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Response Surface Designs, Symmetrical and Asymmetrical, Rotatable and Modified

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Summary

This article gives some modified and/or rotatable response surface, symmetric as well as asymmetric designs that are more precise than the usual rotatable response surface designs for a quadratic response surface. The article restricts the discussions to the fitting of a quadratic response surface though the generalisation is straightforward. The methods of construction given are very simple and work for factors with three and more levels. The designs are small in the sense that the number of design points are within the reach of the experimenters.

Some key words and phrases : Modified rotatable designs, quadratic response surface, symmetric designs, asymmetric designs.

1. Introduction

Investigation of input-output relationship is a useful activity in many situations. Fitting input-output relations to unorganised data involves complex computations and control of precision of estimates of response at desired points is not possible. An alternative is to use for fitting planned data obtainable through appropriate designs. There are some series of such designs in literature. Data from symmetrical factorial experiments with quantitative and equispaced factor levels can be used for fitting such relations conveniently. Box and Hunter (1957) introduced a series of response surface designs with the property that the variances of estimates of response at points equidistant from the centre of the design are all equal. They called these designs Rotatable designs when the relationship between the response variable and several input variables is a quadratic or cubic polynomial. Considerable research activities followed the introduction of these designs though mainly for construction of these designs. For an excellent review on this subject a reference may be made to the text books by Box and Draper (1987), Khuri and Cornell (1987) and Myers and Montgomery (1995) besides two excellent reviews by Hill and Hunter (1966) and Myers, Khuri and Carter (1989). Very little work exists in literature to obtain further series of response surface designs which may provide more precision of estimated response at specific points of interest. Another area that has received little attention is the investigation of the more flexible asymmetrical response surface designs. Some useful references on this aspect are Ramchander (1963), Mehta and Das (1968), Draper and Stoneman (1968) and Dey (1969). Ramchander (1963) gave

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two series of response surface designs for asymmetrical factorials of the type $3x5^m$, but no systematic method of construction was developed. Mehta and Das (1968) gave a general method of construction of rotatable response surface designs for asymmetrical factorials by applying orthogonal transformations on the design points of a suitably chosen symmetrical rotatable design. Although these methods of construction control the degree of asymmetry, it appears that there could be no control on the number of levels of the resulting design. Draper and Stoneman (1968) also studied the response surface designs for asymmetrical factorials when some factors are at two levels and other factors are at *3* or *4* levels each. However, all these methods are for situations with unequispaced factor levels. Dey(1969) gave methods of construction of both rotatable and non-rotatable designs when levels of factors are equispaced or have unequidose ranges. The non-rotatable type of designs have a special feature that a part of the design retains the property of rotatability and as such these designs have been called as partially rotatable designs. The analysis of such designs and their blocking has also been discussed. A direct and straight forward method of construction of asymmetrical rotatable designs is also given. But the method yields response surface designs when some factors are at three levels and others are at five levels.

In this paper we introduce some series of symmetrical response surface designs that provide more precise estimates of response at specific points. We also obtain several series of asymmetrical response surface designs both rotatable and/or modified. We restrict the present investigation to quadratic response surfaces only, although the generalisations are straightforward.

Quadratic polynomials for response surface involve $(v + 1)(v + 2)/2$ parameters when there are v input factors. For $v = 3$, the number of parameters is 10, and for $v = 4$, the number of parameters is 15. These designs contain large number of design points and thus large number of observations are generated through the designs. All such things make estimation of parameters in the response relation very much complex unless proper care is taken to obtain the designs using appropriate spacing of the levels of each of the input factors. When the level codes of each of the factors are the same and equispaced and the level combinations of the factors that form the design are properly selected, computations for estimating the parameters in the polynomial becomes very much simple and the design can be made to possess some useful properties.

It may be interesting to note that Draper and John (1988), Aggarwal and Bansal (1998) and Wu and Ding (1998) gave some designs for fitting response surfaces where the factor levels are both qualitative and quantitative or quantitative alone.

In agricultural and other similar experiments any number of experimental units are available and any factorial combinations can be applied on them without much restriction. But in industrial experiments machines or some industrial / manufacturing processes are experimental units. The number of such units are limited. There is also limitation on the choice of number of levels of factors involved in such experiments. Certain factors may not be allowed to have more than 3 levels while others also may have restrictions on number of levels. For example, if temperature is factor under study , may be that this factor is not allowed to have not more than 3 levels. This type of situations has been pointed out by Draper and John (1998). Asymmetrical factorial response surface designs with control on choice of numbers of levels of different factors are needed in such situations. Some series of response surface designs obtained in this paper are suitable for experiments in such situations.

