

OPTIMAL BLOCK DESIGNS WITH UNEQUAL BLOCK SIZES FOR MAKING TEST TREATMENTS-CONTROL COMPARISONS UNDER A HETEROSCEDASTIC MODEL*

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SUMMARY. This article obtains some sufficient conditions to establish the A -optimality of block designs with unequal block sizes for making test treatments control comparisons under a suitable heteroscedastic model. Balanced test treatments incomplete block designs with unequal block sizes of type G are defined. Some general methods of constructing these designs are given and their A -efficiencies are investigated. These designs have got high A -efficiencies. Some of these designs are also A -optimal.

1. INTRODUCTION

Suppose an experimenter desires to compare $v(\geq 3)$ test treatments with a control treatment. The $(v + 1)$ treatments are indexed as $0, 1, 2, \dots, v$ with 0 denoting the control treatment and $1, 2, \dots, v$ the test treatments. The treatments are applied to n plots, arranged in b blocks, where each of the b_h blocks contains k_h plots, $1 \leq h \leq p$, $\sum_{h=1}^p b_h = b$, $\sum_{h=1}^p b_h k_h = n$. It is also assumed that $k_h \leq v$ and that only one treatment is applied to each plot. The i -th treatment is applied in n_{ihm} plots in the m -th block out of b_h blocks of size k_h each, $1 \leq m \leq b_h$; $0 \leq i \leq v$; $1 \leq h \leq p$: $\sum_{i=0}^v n_{ihm} = k_h$, for all h and m .

An observation Y_{ihmu} is to be taken on the treatment i in the plot u located in the block m of size k_h , where treatment i is the treatment applied in this plot, $1 \leq u \leq n_{ihm}$. We consider modelling Y_{ihmu} by two-way classified, additive, fixed effects, heteroscedastic and linear model,

$$Y_{ihmu} = \mu + t_i + \beta_{hm} + e_{ihmu}, \quad \dots (1.1)$$

Paper received. February 1992; revised May 1993.

AMS (1980) subject classification. Primary 62K05, secondary 62K10.

Key words and phrases. One way elimination of heterogeneity, BIB design, PBIB design, BTIB design, A -optimality, generalised least squares.

Research is part of a Ph.D. thesis of I.A.R.I., New Delhi - 110012.

where μ is the overall mean, t_i the effect of the treatment i , β_{hm} is the effect of the block m of size k_h and e_{ihmu} is the random error present in Y_{ihmu} . We assume e_{ihmu} 's to be uncorrelated random variables with mean 0 and variance-covariance structure as,

$$\begin{aligned} \text{cov}(e_{ihmu}, e_{i'h'm'u'}) &= \sigma^2 k_h^\alpha, \text{ for } i = i'; m = m'; h = h'; u = u' \\ &= 0, \quad \text{otherwise.} \end{aligned}$$

Here $\alpha \in (0, \infty]$. This model is infact a generalisation of Fairfield Smith's Variance Law [see e.g. Sardana *et al* (1967); Bist *et al* (1975); Handa *et al* (1982)]. In these investigations it was found that the intra block variances are proportional to non-negative real power of block sizes. The value of α was estimated by making use of uniformity trial data. In the present study it is assumed that α is a known real number. This model has earlier been studied by Das *et al* (1992). A special case of $\alpha = 1$ has been investigated by Lee and Jacroux (1987) and Gupta *et al* (1991). For $\alpha = 0$ we get the usual homoscedastic model. Associated with a design d is an $(v+1) \times b$ incidence matrix $\mathbf{N}_d^* = ((n_{dihm})) = [\mathbf{n}_d \mathbf{N}'_d]'$, where $\mathbf{n}'_d = (\mathbf{n}'_{d01}, \dots, \mathbf{n}'_{doh}, \dots, \mathbf{n}'_{dop})$, \mathbf{n}_{doh} being the $b_h \times 1$ vector of the integers n_{dohm} and $\mathbf{N}_d = [\mathbf{N}_{d1}, \dots, \mathbf{N}_{dh}, \dots, \mathbf{N}_{dp}]$, \mathbf{N}_{dh} being the $v \times b_h$ incidence matrix of v test treatments versus b_h blocks of size k_h each. The row sums of \mathbf{N}_d^* are (r_{d0}, \mathbf{r}'_d) , where r_{d0} is the replication of the control treatment and $\mathbf{r}'_d = (r_{d1}, \dots, r_{dv})$, are the replications of the test treatments. The column sums of \mathbf{N}_d^* are $\mathbf{k}' = (k_1 1'_{b_1}, \dots, k_p 1'_{b_p})$ and $\sum_{i=0}^v r_{di} =$

$$\sum_{h=1}^p b_h k_h = n.$$

In this paper, we focus our attention only in the contrasts of the test treatments and the control, i.e., contrasts of the form $\mathbf{P}'\mathbf{t}$, where $\mathbf{P}' = [\mathbf{1}_v : -\mathbf{I}_v]$; $\mathbf{t}' = (t_0, t_1, \dots, t_v)$, $\mathbf{1}_t$ a $t \times 1$ vector of ones and \mathbf{I}_t an identity matrix of order t . We study the $(v+1) \times (v+1)$ information matrix

$$\mathbf{C}_d = \begin{bmatrix} e_d & \mathbf{B}_d \\ \mathbf{B}'_d & \mathbf{M}_d \end{bmatrix} \quad \dots (1.2)$$

obtained by using the principle of generalised least squares. It is well known that the matrix \mathbf{C}_d is symmetric, positive semidefinite with zero row sums and $\text{rank}(\mathbf{C}_d) = v$, because we shall study connected designs. Here

$$\mathbf{M}_d = \sum_{h=1}^p k_h^{-\alpha} \sum_{m=1}^{b_h} [\mathbf{R}_{dhm} - k_h^{-1} \mathbf{N}_{dhm} \mathbf{N}'_{dhm}]; \mathbf{B}_d = - \sum_{h=1}^p k_h^{-\alpha-1} \sum_{m=1}^{b_h} n_{dohm} \mathbf{N}'_{dhm}$$

and $e_d = \sum_{h=1}^p k_h^{-\alpha} \sum_{m=1}^{b_h} (n_{dohm} - k_h^{-1} n_{dohm}^2)$. Also $\mathbf{R}_{dhm} = \text{diag}(n_{d1hm}, \dots, n_{dvhm})$.

