ON THE CONSTRUCTION OF SECOND AND THIRD ORDER ROTATABLE DESIGNS

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Dissertation submitted in fulfilment of the requirements for the award of Diploma in Agricultural and Animal Husbandry Statistics of the Institute of Agricultural Research Statistics (I.C.A.R.),

NEW DELHI.

-1961

ACKNOWLEDGEMENT

I have great pleasure in expressing my deep sense of gratitude to Prof. M.N.Das, Professor of Statistics, Institute of Agricultural Research Statistics, for his valuable guidance, keen interest, valuable criticisms and constant encouragement through out the course of the investigation and immense help in writing this thesis.

I am grateful to Dr. G.R. Seth, Statistical Adviser, Institute of Agricultural Research Statistics, for providing necessary facilities to carry out this work.

(V. L. NARASIMHAM)

C O N T E N T S

CHAPTER I

1. Introduction and Summary:-

Rotatable designs were introduced by Box and Hunter (1954, 1957) for the exploration of response surfaces. They constructed these designs through geometrical configurations and obtained several second order designs. Afterwards, Gardiner and others (1959) obtained some third order designs through the same technique for two and three factors and a third order design for four factors. Bose and Draper (1959) obtained some second order designs by using a different method. Draper (1960) gave a method of construction of an infinite series of second order designs in three and more factors. Recently, Box and Behnken (1960) have obtained a class of second order rotatable designs from those of first order. Draper (1960b) has obtained some third order rotatable designs in three dimensions and a sequential third order rotatable design in four dimensions. Das (1960) has obtained such designs, both second and third order upto 8 factors as fractional replicates of factorial designs. Thaker (1960) has obtained series of second and third order rotatable designs by a different method.

In the present work a method of constructing second order rotatable designs with any number of factors, by using balanced incomplete block designs has been presented in chapter II. Another method of constructing such designs through a particular class of balanced incomplete block designs with unequal block sizes has been presented in chapter III. These two methods have been found to be useful for constructing second order rotatable designs with reasonably small number of design points. Second order rotatable designs upto eleven factors with the minimum number of design points obtainable through these two methods have been tabulated below. Second order designs with smallest number of design points obtained by other authors by different methods and available in literature have also been included in the last column of the table given below.

Number of factors	Number of coefficients to be estimated in the second order surface	Minimum Number of design points for the designs obtainable through B.I.B.designs	Minimum number of design points for the designs available in the literature.
3	10	18	12 (Bose, Draper & Das)
4	15	24*	24* (Draper, Das Gardiner & others)
5	21	44	32 (Box & Behnken)
6	28	44	48* (Das)
7	36	56*	56* (Das, Box & Behnken)
8	45	144	144 (Das)
9	55	194	Not available in literature
10	66	196	-do-
11	78	198	-do-

* denotes at least one central point is to be added. Designs are available in the literature only upto 8 factors.

By extending the above method of construction, third order rotatable designs both sequential and nonsequential, upto fifteen factors have been obtained with the help of doubly balanced incomplete block designs and complementary B.I.B.designs.

This has been presented in chapter IV. Sequential third order rotatable designs upto 8 factors with minimum number of design points obtainable through these methods have been presented below. Other designs with minimum number of points obtained by others by different methods have also been given below. Only a few third order rotatable designs are available in the literature.

Number of factors.	Number of coefficients to be estimated in the third order surface	Number of noncentral design points for the designs obtained in the present work.	Minimum number of noncentral design points for the designs available in the literature.
3	20	40	40 (Das)
4	35	72	72 (Das & Thaker)
5	56	192	114 (Das)
6	84	180	200 (Das)
7	120	182	226 (Das)
8	165	480	752 (Thaker)

Except in case of five factors these designs contain reasonably small number of design points. However, a non-sequential third order rotatable design for five factors in 100 design points has been obtained by the above methods.

In Chapter V multifactorial designs suitable for studying simultaneously (i) the main effects and first order interactions of the different factors and (ii) the second degree response surfaces have been evolved. By adding some more design points to the second order rotatable designs if necessary, such designs have been obtained.

2. Rotatable designs:-

Let there be 'v' variates each at 's' levels. If a design be formed with N of the s^v treatment combinations, it can be written as the following N x V matrix, which we call as the design matrix.

$$\begin{bmatrix} x_{11} & x_{21} & x_{31} & \cdots & x_{v1} \\ x_{12} & x_{22} & x_{32} & \cdots & x_{v2} \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdots & \cdot \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdots & \cdot \\ x_{1N} & x_{2N} & x_{3N} & \cdots & x_{vN} \end{bmatrix}$$

For convinience a variate x_i has been associated with the ith factor. The treatment combinations will hereafter be called as points of the design. According to Box and Hunter (1957) a design of the above form will be a rotatable design of order 'd' if a regression surface of degree 'd' of the response 'y' as obtained from treatments on the variates $x_i (i = 1,2, \ldots v)$ with some suitable origin and scale, can be so fitted that the variance of the estimated response from any treatment is a function of the sum of squares of the levels of the factors in that treatment combination. In other words, the variance of the estimated response is a function of the square of the distance of the point frmm a suitable origin, so that the variance of all responses at points equidistant from the origin is the same. When the surface is of second degree (i.e.) d = 2, such constancy of variance is possible if the design points are so selected as to satisfy the following relations.

Relations:-

(A) $\sum x_i = 0 \quad \sum x_i x_j = 0, \quad \sum x_i x_j^2 = 0$

$\sum x_i^3 = 0 \quad \sum x_i x_j^3 = 0 \quad \sum x_i x_j x_k^2 = 0$

$\sum x_i x_j x_k x_l = 0$ for all $i \neq j \neq k \neq l$.

