097	4.760	0.093	– 0.074	2.694	0.101
2	– 0.025	4.805	0.187	0.029	2.752	0.253
3	0.064	5.093	0.278	– 0.007	2.755	0.431
4	0.034	5.178	0.395	– 0.063	3.035	0.552
5	0.158	7.023	0.319	0.028	3.089	0.686
6	0.031	7.094	0.419	0.128	4.268	0.640
7	– 0.115	8.102	0.424	0.172	6.460	0.487
8	– 0.222	11.947	0.216	– 0.231	10.478	0.233
9	0.170	14.252	0.162	– 0.142	12.032	0.212
10	– 0.085	14.843	0.190	0.202	15.229	0.124
11	0.008	14.849	0.250	– 0.061	15.527	0.160
12	– 0.081	15.400	0.283	0.087	16.141	0.185
13	0.189	18.493	0.185	– 0.159	18.246	0.148
14	– 0.023	18.541	0.235	0.042	18.395	0.189
15	– 0.082	19.141	0.261	0.112	19.493	0.192
16	– 0.103	20.120	0.268	– 0.046	19.678	0.235
17	– 0.074	20.634	0.298	– 0.119	20.979	0.227
18	– 0.061	20.995	0.337	0.035	21.094	0.275
19	– 0.113	22.259	0.327	0.021	21.135	0.329
20	– 0.017	22.288	0.383	– 0.057	21.460	0.371

the best selected model is also found to be the ARIMA(1,1,0) model and estimates of parameters along with their standard errors are reported in Table 1.

In order to assess the presence of volatility, in the first instance, ACF of squared residuals of fitted ARIMA(1,1,0) model is computed as -0.22 at lag 6, which is reasonably high. Therefore, proceeding along similar lines as [17], ARCH-Lagrange Multiplier test statistic at lag 6 is computed as 1.90, which is found to be significant at 5% level. This indicates the presence of heteroscedasticity of errors. Therefore, GARCH family of models along with its variants are fitted to the differenced series $\{\Delta x_t\}$. The minimum AIC and BIC values for GARCH, EGARCH, and TGARCH families are reported in Table 2. The best fitted AR(1)-GARCH (1,1) model using Gaussian maximum likelihood estimation procedure is computed as

$$\Delta x_t = 593.75 + 0.91\Delta x_{t-1} + \varepsilon_t,$$
$$h_t = 14683.53 + 0.31\varepsilon_{t-1}^2 - 1.02h_{t-1}.$$

Furthermore, the best fitted AR(1)-EGARCH(1,1) model with t-distributed errors is obtained as

$$\Delta x_t = 689.88 + 0.92\Delta x_{t-1} + \varepsilon_t,$$
$$h_t = 1.79 - 0.58h_{t-1} + \frac{0.33\,(|\varepsilon_{t-1}| + 2.55\varepsilon_{t-1})}{\sigma_{t-1}},$$

while the best fitted AR(1)-TGARCH(1,1) model with normally distributed errors is computed as

$$\Delta x_t = 562.20 + 0.91\Delta x_t + \varepsilon_t,$$
$$h_t = 45102.94 + 0.63\varepsilon_{t-1}^2 - 0.27s_{t-1}^*\varepsilon_{t-1}^2 - 0.92h_{t-1}.$$

We now fit the AR-SV and AR-SVT models to the series $\{\Delta x_t\}$ using MATLAB, Ver. 7.2 software package. To this end, σ_η^2 is estimated from Equation (12) and α^* or equivalently σ_*^2 is estimated by the trial-and-error approach described in Section 3. In case of kernel estimate of error distribution, parameter α in the basic form of the SV model for AR-SV and AR-SVT models is estimated using the relation $\alpha = (1-\varphi)\,\alpha^*$. The fitted AR-SV model given by Equations (1), (2), and (5) is obtained as

$$\Delta x_t - 0.33\Delta x_{t-1} = y_t,$$

where

$$y_t = \varepsilon_t \exp\left(\frac{h_t^*}{2}\right), \quad \hat{\sigma}_\varepsilon^2 = 3734.93,$$
$$h_t^* = -0.05 + 0.49h_{t-1}^* + \eta_{t-1}, \quad \hat{\sigma}_\eta^2 = 0.2209.$$