Remark 1.1. It is interesting to note that a design connected under a homoscedastic model is also connected under the setting (1.1), and hence the connectedness property does not depend on α .

The covariance matrix of the BLUE $\mathbf{P}'\hat{\mathbf{t}}$ of $\mathbf{P}'\mathbf{t}$ is

$$\sigma^{-2}var(\mathbf{P}'\hat{\mathbf{t}}) = \mathbf{P}'\mathbf{C}_d^{-1}\mathbf{P} = \mathbf{M}_d^{-1}, \quad \dots(1.4)$$

and the, average variance of $\mathbf{P}'\hat{\mathbf{t}}$ is proportional to the trace (\mathbf{M}_d^{-1}). Let $\mathcal{D} = \mathcal{D}(v, b_1, \dots, b_p, k_1, \dots, k_p, \alpha)$ denote the class of all connected designs in v test treatments and a control, with n experimental units arranged in b blocks such that b_h blocks contains k_h experimental units each, $\sum_{h=1}^p b_h = b, \sum_{h=1}^p b_h k_h = n$ and given α , a non-negative real number. We shall use $\mathbf{D} = \mathbf{D}(v, b_1, \dots, b_p, k_1, \dots, k_p, \alpha)$ to denote the subclass of all designs in \mathcal{D} which are binary in test treatments. In this paper we shall obtain some results to find designs which minimize trace (\mathbf{M}_d^{-1}) over \mathcal{D} or \mathbf{D} . Such designs are called A -optimal designs. The results obtained in this paper are of general nature as p may vary from 1 to b . The results for $\alpha = 0$ and $p = 1$ are same as Theorem 2.1 of Hedayat and Majumdar (1984) and $p = b$, implies that all the blocks are of different sizes. The results will be used to find out A -optimal designs in the class of connected designs with two distinct block sizes i.e., $\mathbf{D}(v, b_1, b_2, k_1, k_2, \alpha)$ or $\mathbf{D}(v, b, k_1, k_2, n, \alpha)$.

The problem of studying the optimality of designs for test treatment-control comparisons under model (1.1) for $\alpha = 0$ and $p = 1$ is not new. For an excellent review on this topic reference may be made to Hedayat, Jacroux and Majumdar (1988). The class of designs studied here is $\mathbf{D}(v, b, k)$, where $\mathbf{D}(v, b, k)$ is the class of all connected proper designs which are binary in test treatments. Some new investigations in this direction are by Cheng, Stufken, Majumdar and Ture (1988), Jacroux (1989) and Stufken (1991).

Angelis and Moyssiadis (1991) have initiated work on determining the structure of optimal designs with unequal block sizes under a two-way additive, homoscedastic, fixed effects model. When the block sizes differ widely, it is not reasonable to assume intra block variances as constant. The present work, therefore, investigates the optimality aspects of designs for making test treatment-control comparisons under the setting (1.1). Some methods of construction of balanced test treatment incomplete block designs with unequal block sizes for setting (1.1) have been given. The A -optimality of these designs is investigated. For designs which are not A -optimal, A -efficiency has been worked out. A catalogue of designs alongwith the A -efficiencies is also prepared.

2. A-OPTIMAL DESIGNS

In this section we shall obtain conditions for a design to be A -optimal over \mathcal{D} or \mathbf{D} . We begin by giving some definitions and lemmas which culminate in a

general theorem from which A -optimal designs may be obtained as a special case. To begin with the treatment of the problem would be general in nature, but later on we shall confine our discussions to only those designs which have two distinct block sizes and different block sizes are known or number of experimental units is fixed.

Definition 2.1. A design $d \in \mathcal{D}$ is called a Balanced Test Treatment Incomplete Block Design with Unequal Block sizes (BTIUB) of type G , under setting (1.1), if and only if the matrix \mathbf{M}_d in \mathbf{C}_d as given in (1.2) is completely symmetric.

Remark 2.1. When $\alpha = 0$ and $p = 1$, the BTIUB design of type G will simply be a BTIB design of Bechhofer and Tamhane (1981) and for $\alpha = 0$ and $p \neq 1$, the BTIUB design of type G , will simply be a BTIUB design of Angelis and Moysiadis (1991).

Suppose $d \in \mathcal{D}$ is arbitrary. Let \sum be the set of all $v!$ permutations of test treatments 1, 2, ..., v . For $\sigma \in \sum$, let σ_d be the design resulting from d by the permutation σ of the treatments in d . We define

$$\bar{\mathbf{M}}_d = \sum_{\sigma \in \sum} \mathbf{M}_{\sigma_d} / v! = \sum_{\pi \in \Pi} \pi' \mathbf{M}_d \pi / v!, \quad \dots (2.1)$$

where Π is the set of all $v \times v$ permutation matrices. We then have the following lemmas :

Lemma 2.1 (Majumdar and Notz (1983)). *The matrix $\bar{\mathbf{M}}_d$ obtained from \mathbf{M}_d as given in (2.2), is of the form $(a_1 - a_2)\mathbf{I}_v + a_2\mathbf{1}_v\mathbf{1}'_v$ and*

$$\begin{aligned} a_1 &= \left[\sum_{i=1}^v \sum_{h=1}^p k_h^{-\alpha} \{ r_{dih} - k_h^{-1} \sum_{m=1}^{b_h} n_{dihm}^2 \} \right] / v; \\ a_2 &= - \left[\sum_{1 \leq i \neq i' \leq v} \sum_{h=1}^p k_h^{-\alpha-1} \lambda_{dii'h} \right] / v(v-1). \quad \dots (2.2) \end{aligned}$$

Here r_{dih} is the replication of the i -th test treatment in the b_h blocks of size k_h and $\lambda_{dii'h}$ is the number of blocks of size k_h in which treatments i and i' concur. It is easily seen that when $\bar{\mathbf{M}}_d$ is the information matrix of some design in \mathcal{D} , under setting (1.1), then this design is a BTIUB design of type G .