2. Some Preliminaries Regarding Symmetrical Response Surface Designs

A design for fitting response surface consists of a number of suitable combinations of levels of several input factors. We shall use v for number of factors and N for number of combinations in the design each factor having a constant number of levels.

Users of such designs for applied activities usually provide range of real physical level for each factor under investigation with the origin of levels at zero for most factors. Designs, on the other hand, are usually constructed using coded levels and not the physical levels. The level codes are obtained as below. First the origin of the levels of each factor is shifted at or near the middle of the level range of the factor. This level generally corresponds to the approximate optimum level of the factor. The code for the changed origin is taken as zero. Further level codes of a factor are taken in pairs like ka and $-ka$ one on each side of the changed origin where k is a positive constant and a is a scaling constant for the factor.

The values of k have to be so taken that the physical doses corresponding to the maximum value of k remain within the range. Such pairs of codes have been called equidistant codes.

The physical levels can be obtained from the above level codes as discussed below. Let MN and MX denote the minimum and maximum physical levels of a factor and the level codes corresponding to these physical levels are denoted by km and $-km$. Treating the values of a physical level and the corresponding coded level as the co-ordinates of a point, the different points from possible physical levels within the range lie on a straight line. Taking the equation of the line as

$$y = A + Bx$$

and with the points $(MN, -km)$ and (MX, km) on the line it is found that

$$A = (MN + MX)/2 \text{ and } B = (MX - A)/km.$$

Thus, the equation of the line is known. Now, given any level code, the corresponding physical level is obtained from y by substituting the code value k for x in the equation of the line.

A response surface design can be written as N rows of v columns each. Each row is a combination of v level codes one from each of v ordered factors. x_{iu} denotes the level code of the i-th factor in the u-th combination in the design $(u = 1,2,...N;\ i = 1,2,...v)$. A combination of level codes is also called a design point. The combination with 0 code for each factor is called central point.

Choice of a proper set of combinations for inclusion in a design aims at satisfying several conditions. One of them is $S(pqrt) = 0$ where $S(pqrt)$ stands for

$$\sum_{u=1}^{N} x_{iu}^{p} x_{ju}^{q} x_{ku}^{r} x_{mu}^{t}$$

where the summation is over the design points, u; and p, q, r, t can take integral values from 0 to 4; $p + q + r + t < 5$;

(i) $S(pqrt) = 0$ when at least one of p, q, r, t is odd and i, j, k, m stands for any set of factors.

Three more conditions are that S_2, S_{22}, S_4 are constants, where

$$(ii)\ S_2 = \sum_{u=1}^{N} x_{iu}^2 = R(a \text{ constant}),$$

$$(iii)\ S_{22} = \sum_{u=1}^{N} x_{iu}^2 x_{ju}^2 = L(a \text{ constant}) \text{ and}$$

$$(iv)\ S_4 = \sum_{u=1}^{N} x_{iu}^4 = CL(a \text{ constant}).$$

These restrictions are known as conditions of symmetry and are satisfied by proper choice of level codes of the factors as discussed subsequently.

The following quadratic polynomial will be used:

$$y_u = \beta_0 + \sum_{i=1}^{v} \beta_i x_{iu} + \sum_{i=1}^{v} \beta_{ii} x_{iu}^2 + \sum_{i=1}^{v} \sum_{j>i=1}^{v} \beta_{ij} x_{iu} x_{ju} + e_u .$$

Here β_0, β_i, β_{ii}, β_{ij} are the parameters of the model and y_u is the response observed at the *uth* design point, $u = 1, \ldots, N$.

3. Construction Of A Series Of Symmetrical Modified Response Surface Designs

The usual method of construction of symmetrical designs is to take some combinations with unknown constants, associate a 2^V factorial combinations or a suitable fraction of it with factors each at $+1$ and -1 levels to make the level codes equidistant. All such combinations form a design. Generation of design points this way ensures satisfaction of all the conditions even though the design points contain unknowns. Fixing the unknowns arbitrarily also gives a design without associating the design with any property.

Alternatively, by putting some restrictions indicating some relation among S_2, S_{22}, S_4, some equations involving the unknowns are obtained and their solution gives some of the unknowns and the rest, if any, are fixed arbitrarily. In rotatable designs the restriction used is $S_4 = 3S_{22}$ i.e. $C=3$. Other restrictions are also possible though, it seems, not yet exploited. We shall investigate the restriction $S^2_2 = NS_{22}$ i.e. $R^2 = NL$ to get another series of symmetrical response surface designs which provide more precise estimates of response at specific points of interest than what is available from the corresponding existing designs.