Furthermore, the fitted AR-SVT model given by Equations (2), (5), and (8) is computed as

$$\Delta x_t - 0.27\Delta x_{t-1} = y_t,$$

where

$$y_t - 36.54 = \varepsilon_t \exp\left(\frac{h_t^*}{2}\right), \quad \hat{\sigma}_\varepsilon^2 = 3450.53,$$
$$h_t^* = 0.155 + 0.097h_{t-1}^* + \eta_{t-1}, \quad \hat{\sigma}_\eta^2 = 1.9729.$$

In order to examine whether the underlying condition for AR-SV and AR-SVT models that the distribution of standardized residuals $\{\varepsilon_t\}$ is symmetric around zero is satisfied, graphs of their kernel density estimates are drawn and are exhibited in Figure 1(a) and 1(b). The former figure indicates that the condition of symmetry is not satisfied for fitted AR-SV model, implying thereby that this model may not be very proper for describing given data. On the other hand, in view of symmetry, Figure 1(b) indicates that the AR-SVT model may be appropriate.

Another desirable feature is that estimate of parameter φ should be close to φ_R, which is the one estimated from filtered value of h_t, given in Equation (7). Under regression set up, the time-update and measurement-update of h_t are respectively considered as dependent and independent variables. Estimate of regression coefficient, φ_R, is computed as 0.093, which is very close to estimate of parameter φ, namely 0.097. Furthermore, using Equation (11), two regime-specific dynamic systems are constructed to obtain regime-specific error variances and are computed as 0.623 and 0.129, which are found to be quite close to the estimated values of $\left(\sigma_1^2 - \gamma^{*2}/\sigma_\xi^2\right)$, that is, $0.71 - (-0.54)^2/4.79$, that is, 0.65 and $\left(\sigma_2^2 - \gamma^{*2}/\sigma_\xi^2\right)$, that is, $0.21 - (-0.54)^2/4.79$, that is, 0.15. All this implies that the UQML method of estimation based on the conditional state space model with conditional transition error variance is appropriate for fitting the AR-SVT model to given data.

Furthermore, in order to assess appropriateness of correct specification of the mean and variance equations of the fitted AR-SVT model, ACF of standardized residuals $\{\varepsilon_t\}$ and squared standardized residuals $\left\{\varepsilon_t^2\right\}$ respectively need to be computed to test the null hypotheses that the series $\{\varepsilon_t\}$ and $\left\{\varepsilon_t^2\right\}$ are independent and are reported in Table 2. Using Box–Ljung statistics based on ACF values at various lags, it is found that, in both situations, null hypotheses of