Lemma 2.2. *If $d \in \mathcal{D}$, then $\bar{\mathbf{M}}_d$ has eigen values $\mu_{1d}, \mu_{2d} = \dots = \mu_{vd}$ with*

$$\begin{aligned} \mu_{1d} &= \left[\sum_{h=1}^p k_h^{-\alpha} \{ r_{doh} - k_h^{-1} \sum_{m=1}^{b_h} n_{dohm}^2 \} \right] / v; \\ \mu_{2d} &= \left[\sum_{i=1}^v \sum_{h=1}^p k_h^{-\alpha} \{ r_{dih} - k_h^{-1} \sum_{m=1}^{b_h} n_{dihm}^2 \} - \mu_{1d} \right] / (v-1) \quad \dots (2.3) \end{aligned}$$

Here r_{doh} is the replication of control treatment in b_h blocks of size k_h . In addition if d is binary in test treatments, then μ_{2d} simplifies to

$$\mu_{2d} = \left[\sum_{h=1}^p k_h^{-\alpha} \{b_h(k_h - 1) - k_h^{-1}(k_h - 1)r_{doh}\} - \mu_{1d} \right] / (v - 1) \quad \dots (2.4)$$

Lemma 2.3. For $d \in \mathcal{D}$, $\text{trace}(\overline{\mathbf{M}}_d^{-1}) = 1/\mu_{1d} + (v - 1)/\mu_{2d}$. $\dots (2.5)$

Lemma 2.4 (Majumdar and Notz (1983)). Suppose ϕ is a convex real valued possibly infinite function on the set of all $v \times v$ non-negative definite matrices and ϕ is permutation invariant, then for $d \in \mathcal{D}$, $\phi(\mathbf{M}_d) \leq \phi(\overline{\mathbf{M}}_d)$. As a consequence of Lemma 2.4

$$\text{trace}(\overline{\mathbf{M}}_d^{-1}) \leq \text{trace}(\mathbf{M}_d^{-1}).$$

Lemma 2.5 (Majumdar and Notz (1983)). Suppose ϕ is a convex real valued possibly infinite function on the set of all non-negative definite $v \times v$ matrices with the property that if \mathbf{M} and \mathbf{N} are non-negative definite $v \times v$ matrices with eigenvalues $\mu_1 \leq \mu_2 \leq \dots \leq \mu_v$ and $\nu_1 \leq \nu_2 \leq \dots \leq \nu_v$ respectively, which satisfies $\mu_i \geq \nu_i$ for $1 \leq i \leq v$, then $\phi(\mathbf{M}) \leq \phi(\mathbf{N})$. Let $d \in \mathcal{D}$, which is not binary in test treatments. Then there exists a design $d^* \in \mathbf{D}$, which is binary in test treatments with $r_{d0} = r_{d^*0}$ and satisfies

$$\phi(\overline{\mathbf{M}}_{d^*}) \leq \phi(\overline{\mathbf{M}}_d).$$

From now onwards, we shall deal with $d \in \mathbf{D}$ which is binary in test treatments.

Lemma 2.6. Suppose ϕ is as in lemma 2.5. Suppose $d \in \mathbf{D}$ which is binary in test treatments and has $r_{doh} > \frac{b_h k_h}{2}$, then there exists $d^* \in \mathbf{D}$ which is binary in test treatments and has $r_{d^*oh} < \frac{b_h k_h}{2}$ and satisfies $\phi(\overline{\mathbf{M}}_{d^*}) \leq \phi(\overline{\mathbf{M}}_d)$.

Remark 2.2. All the above lemmas can be proved on the lines of proof of Lemma 2.1 to Lemma 2.5 of Majumdar and Notz (1983).

Lemma 2.7. For fixed r_{doh} , the total number of control replications in the blocks of size k_h , and for $1 \leq i \leq v$; $1 \leq m \leq b_h$ and $1 \leq h \leq p$, $1/\mu_{1d} + (v - 1)/\mu_{2d}$ is minimised when n_{dohm} are either $\left[\frac{r_{doh}}{b_h} \right]$ or $\left[\frac{r_{doh}}{b_h} \right] + 1$. Here $[.]$ denotes the greatest integer function.

Proof. Follows on the lines of Lemma 2.5 of Notz (1985).