The parameters in the response relation are estimated using least squares technique. Solving the resulting normal equations the estimates of the parameters are obtained as below:

$$b_0 = \left[L(C+v-1)\sum_u y_u - R\sum x_{iu}^2 y_u \right] / D$$

$$b_i = \sum_{u=1}^{N} x_{iu} y_u / R,$$

$$b_{ii} = \sum_u x_{iu}^2 y_u \{1+(R^2-NL)/D\} - \sum_u y_u \{RL(C-1)/D\} + \sum_m \sum_u x_{mu}^2 y_u /(R^2-NL)$$

$$b_{ij} = \sum_{u=1}^{N} x_{iu} x_{ju} y_u / L$$

where $D = v(NL - R^2) + NL(C - 1)$.

Using these solutions variances and co-variances of these estimates are obtained as below:

$Var(bo) = \{L(C + v - 1)/D\}\sigma^2$

$var(b_i) = \sigma^2/R$

$var(b_{ij}) = \sigma^2/L$

$var(b_{ij}) = \{1+(R^2 - NL)/D\} / \{L(C-1)\}\sigma^2$

$co\text{-}var(bo, b_{ii}) = (-R/D)\,\sigma^2$

$covar\,(b_{ii}, b_{jj}) = (R^2 - NL)\,\sigma^2 / \{DL(C-1)\},$

It is seen that if $R^2 = NL$, then $covar(b_{ii}, b_{jj}) = 0$.

Further, $var(b_{ii})$ becomes $\sigma^2/\{L(C-1)\}$ and D becomes $NL(C-1)$.

These modifications of the variances and co-variances affect the variance of estimated response at specific points considerably as will be discussed subsequently.

Using these variances and co-variances, variance of estimated response at any point can be obtained.

Let $\hat{y}_0$ denote the response at the point $(x_{10} \quad x_{20} \quad x_{30} \ldots\ldots x_{v0})$ as estimated using the response relation. Then

$$Var(\hat{y}_0) = [L(C + v - 1)/D + d^2(D - 2R^2)/RD + d^4\{1 + (R^2 - NL)/D\}/(L(C-1)$$

$$+ B_o(C-3)/(C-1)L]\,\sigma^2, \quad where\ B_0 = \sum x_{i0}^2 x_{j0}^2 .$$

Construction of a series of modified response surface designs is the same as for rotatable designs except that instead of taking $C = 3$ the restriction $R^2 = NL$ is to be used and this will provide different values of the unknowns involved.

Remark 1: Besides rotatability, D-optimality criterion has also been widely advocated in the literature for selection of a response surface design. D-optimal design is one which minimises determinant of **X'X** in a specified experimental region, where **X** is the design matrix for the response surface design. Another criterion for selection of a design is the minimisation of variance of predicted response at a given point. It may be seen easily that $R^2 = NL$ maximises the determinant and minimises the variance of the predicted response to a reasonable extent, if not the absolute maximisation and minimisation. For a rotatable design, *i.e.*, $C = 3$ and also if $R^2 = NL$ is satisfied, then $D = 2NL$ and

$$Var(\hat{y}_0) = \left[\frac{C + v - 1}{2N} + \frac{d^4}{4NL^2}\right]\sigma^2 .$$

Therefore, the application of the condition $R^2 = NL$ in obtaining D-optimal designs needs further attention.

After a design is obtained the expressions R, L and CL can be obtained as functions of level codes of the factors and other parameters of the design as discussed below.

Let n_k denote the number of levels of the *k-th* factor and $M(kp) = g_{km}^p$, where g_{km} is the *m-th* level code of the of *k-th* factor and p is an even integer less than 5, that is, $M(kp)$ is the sum of the *p-th* power of the level codes of the *k-th* factor where p is even and takes values from 0 to 4.

$M(pqim) = \sum x_{iu}^p x_{mu}^q$, where i and m indicate factors.

Then $M(pqim) = Na_i^p a_m^q M_{ip} M_{mq} n_i n_m$ where a_i, a_m are the scaling constants of the *i-th* and *m-th* factors respectively.

Accordingly, $R = M(20i0) = \sum_u x_{iu}^2$, $L = M(22im)$ and $CL = M(40i0)$. These being constants are independent of i and m. In asymmetrical designs these are not constant but vary with i and m and as such we shall use symbols like R^1, R^2, R^3, etc for different factors.

Further discussion is based on the following example with $v=3$ and $N=14$ *and* $N=15$.