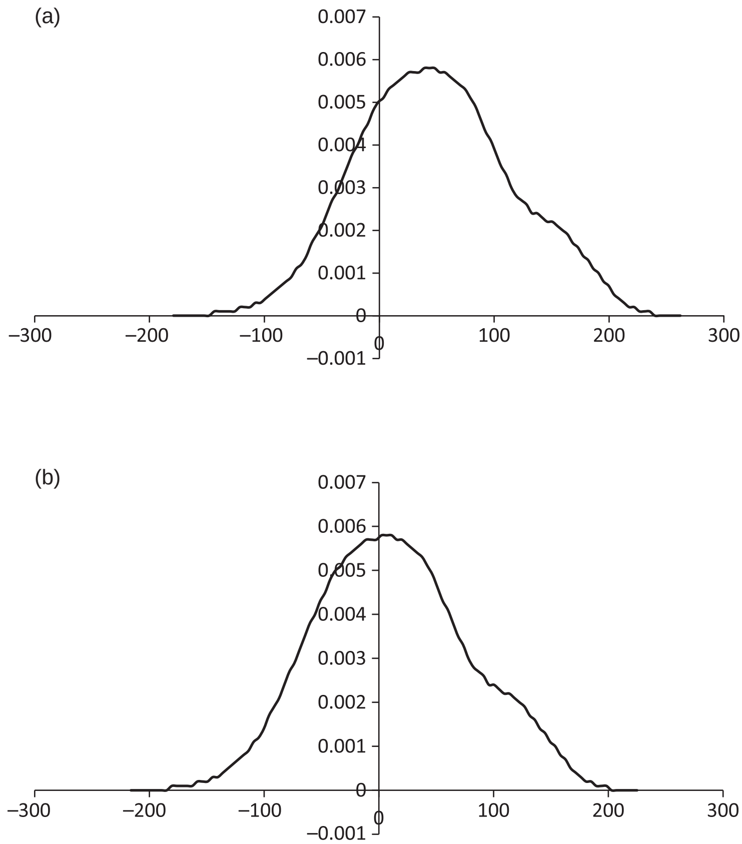

Figure 1. Kernel density of standardized residuals of fitted (a) AR-SV model and AR-SVT model.

Table 3. Goodness of fit of models.

Criterion	Model				
	AR-GARCH	AR-TGARCH	AR-EGARCH	AR-SV	AR-SVT
AIC	536.30	532.33	536.18	522.20	517.21
BIC	547.01	543.05	547.03	526.48	521.50

their independence are not rejected at the 5% level, thereby implying that the mean and variance equations of the fitted AR-SVT model are correctly specified.

The AIC and BIC values for the fitted AR-SV and AR-SVT models are computed using Equations (26) and (27) and are reported in Table 3. A perusal of this table indicates that the AR-SVT model performs best, followed by AR-SV model and then by AR-Threshold GARCH (AR-TGARCH) model so far as modelling of the data is considered. Another heartening aspect of the fitted AR-SVT model is that it is able to identify the underlying correlation structure in a satisfactory manner as shown by the closeness of observed ACF of squared residuals $\{(y_t - \delta)^2\}$ with theoretical autocorrelations computed using Equation (9) and the same are reported in Table 4. This implies that the stochastic variance equation assumed in the AR-SVT model is able to identify volatility present in the data.

Table 4. ACF of squared residuals of the AR-SVT model.

Lags	Empirical	Theoretical
1	0.832	0.831
2	0.737	0.738
3	0.624	0.626
4	0.552	0.551
5	0.499	0.500
6	0.454	0.455
7	0.457	0.459
8	0.428	0.426
9	0.391	0.393
10	0.290	0.293
11	0.230	0.232
12	0.215	0.213
13	0.163	0.165
14	0.157	0.158
15	0.128	0.130
16	0.119	0.121
17	0.057	0.058
18	0.011	0.012
19	– 0.012	– 0.011
20	– 0.017	– 0.016

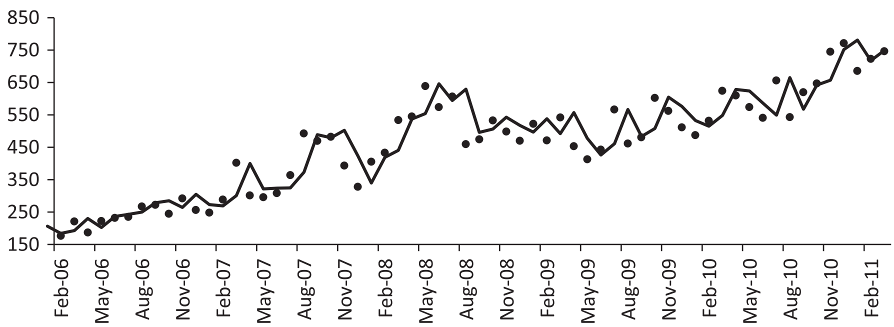

Figure 2. Fitted AR-SVT model along with data points.

In view of all the above, the AR-SVT model is found to be quite appropriate for modelling volatile data under consideration. To get a visual insight, graph of the fitted AR-SVT model along with data points is exhibited in Figure 2.

5.1 *Forecasting performance for hold-out data*

All-India monthly exports of spices from April 2011 to January 2012 are considered as 10 hold-out data points. For all the fitted 05 models, one-step-ahead forecasts along with corresponding forecast standard errors in brackets () are computed and the same are reported in Table 5. It may be pointed out that, for the fitted AR-SVT model, all actual values lie within the prediction intervals corresponding to 'Forecast ± Standard error'. However, this desirable feature does not hold for any of the other four models, namely AR-Generalized autoregressive conditional heteroscedastic (AR-GARCH), AR-Exponential GARCH (AR-EGARCH), AR-TGARCH and AR-SV models. The MAPE, RMSE, and RMAPE criteria are computed for all fitted models using, respectively, Equations (28)–(30), and the results are reported in Table 6. Evidently, the AR-SVT model is found to be the best as all the three criteria have minimum values for

Table 5. One-step-ahead forecasts of spices export data (in Rs. Crores).