In further investigations, we shall use the following definition. *Definition* 2.2. For $t_h \in \{0, 1, \dots, k_h - 1\}$, $s_h \in \{0, 1, \dots, b_h - 1\}$ and at least one $s_h > 0$, for all $1 \leq h \leq p$, a design $d \in \mathbf{D}$, is a BTIUB design of type G with parameters $v, b_1, \dots, b_p, k_1, \dots, k_p, t_1, \dots, t_p, s_1, \dots, s_p, n, \alpha$ if and only if it is a BTIUB design of type G with the additional property that

$$n_{dihm} \in \{0, 1\}, \quad 1 \leq i \leq v; 1 \leq h \leq p; 1 \leq m \leq b_h,$$

$$n_{doh1} = \dots = n_{doh,s_h} = t_h + 1; \quad n_{doh,s_{h+1}} = \dots = n_{doh,b_h} = t_h. \quad \dots (2.6)$$

We now have the main theorem :

Theorem 2.1. Under the setting (1.1) for given integers $v, b_1, \dots, b_p, k_1, \dots, k_p, n, \alpha$, a BTIUB design of type G with parameters $v, b_1, \dots, b_p, k_1, \dots, k_p, t_1, \dots, t_p, s_1, \dots, s_p, n, \alpha$, as in Definition 2.2, is A -optimal over \mathbf{D} , if

$$g(t_1, \dots, t_p, s_1, \dots, s_p, \alpha) = \min\{g(x_1, \dots, x_p, z_1, \dots, z_p, \alpha); \quad \dots (2.7)$$

$$(x_h, z_h) \in \Delta\}$$

where $\Delta = \{(x_h, z_h); x_h = 0, 1, \dots, [k_h/2] - 1; z_h = 0, 1, \dots, b_h \text{ with at least one } z_h > 0 \text{ when } \sum_{h=1}^p x_h = 0, 1 \leq h \leq p\}$.

$$g(x_1, \dots, x_p, z_1, \dots, z_p, \alpha) = v(v - 1)^2/A(x_1, \dots, x_p, z_1, \dots, z_p, \alpha) +$$

$$v/B(x_1, \dots, x_p, z_1, \dots, z_p, \alpha)$$

$$c_h = vb_h k_h(k_h - 1); a_h = v(k_h - 1) + k_h; e_h = b_h x_h + z_h;$$

$$f_h = b_h x_h^2 + 2x_h z_h + z_h,$$

$$A(x_1, \dots, x_p, z_1, \dots, z_p, \alpha) = \sum_{h=1}^p k_h^{-\alpha-1} \{c_h - a_h e_h + f_h\},$$

$$B(x_1, \dots, x_p, z_1, \dots, z_p, \alpha) = \sum_{h=1}^p k_h^{-\alpha-1} \{k_h e_h - f_h\}.$$

Proof. Follows on the lines of Theorem 2.1 of Hedayat and Majumdar (1984).

Let d_h be the part of the design with b_h blocks of size k_h each. The first s_h blocks in d_h , referred to as d_{1h} , have $k_h - t_h - 1$ units each for test treatments. The remaining $b_h - s_h$ blocks in d_h , referred to as d_{2h} , have $k_h - t_h$ units each for test treatments. It is easily seen that in both d_{1h} and d_{2h} , the test treatments are equally replicated. Then

$$\sum_{m \in d_h} n_{dihm} = r_{dh} \quad \text{and} \quad \sum_{m \in d_{1h}} n_{dihm} = T_h.$$

Then it is easy to establish the following lemma :

Lemma 2.9. In a BTIUB design of type G with property (2.6), the following relations must hold

$$vr_{dh} = b_h(k_h - t_h) - s_h,$$

$$vT_h = s_h(k_h - t_h - 1) \quad \dots (2.8)$$

and off diagonal element of $\overline{\mathbf{M}}_d$ and hence of \mathbf{M}_d will be equal to

$$\left(\sum_{h=1}^p k_h^{-\alpha-1} [T_h(k_h - t_h - 2) + (r_{dh} - T_h)(k_h - t_h - 1)]\right) / (v - 1).$$

In search of an optimal design with parameters $v, b_1, \dots, b_p, k_1, \dots, k_p, n$ and α we are led to the following algorithm by combining the preceding results :

Step 1. Find integers $t_1, \dots, t_p, s_1, \dots, s_p, \alpha$ which minimise the function $g(x_1, \dots, x_p, z_1, \dots, z_p, \alpha)$ given in 2.7.

Step 2. Verify the necessary conditions, i.e., check whether the following quantities are integers.

$$q_{1h} = \{b_h(k_h - t_h) - s_h\} / v, \quad \forall h = 1, 2, \dots, p$$

$$q_{2h} = s_h(k_h - t_h - 1) / v, \quad \forall h = 1, 2, \dots, p$$

and $q_3(v - 1) = \left\{ \prod_{h=1}^p k_h^{\alpha+1} \right\} \left[\sum_{h=1}^p \{q_{2h}(k_h - t_h - 2) + (q_{1h} - q_{2h})(k_h - t_h - 1) / k_h^{\alpha+1}\} \right]$

is constant.

Proceed if q_{1h}, q_{2h} are integers and q_3 is constant. Otherwise Theorem 2.1 cannot be applied.

Remark 2.3. Using Theorem 2.1, if we check whether a given BTIUB design of type G is A -optimal or not, then these conditions are automatically satisfied, and do not require checking.

Henceforth we shall restrict our search to only those designs which have two distinct block sizes.

It is difficult to give general methods of construction of designs which satisfy the condition of Theorem 2.1 and are A -optimal. We shall, therefore, handle this problem through an alternate approach which involves the use of A -efficiency criterion given by Stufken (1988). Some general methods of construction of BTIUB designs of type G will be given in next section and A -efficiency of these designs defined as follows will be computed.

Definition 2.3. The efficiency $E(d^*)$ of a design $d^* \in \mathbf{D}$, a class of competing designs, is defined as :

$$E(d^*) = \frac{\min_{d \in \mathbf{D}} \text{trace}(\mathbf{M}_d^{-1})}{\text{trace}(\mathbf{M}_{d^*}^{-1})}$$

Here d^* is a BTIUB design of type G and class of competing designs is \mathbf{D} . We shall attempt to find designs with an efficiency close to one. A design with A -efficiency exactly equal to one will be termed as A -optimal.