The design is obtained by using the sets *(1) (a a a)* , *(2) (b 0 0)* , *(3) (0 b 0)* and *(5) (0 0 b)*, where a and b are unknowns. Associating the factorial 2^3 with these sets the distinct combinations give the design. The above method of construction is on the similar lines to that of central composite designs of Box and Wilson(1951). The points obtained using (a a a) are factorial points, with (±b 0 0) as axial points. Some more points of the type (0 0 0) can be added to the design which are known as central points.

The design points are the following

a	*a*	*a*
a	*a*	*-a*
a	*-a*	*a*
a	*-a*	*-a*
-a	*a*	*a*
-a	*a*	*-a*
-a	*-a*	*a*
-a	*-a*	*-a*
b	*0*	*0*
-b	*0*	*0*
0	*b*	*0*
0	*-b*	*0*
0	*0*	*b*
0	*0*	*-b*

$R = 8a^2 + 2b^2, \mathrm{L} = 8\mathrm{a}^4, \mathrm{CL} = 8\mathrm{a}^4 + 2\mathrm{b}^4$.

Using the condition $R^2 = NL$ the following equation is obtained

$(8a^2 + 2b^2)^2 = 14 \times 8a^4$: or $8a^2 + 2b^2 = 4\times 2.645751311\ a^2$: or $b^2 = 1.291503\ a^2$

Now fixing a conveniently b is known. Thus the design as combinations of level codes is obtained along with R, L and CL. For $a=1$, $b=1.136443$. For a rotatable design *i.e.*, for $C=3$, $b^4=8a^4$. *For* $a=1$, $b=1.682$. It may be seen easily that as N changes for a modified response surface design, the ratio b/a also changes, *e.g.*, with the addition of one central point in the above design $N = 15$ and for $a = 1$, $b = 1.21541169$.

Taking $a = 1$ the variances of estimated responses at the central, axial and factorial points of interest for modified and rotatable designs are presented in the following table.

Number of Design points	**Nature of point**	**Variance of the estimated response (Var ($\hat{Y}_0$) $/\sigma^2$)** **Modified**	 **Rotatable**
14	Central	0.58531	85.65518
	Axial	0.62203	0.70716
	Factorial	0.78347	0.71966
15	Central	0.43327	0.98846
	Axial	0.50113	0.60831
	Factorial	0.76553	0.67021

It can easily be seen that for this design both the conditions *viz.* $C = 3$ and $R^2 = NL$ cannot be satisfied simultaneously. There can be a further series of designs which are both modified in the above sense and rotatable using both the restrictions $C = 3$ and $R^2 = NL$ together for fixed $N = 14$ or $N = 14 + n_0$, where n_0 is the number of central points. To construct these designs one more unknown is to be introduced as there will be two equations. For example, the initial combinations (1) $(a\ a\ a\ a)$, (2) $(b\ b\ b\ b)$ and (3) $(\alpha\ 0\ 0\ 0)$ and three more with α in different positions give a modified rotatable design. This design has 40 points. The equations can be solved conveniently as below by first taking $\alpha = 1$.

$R = 16a^2 + 16b^2 + 2\alpha^2; CL = 16a^4 + 16b^4 + 2\alpha^4; L = 16a^4 + 16b^4$. Using $C = 3$, $R^2 = NL$, and $\alpha = 1$, one gets $16a^2 = 4.3245 - 16b^2$. Now selecting $b \leq 0.51$, a may be obtained.

The above method can easily be applied for obtaining response surface designs from balanced incomplete block (BIB) designs as given by Das and Narsimham (1962).

In agricultural experiments, the factors may have equispaced doses (levels) as discussed below. The response surface designs with v factors each having equispaced doses may be obtained through a central composite design by using the following procedure. Get 2^k points by associating $(a,a,\ldots,a)$ with 2^k factorial ($k \leq v$, such that no interaction with less than 5 factors is confounded), and call these points as factorial points. Then add $2v$ axial points $(\pm b, 0,\ldots,0),\ldots,(0, 0,\ldots,\pm b)$ and n_0-central points to the factorial points. Take s copies that is, repetition, of the factorial points, and t copies of axial points such that $N = s.2^k + 2tv + n_0$. Let the dose codes be $-2, -1, 0, 1, 2$, *i.e.*, $a = 1$, $b = 2$. These are equispaced doses. Subsequently we shall use w for 2^k.