Months	Actual	AR-GARCH	AR-EGARCH	AR-TGARCH	AR-SV	AR-SVT
April, 2011	758.45	733.03 (72.47)	742.05 (85.86)	729.31 (164.77)	801.72 (65.94)	762.14 (65.88)
May 2011	890.10	743.83 (118.21)	752.98 (103.04)	740.07 (140.12)	872.83 (67.21)	837.18 (69.37)
June 2011	876.86	863.82 (99.55)	874.13 (104.31)	859.39 (143.75)	903.91 (74.88)	894.93 (94.13)
July 2011	1007.94	851.74 (96.89)	861.94 (123.98)	847.39 (160.66)	998.92 (77.00)	959.97 (59.81)
August 2011	1222.66	971.20 (121.62)	982.57 (124.29)	966.20 (118.41)	1040.95 (84.93)	1116.98 (110.72)
September 2011	1248.52	1166.89 (137.33)	1180.16 (113.40)	1160.81 (118.23)	1225.94 (99.33)	1208.46 (88.99)
October 2011	1266.68	1190.44 (23.03)	1203.96 (123.83)	1184.25 (173.18)	1297.28 (106.30)	1252.94 (70.46)
November 2011	1160.27	1207.00 (137.53)	1220.67 (134.99)	1200.71 (123.92)	1229.62 (109.83)	1198.28 (71.89)
December 2011	1256.98	1110.03 (58.56)	1122.75 (125.65)	1104.27 (171.30)	1179.96 (105.07)	1201.72 (57.34)
January 2012	1071.73	1198.15 (137.76)	1211.75 (127.44)	1191.92 (106.83)	1158.72 (108.84)	1193.57 (129.47)

Table 6. Performance of one-step-ahead forecasts.

	Model				
Criterion	AR-GARCH	AR-EGARCH	AR-TGARCH	AR-SV	AR-SVT
MAPE	116.40	100.80	109.71	114.25	96.60
MSPE	19,607.29	14,790.87	16,909.37	17,405.27	12,914.47
RMAPE	10.55	9.14	9.99	10.42	8.65

this model. This indicates superiority AR-SVT model over the other competing models for forecasting purpose also.

The best-fitted model, namely the AR-SVT model, is now used for generating up to two-step-ahead forecasts of India's spices exports along with forecasts of conditional variances. Using Equations (32) and (33), one-step-ahead out-of-sample forecast for February 2012 and its conditional variance are obtained as Rs. 1185.34 crores and 59.94 (Rs. crores)^2, respectively. Equations (35) and (37) give the two-step-ahead out-of-sample forecast for March 2012 and its conditional variance as Rs. 1201.20 crores and 84.21 (Rs. crores)^2, respectively. Finally, out-of-sample forecasts of log-volatility h_t^* are computed as Rs. 12.54, 1.36, 0.27, 0.17, 0.16 (Rs. crores)^2, respectively. It may be noted that these values are converging to 0.155, which is the estimate of parameter α, implying thereby that the realistic feature that impact of present volatility should decrease with increase in the forecast period is also satisfied.

To sum up, it may be concluded that the AR-SVT model has performed best for modelling as well as forecasting of the volatile data under consideration.

6. Concluding remarks

In this article, univariate SV with threshold (SVT) model was proposed. As an illustration, its superiority over other competing models for describing and forecasting purposes for the data under consideration is clearly demonstrated. As future work, the exact analytical expression for *k*-step-ahead forecast of volatility needs to be derived. Further, attempts may be made to extend the developed methodology for fitting Stochastic volatility model with long-memory. Efforts may be directed towards generalizing these results for multivariate situations. Finally, the possibility of application of particle-filtering approach using Monte Carlo technique may also be explored for parameter estimation.

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