3. CONSTRUCTION OF BTIUB DESIGNS OF TYPE G

In this section we give some methods of construction of BTIUB designs of type G . The optimality aspects of these designs are investigated using A -efficiency criterion defined in section 2. For $\alpha = 0$, we get BTIUB designs under a homoscedastic model, while, for any known $\alpha \in (0, \infty]$, we get BTIUB designs of type G , when the intra block variances are proportional to non-negative real power of block sizes. Methods of construction of BTIUB designs of type G are given as follows :

Method 1. For $i = 1, 2, \dots, p$, let \mathbf{N}_i denote the incidence matrix of a balanced incomplete block (BIB) design with parameters $v, b_i, r_i, k_i, \lambda_i$ { see Dey (1986)}. Let at least two $k_i + a_i$ be different, where $a_i, i = 1(1)p$ are respectively the number of times the control treatment is added to the blocks of size k_i and $0 \leq a_i \leq k_i$ and satisfy $\sum_i a_i \neq 0$. Then the design with incidence matrix

$$\mathbf{N}^* = \begin{bmatrix} \mathbf{N}_1 & \mathbf{N}_2 & \dots & \mathbf{N}_p \\ a_1 \mathbf{1}' & a_2 \mathbf{1}' & \dots & a_p \mathbf{1}' \end{bmatrix} \quad \dots (3.1)$$

is a BTIUB design of type G with parameters

$$v^* = v + 1, b^* = \sum_{i=1}^p b_i, r_0^* = \sum_{i=1}^p a_i b_i, \mathbf{k}^{*'} = \{(k_1 + a_1) \mathbf{1}'_{b_1}, \dots, (k_p + a_p) \mathbf{1}'_{b_p}\}.$$

A lot of BTIUB designs of type G can be obtained using this method, but due to paucity of space here we present only some of the designs obtained by taking union of two BIB designs [see *e.g.*, Raghavarao (1971) and Kageyama (1976)] for $a_i \in [0, 1]$ such that $a_1 + a_2 \neq 0$ in Table 1. A -efficiency of these designs changes with the change in the value of α (see Table 1).

TABLE 1. BTIUB DESIGNS OF TYPE G , OBTAINED THROUGH BIBD'S USING METHOD 1 ALONGWITH THEIR A -EFFICIENCIES

v	b_1	b_2	k_1	k_2	r	r_0	source	A -efficiency				
								$\alpha = 0$	$\alpha = 1$	$\alpha = 1.6$	$\alpha = 2$	$\alpha = 2.5$
18	45	38	3	4	18	75	RR24,RR25	0.988	0.985	0.983	0.982	0.980
11	55	55	3	4	25	118	RR31,RR33	0.988	0.985	0.983	0.982	0.979
12	66	44	3	4	22	118	K9,K18	0.978	0.975	0.972	0.971	0.968
18	153	182	3	4	34	255	K19,K28	0.952	0.947	0.944	0.942	0.939
19	57	57	3	5	21	57	RR54,RR57	0.972	0.935	0.898	0.867	0.826

$RR \#$ denotes BIB design from Raghavarao [(1971) p. 91]

$K \#$ denotes resolvable BIB design from Kageyama (1976)

Remark 3.1. This method of construction is robust against the value of $\alpha \in (0, \infty]$ as far as balancing is concerned. However, from Table 1, it is clear that A -efficiency of the design changes with change in the value of α .

Example 3.1. Consider a BIB design with parameters $v = 5, b_1 = 10, r_1 = 6, k_1 = 3, \lambda_1 = 3$. Let another BIB design have parameters $v = 5, b_2 = 10,$

$r_2 = 4, k_2 = 2, \lambda_2 = 1$. Then for $p = 2, a_1 = a_2 = 1$, (3.6) gives a BTIUB design of type G with parameters $v^* = 6, b^* = 20, r_0^* = 20, \mathbf{k}^* = (4 \mathbf{1}'_{10}, 3 \mathbf{1}'_{10})$. The A -efficiency of the design is one for all values of α tried in Table 1.

Method 2. For $i = 1, 2$, let \mathbf{N}_i be a $v \times b_i$ incidence matrix of a two-associate class partially balanced incomplete block (PBIB) design with parameters $v, b_i, r_i, k_i, n_{i1}, n_{i2}, \lambda_{i1}, \lambda_{i2}$ and two designs have same association scheme [see *e.g.*, Dey (1986)]. Then for any two positive integers θ and ϕ and $k_1 + i_1 \neq k_2 + i_2$, where i_1, i_2 are respectively the number of times the control treatment is added to blocks of size k_1 and k_2 , and $0 \leq i_1 \leq k_1; 0 \leq i_2 \leq k_2$ and satisfy $i_1 + i_2 \neq 0$. Then the design with the following incidence matrix

$$\mathbf{N}^* = \begin{bmatrix} \mathbf{N}_1 & \dots & \mathbf{N}_1 & \mathbf{N}_2 & \dots & \mathbf{N}_2 \\ i_1 \mathbf{1}' & \dots & i_1 \mathbf{1}' & i_2 \mathbf{1}' & \dots & i_2 \mathbf{1}' \\ & \theta \text{ times} & & & \phi \text{ times} & \end{bmatrix} \dots (3.2)$$

is a BTIUB design of type G if θ and ϕ are so chosen that

$$\theta/\phi = [(\lambda_{22} - \lambda_{21})/(\lambda_{11} - \lambda_{12})]\{(k_1 + i_1)/(k_2 + i_2)\}^{\alpha+1} \dots (3.3)$$

Since θ and ϕ are positive integers, $\lambda_{22} - \lambda_{21}$ and $\lambda_{11} - \lambda_{12}$ must be of the same sign. The BTIUB design of type G has the parameters as $v^* = v + 1; b^* = \theta b_1 + \phi b_2; r_0^* = i_1 \theta b_1 + i_2 \phi b_2; \mathbf{k}_0^* = ((k_1 + i_1) \mathbf{1}'_{\theta b_1}, (k_2 + i_2) \mathbf{1}'_{\phi b_2})$.