In this situation

$R = s.w. + 8\,t$

$L = s.w.,\ CL = sw + 32\,t$

To make the design rotatable, we take C=3 and get the following equation

$sw + 32\,t = 3\,sw$

or $16\,t = sw$

or $\frac{s}{t} = \frac{16}{w}$

Therefore, a central composite type rotatable design with equispaced doses can now be obtained by taking 's' and 't' in the ratio $16 : w$. Some central points can also be added when required. We know that for a modified and rotatable design $R^2 = NL$ and $C = 3$. These two conditions are satisfied simultaneously by adding n_0 central points where $n_0 = 2t\,(10\text{-}v)$. This is so because of the following :

$s^2w^2 + 64t^2 + 16\,tsw = (sw + 2tv + n_0)\,sw$

If $C = 3$, then $sw = 16\,t$. Substituting for s.w in the above, we get

$256\,t^2 + 64\,t^2 + 256\,t^2 = 16\,t\,(16\,t + 2tv + n_0)$

i.e. $36\,t = 16\,t + 2\,tv + n_o$

i.e. $n_0 = 2\,t\,(10 - v)$

Thus, choosing s, t and n_0 as above, we get modified and rotatable designs when each of the factors is at 5 equispaced levels. For number of factors (v) ranging from 3 to 6, the values of s, t, n_0 and N are shown in the following table.

v	k	S	t	n_o	N
3	3	2	1	14	36
3	4	1	1	12	36
5	5	1	2	20	72
6	5	1	2	16	72

To obtain a modified design only, the condition to be satisfied is :

$S^2w^2 + 64\,t^2 + 16\,tsw = (sw + 2\,tv + n_o)\,sw$

By fixing $s = 1$ and $t = 1$, we get

$$n_0 = \frac{64 + (162v)2w}{w}$$

For. v = 3 to 6, the values of n_0 and N are given in the following table

V	k	n_0	N
3	3	18	32
4	4	12	36
5	5	8	50
6	5	6	50

It will be seen that total number of design points required for a modified design in different cases are less as compared to a modified and rotatable design.

4. Asymmetrical Response Surface Designs

The technique used for construction of asymmetrical response surface designs is first to take v factors with number of levels, $n_1, n_2, \ldots, n_v$ where n_i's are not all equal. For each factor equidistant level codes like ka and $-ka$ are taken in pairs where some of the k's may be unknown. Using such codes the complete factorial with $N = \prod_{j=1}^{v} n_i$ level combinations is written. Some of the level codes in these combinations are unknown. Denoting the level codes in the design by x_{iu} for the level codes of the i-th factor in the u-th combination of the design as used for the symmetrical designs and taking the same quadratic polynomial, the expressions for S_2, S_{22}, S_4 are obtained for each factor.

In such designs the condition $S(pqrt)=0$ holds when the dose codes for each factor are equidistant and the factorial is complete. The condition $S_2 = \sum_u x_{iu}^2$ = constant for different factors and similar others do not hold as such in these designs. We shall denote for these designs expressions like $\sum_u x_{iu}^2$ by R_i' $\sum_u x_{iu}^2 x_{mu}^2$ by L_{im} and $\sum_u x_{iu}^4$ by CL_i. The unknowns in the level codes will be obtained by solving equations like $R_i = R_m$, $L_{im} = L_{ik}$,and $CL_i = CL_m$ for different values of i, m , k etc.

The main problem is how best to place the unknowns among the level codes of each factor and how many of them. This problem is discussed first and then the problems of forming the eqaations and their solutions are taken.

4.1 Choice of level codes : Scheme A

For factors with 3 levels the codes are taken as $-a$ 0 a where a is an unknown constant and same for all factors with 3 levels. For factors with 4 levels the codes are $-k_2a$ $-k_1a$ k_1a k_2a where k_1 and k_2 are unknowns. For factors with 5 levels the codes are similar as for 4 levels and one code is taken as 0. The unknown constants are , however, different from those for 4 levels. For factors with 6 levels there are likewise 3

unknowns. Actually, the scaling constant is the same for all factors in this scheme and for factors with same number of levels the codes are the same.

The response surface design is now obtained from the complete factorial obtained by using such level codes. For example, let there be *3* factors *A*, *B* and *C* with numbers of levels as *3*, *4* and *5* respectively.

The following level codes are used.

Factor *A*	$-a$	0	a		
Factor *B*	$-k_2a$	$-k_1a$	k_1a	k_2a	
Factor *C*	$-p_2a$	$-p_1a$	0	p_1a	p_2a ,

Number of combinations in the design is $N=60$.

Using this design and method of obtaining sum of squares and products as discussed in section 3 the following are obtained:

$R_1 = N\,2a^2/3$, $R_2 = N\,2a^2\,(k_1^2+k_2^2)/4$,

$R_3 = N2a^2(p_1^2+p_2^2)/5$,

$L_{12} = N2a^2\,2(k_1^2+k_2^2)a^2/(3x4) = 4Na^4(k_1^2+k_2^2)/12$,

$L_{13} = 4Na^4(p_1^2+p_2^2)/15$,

$L_{23} = Na^4(k_1^2+k_2^2)(p_1^2+p_2^2)/20, \mathrm{CL}_1 = N2a^4/3$,

$CL_2 = N2a^4(k_1^4+k_2^4)/4, CL_3 = N2a^4(p_1^4+p_2^4)/5$,

Restriction $R_1 = R_2$ gives the equation

$$N2a^2/3 = N2a^2(k_1^2+k_2^2)/4$$
$$or K_1^2+k_2^2 = 4/3 \qquad (4.1)$$

Restriction $R_1 = R_3$ gives the equation

$$N2a^2/3 = N2a^2(p_1^2+p_2^2)/5$$

$$\text{or } p_1^2+p_2^2 = 5/3 \qquad (4.2)$$

It will be noticed that R_1, that is, the expression for the factor *A* without any unknown constant in its codes beside the scaling constant has been necessarily used in each such restriction.

Restriction $L_{12} = L_{13}$ gives the equation

$$4Na^4(k_1^2 + k_2^2)/12 = 4Na^4(p_1^2 + p_2^2)/15$$

or $$(k_1^2 + k_2^2) \Big/ (p_1^2 + p_2^2) = 4/5 \qquad (4.3)$$

Restriction $L_{12} = L_{23}$ gives the equation

$$4Na^4(k_1^2 + k_2^2)/12 = 4Na^4(k_1^2 + k_2^2)(p_1^2 + p_2^2)/20$$

or $$p_1^2 + p_2^2 = 5/3 \qquad (4.4)$$

It will be seen that when conditions (4.1) and (4.2) hold then conditions (4.3) and (4.4) automatically hold. This fact is true in general for all designs constructed as discussed above.

Restriction $CL_1 = CL_2$ gives the equation

$$N2a^4/3 = N2a^4(k_1^4 + k_2^4)/4$$

or $$k_1^4 + k_2^4 = 4/3 \qquad (4.5)$$

Restriction $CL_1 = CL_3$ gives the equation

$$N2a^4/3 = N2a^4(p_1^4 + p_2^4)/4$$

or $$p_1^4 + p_2^4 = 5/3 \qquad (4.6)$$

Solving the biquadratic equations at (4.1) and (4.5) k_1 and k_2 are obtained. Again solving similar equations at (4.2) and (4.6) p_1 and p_2 are obtained. Putting $x = k_1^2$ and $y = k_2^2$ the equations at (4.1) and (4.5) become

$$x + y = 4/3$$

$$x^2 + y^2 = 4/3$$

The equations at (4.2) and (4.6) become

$$x + y = 5/3$$
$$x^2 + y^2 = 5/3$$

where $x = p_1^2$ and $y = p_2^2$

The solutions are given below.

$k1 = 0.4419$

$k2 = 1.0668$

$p1 = 0.6787$

$p2 = 1.0982$

These solutions for each of *4* and *5* levelled factors remain the same whatever the design.

For *6* levelled factors *3* unknowns are involved in the codes. But there will be only two equations to solve them out *viz.*

$x + y + z = 6/3$
$x^2 + y^2 + z^2 = 6/3$

To get unique solutions one of x, y or z has to be fixed conveniently.

Now in the level codes only the scaling constant a remains and this has to be fixed conveniently.

At this stage the design is asymmetrical response surface design but without any added property like the modified designs or rotatable designs although conditions of symmetry are satisfied. But these designs can be converted to them by taking some more initial sets of level combinations and the unknowns in them appear in the equations to satisfy $C = 3$ or $R^2 = NL$ or both. We shall discuss an example in this regard subsequently.

It will be seen that the level codes of one of the factors in the above design *viz.* factor A do not involve any unknown beside the scaling constant and the expression R for this factor has been used in each restriction for forming equation.

It is necessary to have a factor with known constant like the factor A in the above example and in all the restrictions R and CL corresponding expressions of this factor have to be used.

4.2 Choice of level codes : Scheme B

In this scheme also a factor with conveniently chosen known codes have to be taken along with a scaling constant. If there is a factor with *3* levels then this is the factor with known constant *viz. 1* along with a scaling constant.

For other factors only one pair of equidistant codes need involve an unknown along with its own scaling constant and the rest can be fixed suitably in equidistant pairs.

The codes for some number of levels of factors are shown below:

A	$-a$	0	a			
B	$-k_1b_1$	$-s_1b_1$	s_1b_1	k_1b_1		
C	$-k_2b_2$	$-s_2b_2$	0	s_2b_2	k_2b_2	
D	$-k_3b_3$	$-s_{32}b_3$	$-s_{31}b_3$	$s_{31}b_3$	$s_{32}b_3$	k_3b_3

In all these factors except A the extreme codes involve one unknown for each factor and the rest codes are known except the scaling constants, that is, all k's are unknowns and s's are known.