Using this method a large number of BTIUB designs of type G can be obtained. But for illustration purpose, we give some examples by taking α to be a non-negative real number and choosing θ and ϕ according to condition (3.3).

Example 3.2 (1). Let \mathbf{N}_1 be the incidence matrix of a semi regular group divisible (SR9) design with parameters $v = 8, b_1 = 16, r_1 = 4, k_1 = 2, \lambda_{11} = 0, \lambda_{12} = 1$. Let \mathbf{N}_2 be the incidence matrix of another group divisible (R133) design with parameters $v = 8, b_2 = 8, r_2 = 5, k_2 = 5, \lambda_{21} = 4$ and $\lambda_{22} = 2$.

Let $\alpha = 1.6$ and $i_1 = i_2 = 1$. Then $\theta = 1$ and $\phi = 3$ gives a BTIUB design of type G with parameters $v = 8, b_1 = 16, b_2 = 24, k_1 = 3, k_2 = 6, r = 19, r_0 = 40$. A -efficiency of this design is one and is therefore A -optimal.

Example 3.2 (2). Let \mathbf{N}_1 be the incidence matrix of a semi regular group divisible (SR21) design with parameters $v = 6, b_1 = 16, r_1 = 8, k_1 = 3, \lambda_{11} = 0, \lambda_{12} = 4$. Let \mathbf{N}_2 be the incidence matrix of another group divisible (R22) design with parameters $v = 6, b_2 = 24, r_2 = 8, k_2 = 2, \lambda_{21} = 4$ and $\lambda_{22} = 1$.

Let $\alpha = 2.4$ and $i_1 = i_2 = 1$. Then $\theta = 2$ and $\phi = 1$ gives a BTIUB design of type G with parameters $v = 6, b_1 = 32, b_2 = 24, k_1 = 4, k_2 = 3, r = 24, r_0 = 56$. A -efficiency of the design is one and hence is A -optimal.

Remark 3.1. For $\alpha = 1$, we have reported BTIUB designs of type G in Table 2 along with their A -efficiencies for which $\theta = \phi = 1$.

Remark 3.2. For $\alpha = 0$, this method gives BTIUB designs under a homoscedastic model and is in fact an improvement on method 2 of Angelis and

Moyssiadis (1991) in the sense that they consider the union of only Group Divisible Treatment Designs (GDTD'S), *i.e.*, a particular case of our method for which $i_1 = i_2 = 1$.

TABLE 2. BTIUB DESIGNS OF TYPE G , OBTAINED THROUGH PBIBD'S USING (METHOD 2) FOR SPECIFIED VALUE OF $\alpha = 1$

v	b_1	b_2	k_1	k_2	r	r_0	n	source	A -efficiency
6	16	18	4	2	14	16	100	SR21,R19	0.812
6	9	18	2	4	12	18	90	SR6,R49	0.913
6	24	18	2	4	17	18	120	R24,R49	0.780
8	24	24	2	4	15	24	144	R29,R56	0.891
9	27	24	2	4	14	24	150	R34,R63	0.899
12	48	18	3	6	17	48	252	R38,R167	1.000
12	12	54	6	3	15	54	234	S29,R39	0.994
12	15	20	8	4	15	20	200	S58,R70	0.930
12	24	24	3	6	16	24	216	R71,R147	0.808
15	30	15	5	10	17	45	300	R117,R194	0.994
15	75	15	3	6	16	75	315	R41,R168	0.981
20	60	20	4	8	16	80	400	R90,R180	0.994
20	15	60	8	4	15	60	360	S67,R90	0.999
21	105	28	3	6	18	105	483	T8,T68	0.974
21	105	42	3	6	20	147	567	T8,T53	0.953
24	15	80	8	4	15	80	440	S69,R93	0.993
32	64	32	5	10	18	64	640	SR49,R208	0.995

3.1 Particular cases. We now give some methods of construction of BTIUB designs of type G when α is a non-negative integer. For illustration purpose we shall prepare a catalogue of A -optimal BTIUB designs of type G for any non-negative integer α . Putting $\alpha = 1$ in this catalogue we get the designs when the intra block variances are proportional to block sizes. Similarly $\alpha = 0$ gives BTIUB designs under a homoscedastic model.

Remark 3.3. The design as given in (3.2) with θ and ϕ chosen in the ratio $\alpha_1(k_1 + i_1)^\alpha : \alpha_2(k_2 + i_2)^\alpha$ where α_1 and α_2 are positive integers satisfying

$$\alpha_1/\alpha_2 = [(\lambda_{22} - \lambda_{21})/(\lambda_{11} - \lambda_{12})]\{(k_1 + i_1)/(k_2 + i_2)\} \quad \dots (3.4)$$

is a BTIUB design of type G with parameters, $v^* = v + 1, b^* = \theta b_1 + \phi b_2, r_0^* = i_1 \theta b_1 + i_2 \phi b_2, \mathbf{k}^{*'} = ((k_1 + i_1)\mathbf{1}', (k_2 + i_2)\mathbf{1}')$.

Using this method a large number of BTIUB designs of type G can be obtained. But in this paper we shall report only the designs obtained by taking $\alpha_1 = \alpha_2 = 1$ and $i_1, i_2 \in \{0, 1\}$. The designs obtained are given in Table 3.