For these factors

$$R_1 = (N/3)\, 2a^2$$

$$R_2 = (N/4)2(k_1^2 + s_1^2)b_1^2$$

$$R_3 = (N/5)2(k_2^2 + s_2^2)b_2^2$$

$$R_4 = (N/6)2(k_3^2 + s_{32}^2 + s_{31}^2)b_3^2$$

Different restrictions involving R's give the following equations

$$(k_1^2 + s_1^2) = (4/3)(b_1^2 / a^2) \tag{4.7}$$

$$(k_2^2 + s_2^2) = (5/3)(b_2^2 / a^2) \tag{4.8}$$

$$(k_3^2 + s_{31}^2 + s_{32}^2) = (5/3)(b_3^2 / a^2) \tag{4.9}$$

The equations to make L expressions equal come out to be the same as above.

The equations to make CL expressions equal come out as the above equations except that where ever there is power 2 it should be made 4, that is,

$$(k_1^4 + s_1^4) = (4/3)(b_1^4 / a^4) \tag{4.10}$$

$$(k_2^4 + s_2^4) = (5/3)(b_2^4 / a^4) \tag{4.11}$$

$$(k_3^4 + s_{31}^4 + s_{32}^4) = (5/3)(b_3^4 / a^4) \tag{4.12}$$

These equations are biquadratic equations in pairs. (4.7 and 4.10) form one pair. (4.8 and 4.11) another and the remaining two the third pair. The unknowns in each pair are k_i^2 and the ratio $b_i^2 / a^2 (i = 1,2,3)$. All s's are constants as given while writing the codes. After the codes are known through positive solutions of the equations these can be used in any design provided one of the factors has 3 levels. One of the scaling constants can be fixed conveniently and the other is worked from the solution of their ratio.

The design is now an asymmetrical response surface design where *R*, *L* and *CL* are constants and as such can be treated just like symmetrical response surface designs regarding parameter estimates, variances and co-variances. Taking further sets of combinations with fresh unknowns these designs can be converted into rotatable or modified designs.

4.3 Fractional asymmetrical response surface designs

The designs in previous two sections are based on complete factorial. When there are more than *4* factors suitable fractions of the complete asymmetrical factorial where no interaction with less than *5* factors is confounded can be used without any change in procedure and solutions except for change of *N*.

Another procedure of getting fractional designs is first to take some initial sets with unknowns and generate design points as is done for obtaining symmetrical designs with *n* as number of levels of each factor. Let this design be denoted by D. Next each unknown level code of an additional factor *X* with *m* levels, *m* not equal to *n* is associated (pre-fixed) with each combination of the symmetrical design D. The resulting design will have *mN* combinations where *N* is the number of combinations in the symmetrical design D. If factor *X* which we call factor *1*, has *3* levels with unknown scaling constant there will be only two unknowns. The constancy restrictions are three viz. $R_1 = R_2$, $L_{12} = L_{23}$ and $CL_1 = CL_2$ and each one gives a separate equation unlike what happened in design based on complete factorial. Thus a *3*-levelled factor cannot be used as the additional factor *X*. As there are *3* equations there should be at least three unknowns in the sets of D and in the additional factor together. The following illustration clarifies different issues.

The design D is obtained as below:

1. We take the factor *X* at *4* levels involving two unknowns. Design D is obtained from the sets (1) *(a a -a a)*, (2) *(b 0 0 0)*, (3) *(0 b 0 0)*, (4) *(0 0 b 0)* and (5) *(0 0 0 b)*.

The factor *X* has the *4* levels viz. *-p -q q p*. These codes are each associated with the *16* points from set (1) only. Against the points generated from sets (2) to (5) the factor *X* will have levels *1* and *-1* as shown below. The design will thus have *72* points in *5* factors with *X* at *6* levels including +1 and –1 and rest at *5* levels each.