Example 3.1.1. Let \mathbf{N}_1 be the incidence matrix of a group divisible design SR9 (Clatworthy, 1973) with parameters $v = 8, b_1 = 16, r_1 = 4, k_1 = 2, n_{11} = 3, n_{12} = 4, \lambda_{11} = 0, \lambda_{12} = 1$. Let \mathbf{N}_2 be the incidence matrix of another group divisible design with the same association scheme R133 (Clatworthy, 1973) and with parameters $v = 8, b_2 = 8, r_2 = 5, k_2 = 5, n_{21} = 3, n_{22} = 4, \lambda_{21} = 4, \lambda_{22} = 2$. Now for $\alpha_1 = \alpha_2 = 1$ and $i_1 = 1$ and $i_2 = 1$, from (3.4), we have the design d^* with incidence matrix \mathbf{N}_d^* of (3.2) obtained by taking copies of \mathbf{N}_1

and \mathbf{N}_2 in the ratio $1 : 2^\alpha$, a BTIUB design of type G with the parameters $v^* = 9, b^* = 16 + 8.2^\alpha, r_0^* = 16 + 8.2^\alpha, \mathbf{k}^{*'} = (3 \mathbf{1}'_{16}, 6 \mathbf{1}'_{8.2^\alpha})$. This design as reported in Table 3 is A -optimal.

TABLE 3. A -OPTIMAL BTIUB DESIGNS OF TYPE G , OBTAINED THROUGH PBIBD'S USING REMARK 3.3 AND METHOD 2 FOR ANY SPECIFIED NON-NEGATIVE INTEGER α

v	b_1	b_2	k_1	k_2	r	r_0	source
6	27.3^α	18.4^α	3	4	$9.3^\alpha + 9.4^\alpha$	$b_1 + b_2$	3^α .SR8, 4^α .R49
6	16.4^α	24.3^α	4	3	$8.4^\alpha + 8.3^\alpha$	$b_1 + b_2$	4^α .SR21, 3^α .R22
8	16	8.2^α	3	6	$4 + 5.2^\alpha$	$b_1 + b_2$	SR9, 2^α .R133
8	32	16.2^α	3	6	$8 + 10.2^\alpha$	$b_1 + b_2$	SR10, 2^α . R135
8	20.5^α	40.3^α	5	3	$10.5^\alpha + 10.3^\alpha$	$b_1 + b_2$	5^α .SR40, 3^α .R33
8	20.5^α	24.4^α	5	4	$10.5^\alpha + 9.4^\alpha$	$b_1 + b_2$	5^α .SR40, 4^α .R56
9	27	9.2^α	3	6	$6 + 5.2^\alpha$	$b_1 + b_2$	R34, 2^α . R137
9	18	18.2^α	3	6	$4 + 10.2^\alpha$	$b_1 + b_2$	LS1, 2^α . LS50
10	30	12.2^α	3	6	$6 + 6.2^\alpha$	$b_1 + b_2$	T1, 2^α .T46
12	32.2^α	12.3^α	4	6	$8.2^\alpha + 5.3^\alpha$	$b_1 + b_2$	2^α .SR27, 3^α .R143
15	27.3^α	45.2^α	6	4	$9.3^\alpha + 9.2^\alpha$	$b_1 + b_2$	3^α .SR57, 2^α .R84

Method 3. Let \mathbf{N}_1 be a $v \times b_1$ incidence matrix of a group divisible design with parameters $v = mn, b_1, r_1, k_1, \lambda_{11} = a, \lambda_{12} = a + p$ [see e.g. Dey (1986)], a being a non-negative integer and p is a positive integer. Let \mathbf{N}_2 be the $v \times b_2$ incidence matrix of another design formed from the group divisible association scheme (m, n) by treating groups as blocks. Taking p copies of \mathbf{N}_2 , we obtain \mathbf{N}_2^* , with parameters $v_2 = mn, b_2 = mp, r_2 = p, k_2 = n, \lambda_{21} = p, \lambda_{22} = 0$. Let $k_1 + i_1 \neq k_2 + i_2$, where i_1, i_2 are respectively the number of times the control treatment is added to blocks of size k_1 and k_2 , and $0 \leq i_1 \leq k_1; 0 \leq i_2 \leq k_2$ and satisfy $i_1 + i_2 \neq 0$; then if \mathbf{N}_1 and \mathbf{N}_2^* are copied in the ratio $(k_1 + i_1)^{\alpha+1} : (n + i_2)^{\alpha+1}$; where α is any non-negative integer; then the design with the incidence matrix

$$\mathbf{N}^* = \begin{bmatrix} \mathbf{N}_1 & \dots & \mathbf{N}_1 & \mathbf{N}_2^* & \dots & \mathbf{N}_2^* \\ i_1 \mathbf{1}' & \dots & i_1 \mathbf{1}' & i_2 \mathbf{1}' & \dots & i_2 \mathbf{1}' \end{bmatrix} \dots (3.5)$$

is a BTIUB design of type G . The parameters of the resulting design will be $v^* = (v + 1), b^* = b_1^* + b_2^*, r_0^* = i_1 b_1^* + i_2 b_2^*, \mathbf{k}^{*'} = ((k_1 + i_1) \mathbf{1}'_{b_1^*}, (k_2 + i_2) \mathbf{1}'_{b_2^*})$; where $b_1^* = (k_1 + i_1)^{\alpha+1} b_1 / c$ and $b_2^* = (k_2 + i_2)^{\alpha+1} b_2 / c$ and c is the highest common factor between $(k_1 + i_1)^{\alpha+1}$ and $(n + i_2)^{\alpha+1}$.

Example 3.1.2. Let \mathbf{N}_1 be the incidence matrix of a semi-regular group divisible, SR18 (Clatworthy 1973) design with parameters $v = 6, b_1 = 4, r_1 = 2, k_1 = 3, m = 3, n = 2, \lambda_{11} = 0, \lambda_{12} = 1$.