The design is shown below:

Design Combinations

Factors

X	A	B	C	D	X	A	B	C	D	X	A	B	C	D
$-p$	a	a	a	a	$-q$	a	a	a	a	1	b	0	0	0
$-p$	a	a	$-a$	a	$-q$	a	a	$-a$	a	1	$-b$	0	0	0
$-p$	a	$-a$	a	a	$-q$	a	$-a$	a	a	-1	0	b	0	0
$-p$	a	$-a$	$-a$	a	$-q$	a	$-a$	$-a$	a	-1	0	$-b$	0	0
$-p$	$-a$	a	a	a	$-q$	$-a$	a	a	a	1	0	0	b	0
$-p$	$-a$	a	$-a$	a	$-q$	$-a$	a	$-a$	a	1	0	0	$-b$	0
$-p$	$-a$	$-a$	a	a	$-q$	$-a$	$-a$	a	a	-1	b	0	0	b
$-p$	$-a$	$-a$	$-a$	a	$-q$	$-a$	$-a$	$-a$	a	-1	0	0	0	$-b$
$-p$	a	a	a	$-a$	$-q$	a	a	a	$-a$					
$-p$	a	a	$-a$	$-a$	$-q$	a	a	$-a$	$-a$					
$-p$	a	$-a$	a	$-a$	$-q$	a	$-a$	a	$-a$					
$-p$	a	$-a$	$-a$	$-a$	$-q$	a	$-a$	$-a$	$-a$					
$-p$	$-a$	a	a	$-a$	$-q$	$-a$	a	a	$-a$					
$-p$	$-a$	a	$-a$	$-a$	$-q$	$-a$	a	$-a$	$-a$					
$-p$	$-a$	$-a$	a	$-a$	$-q$	$-a$	$-a$	a	$-a$					
$-p$	$-a$	$-a$	$-a$	$-a$	$-q$	$-a$	$-a$	$-a$	$-a$					

There are 4 sets of 16 points each set having a different level of X. Two of the sets with levels $-p$ and $-q$ are shown along with the 8 points from the initial sets from (2) to (5). There are two more groups of 16 points which are identical to the above two groups except that $-p$ has to be replaced by p and $-q$ by q. This way all the 72 points in the design are obtained.

Different R, L etc. expressions are shown below:

$$R_1 = 32(p^2 + q^2) + 8$$

$$R_2 = 64a^2 + 2b^2 = R_3$$

$$L_{12} = 32\, a^2 (p^2 + q^2) + 2\, b^2,$$

$$L_{23} = 64\, a^4,$$

$$CL_1 = 32(p^4 + q^4) + 8$$

$$CL_2 = 64\, a^4 + 2b^4$$

The following 3 equations follow from the constancy restrictions:

$$32(p^2 + q^2) + 8 = 64\, a^2 + 2b^2 \tag{4.13}$$

$$32\, a^2 (p^2 + q^2) + 2\, b^2 = 64\, a^4 \tag{4.14}$$

$$32(p^4 + q^4) + 8 = 64\, a^4 + 2b^4 \tag{4.15}$$

Dividing the second by a^2 and then subtracting from the first equation

$$2b^2 + 2b^2 / a^2 = 8.$$

Taking $a = 1$ we get $b^2 = 2$. With these values of a and b the first two equations are satisfied. Substituting these values in equations (4.13) and (4.15) above

$$p^2 + q^2 = 15/8$$
$$p^4 + q^4 = 2.$$

Solving these equations

$$p^2 = (15 + 5.568)/16 = 1.285$$
$$q^2 = (15 - 5.568)/16 = .5895$$

and $p = 1.13$ and $q = 0.76$

All the unknowns are now known and the design is complete except for its conversion to actual levels which can be obtained by following the method given earlier after the level ranges for the factors are known.

4.4 Conversion to rotatable or modified designs

By taking the initial set $(d\ d\ d\ d\ d)$ we obtain 16 design points from it using half fraction of 2^5. The sets $(d\ 0\ 0\ 0\ 0)$ that give 10 additional points can also be taken. Taking these points along with the 72 points of the design obtained above each value of expressions R, L and CL will increase by a constant separately for each category of expressions. Thus the asymmetrical design obtained earlier is not disturbed due to addition of these points except for change of number of levels by increase of 2 or 3 for each factor. Now by using restrictions either $C = 3$ or $R^2 = NL$, d can be obtained and the design will be rotatable or modified.

Another method for converting the design to rotatable or modified response surface design is to generate another equation in addition to the three at (4.13), (4.14), (4.15) by using the restriction $C = 3$ or $R^2 = NL$. This is possible as there are 4 unknowns in the equations. When $C=3$ no positive solution of p^2 is possible.

Using $R^2 = NL$ we get the equation

$$(32(p^2 + q^2) + 8)^2 = 72 \times 64a^4$$

or $$32(p^2 + q^2) + 8 = 8 \times 8.48828\, a^2, \qquad (4.16)$$

Solving these 4 equations

$$a^2 = 1.0607$$
$$b^2 = 2.05888$$
$$p = 1.1679$$
$$q = 0.7975$$

With these values of the unknowns the design becomes modified response surface design with *72* points and with *82* points the design becomes both modified and rotatable as discussed earlier.

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