Then treating the association scheme (3,2) as another design, with incidence matrix \mathbf{N}_2 , with parameters $v = 6, b_2 = 3, r_2 = 1, k_2 = 2, \lambda_{21} = 2, \lambda_{22} = 0$. As $p = 1$, therefore, $\mathbf{N}_2 = \mathbf{N}_2^*$. Then if \mathbf{N}_1 and \mathbf{N}_2 are copied in the ratio $4^{\alpha+1} : 3^{\alpha+1}$

and control treatment is added once to each of the blocks so obtained, we get a BTIUB design of type G with parameters $v^* = 7, b^* = 4^{\alpha+2} + 3^{\alpha+2}, r_0^* = 4^{\alpha+2} + 3^{\alpha+2}, \mathbf{k}^* = (41'_{4^{\alpha+2}}, 31'_{3^{\alpha+2}})$. The design as reported in Table 4 is A -optimal.

TABLE 4. A -OPTIMAL BTIUB DESIGNS OF TYPE G , OBTAINED THROUGH SEMI-REGULAR AND REGULAR GROUP DIVISIBLE DESIGNS USING (METHOD 3) FOR ANY SPECIFIED NON-NEGATIVE INTEGER α .

v	b_1	b_2	k_1	k_2	r	r_0	source
6	$4.4^{\alpha+1}$	$3.3^{\alpha+1}$	4	3	$2.4^{\alpha+1} + 3^{\alpha+1}$	$b_1 + b_2$	SR18
6	$14.4^{\alpha+1}$	$3.3^{\alpha+1}$	4	3	$7.4^{\alpha+1} + 3^{\alpha+1}$	$b_1 + b_2$	R46
6	$18.4^{\alpha+1}$	$6.3^{\alpha+1}$	4	3	$9.4^{\alpha+1} + 2.3^{\alpha+1}$	$b_1 + b_2$	R51
8	$16.3^{\alpha+1}$	$2.4^{\alpha+1}$	3	4	$4.3^{\alpha+1} + 4^{\alpha+1}$	b_1	SR9
8	$8.5^{\alpha+1}$	$8.3^{\alpha+1}$	5	3	$4.5^{\alpha+1} + 2.3^{\alpha+1}$	$b_1 + b_2$	SR36
8	$8.4^{\alpha+1}$	$4.3^{\alpha+1}$	4	3	$3.4^{\alpha+1} + 3^{\alpha+1}$	$b_1 + b_2$	R54
8	$16.4^{\alpha+1}$	$8.3^{\alpha+1}$	4	3	$6.4^{\alpha+1} + 2.3^{\alpha+1}$	$b_1 + b_2$	R55
8	$24.4^{\alpha+1}$	$12.3^{\alpha+1}$	4	3	$9.4^{\alpha+1} + 3.3^{\alpha+1}$	$b_1 + b_2$	R57
8	$24.4^{\alpha+1}$	$2.5^{\alpha+1}$	4	5	$9.4^{\alpha+1} + 5^{\alpha+1}$	$b_1 + b_2$	R58
8	$18.5^{\alpha+1}$	$4.3^{\alpha+1}$	5	3	$9.5^{\alpha+1} + 3^{\alpha+1}$	$b_1 + b_2$	R101
10	$8.2^{\alpha+1}$	10	6	3	$4.2^{\alpha+1} + 2$	$b_1 + b_2$	SR52
12	$9.5^{\alpha+1}$	$4.4^{\alpha+1}$	5	4	$3.5^{\alpha+1} + 4^{\alpha+1}$	$b_1 + b_2$	SR41
12	$12.2^{\alpha+1}$	12	6	3	$5.2^{\alpha+1} + 2$	$b_1 + b_2$	R144
12	$12.3^{\alpha+1}$	$4.2^{\alpha+1}$	6	4	$5.3^{\alpha+1} + 2^{\alpha+1}$	$b_1 + b_2$	R145
12	$24.2^{\alpha+1}$	24	6	3	$10.2^{\alpha+1} + 4$	$b_1 + b_2$	R147
14	$14.5^{\alpha+1}$	$7.3^{\alpha+1}$	5	3	$4.5^{\alpha+1} + 3^{\alpha+1}$	$b_1 + b_2$	R112
15	$25.4^{\alpha+1}$	$3.5^{\alpha+1}$	4	5	$5.4^{\alpha+1} + 5^{\alpha+1}$	b_1	SR28
15	$18.3^{\alpha+1}$	$10.2^{\alpha+1}$	6	4	$6.3^{\alpha+1} + 2.2^{\alpha+1}$	$b_1 + b_2$	SR56
15	$15.5^{\alpha+1}$	$5.4^{\alpha+1}$	5	4	$4.5^{\alpha+1} + 4^{\alpha+1}$	$b_1 + b_2$	R114
15	$30.5^{\alpha+1}$	$10.4^{\alpha+1}$	5	4	$8.5^{\alpha+1} + 2.4^{\alpha+1}$	$b_1 + b_2$	R116
15	$30.3^{\alpha+1}$	$5.2^{\alpha+1}$	6	4	$10.3^{\alpha+1} + 2^{\alpha+1}$	$b_1 + b_2$	R150
20	$16.6^{\alpha+1}$	$5.5^{\alpha+1}$	6	5	$4.6^{\alpha+1} + 5^{\alpha+1}$	$b_1 + b_2$	SR58

Note. Source of PBIB design for Tables 2, 3 and 4 is tables of Clatworthy (1973).

A number of BTIUB designs of type G can be obtained using this method. But here we shall report only A -optimal designs for $i_1, i_2 \in \{0, 1\}$ for any non-negative integer value of α in Table 4.

Remark 3.4. Taking $\alpha = 0$ in Tables 3 and 4, we get A -optimal BTIUB designs under a homoscedastic model and for $\alpha = 1$, we get BTIUB designs when intra block variances are taken as proportional to block sizes.

Acknowledgements. The authors are grateful to the referee whose insightful comments and helpful suggestions improved the paper significantly.

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