(B) $\sum x_i^2 = \text{constant} = N \lambda_2$.

$\sum x_i^4 =$ -do- $= 3N \lambda_4$. for all i.

(C) $\sum x_i^2 x_j^2 = \text{Constant}$ for all $i \neq j$

(D) $\sum x_i^4 = 3 \sum x_i^2 x_j^2$. for all $i \neq j$.

(E) $\dfrac{\lambda_4}{\lambda_2^2} > \dfrac{v}{v+2}$.

In the above relations, the summations are over the design points.

In case of third order designs the following further relations also should be satisfied.

(A_1) Each of the sum of powers or products of powers of x_i's in which at least one power is odd, is zero.

(B_1) $\sum x_i^6 = \text{constant} = 15N \lambda_6$ for all i.

(C_1) (i) $\sum x_i^2 x_j^4 = \text{constant}$

(ii) $\sum x_i^2 x_j^2 x_k^2 = \text{constant}$ for all $i \neq j \neq k$.

(D_1) (i) $\sum x_i^6 = 5 \sum x_i^2 x_j^4$.

(ii) $\sum x_i^2 x_j^4 = 3 \sum x_i^2 x_j^2 x_k^2$.

(E_1) $\dfrac{\lambda_6 \lambda_2}{\lambda_4^2} > \dfrac{v+2}{v+4}$,

the summations being over the design points.

CHAPTER II.

Construction of second order rotatable designs through balanced incomplete block designs:-

Construction of second order rotatable designs is nothing but getting a design matrix in which the relations A,B,C,D,E mentioned earlier, are satisfied. Now we shall discuss a method of getting such design matrices which will give second order rotatable designs.

Each point in a design is essentially a combination of the levels of different factors. We then propose to first take some unknown levels to be denoted by a, b, c, ...etc. excepting that some of these may be zero also, and get a factorial design out of these unknown levels. Thus, if there are four factors each at two levels denoted by 'a' and 'b', the sixteen combinations will be of the form

a a a a

a b b b

b a a b

a b b a

. . . .

. . . . etc.

Next we shall have another design in v factors of the form 2^v where the two levels are +1 and -1 . By multiplying any combination of the first design with all the combinations of the second design 2^v, we shall get 2^p distinct combinations where p denotes the number of non zero unknown levels in the combination considered of the first design. As a matter of fact if there be only p unknowns in a combination together with some zeros, we have to multiply only the nonzero levels in the combination with each of the 2^p combinations of +1 and -1.

For example, in the design with four factors if the combination a b b b be "multiplied" by the 2^4 combinations of the levels +1 and -1, we shall get the 16 combinations as below:

a b b b	when multiplied by	1 1 1 1
a b b -b	-do-	1 1 1 -1
a b -b b	-do-	1 1 -1 1
a b -b -b	-do-	1 1 -1 -1
. . . . .		. . . . .
. . . . . etc.		. . . . . etc.

If one of unknown levels say (a) be zero, all the sixteen combinations will not be distinct but only eight of them will be distinct as by associating +1 and -1 with zero we get the same thing. We shall consider in future only those combinations which are distinct unless otherwise mentioned.

We have by now come across three types of combinations, namely,

(i) Factorial combinations of the unknown levels a, b, ..etc. together with 0.

(ii) Factorial combinations of levels +1 and -1.

(iii) Combinations when each 0, a, b.. etc. is associated with +1 and -1 through multiplication.

The first type of factorial combinations will be called combinations of unknown levels, the second will be called associate combinations. The third combinations will actually constitute the design points and hence they will be referred to as the design points.

It will be seen easily that if a design be formed by including all the distinct points which are got by multiplying any combination of the unknown levels with all the associated combinations, these points will always satisfy relations A and A_1. When $v > 4$, or $p > 4$, these relations A and A_1 will also be satisfied when a suitable fraction of the 2^v or 2^p associate combinations, as the case may be, be so chosen for multiplication to obtain a second order rotatable design, that no interaction with less than five factors is confounded in these 2^v or 2^p associate combinations. In case of third order designs the fraction should be so chosen that no interaction with less than seven factors is confounded. For satisfying the other relations B, C, D, E one or more combinations of the unknown levels will have to be chosen suitably. A method for the choice of such combinations through which second order rotatable designs can be obtained has been described below:

Let there be a balanced incomplete block design with the parameters (v, b, r, k, λ). Let us write the design in the form of b x v matrix, the elements of which are zero and 'a'. If in any block a particular treatment occurs the element in that block corresponding to that treatment will be 'a', otherwise zero. Each row of the matrix or block of the BIBD can be considered to give a combination of zero and the unknown level 'a', By multiplying each of these 'b' combinations thus obtained through the BIBD with 2^k since $p = k$ here, or a suitable fraction of the associate combinations, we shall get a number of design points less than or equal to $b2^k$. These points which we hereafter will denote as $a(v\ b\ r\ k\ \lambda) \times 2^k$ or $a(v\ b\ r\ k\ \lambda) \times$ suitable fraction of 2^k, will satisfy all relations except D and, E, as constancy of replication will satisfy relation B and that of replication of pairs of treatments will satisfy relation C. If $r = 3\lambda$ in the BIBD, then relation D will also be satisfied and hence these points together with at least one central point so as to satisfy E, will give a second order rotatable design in 'v' factors. The unknown level 'a' has to be obtained from the relation $\sum x_i^2 = N$.

For example, in the following designs for 4,7,10,16 factors the relation $r = 3\lambda$ is satisfied and hence through each of these designs a second order rotatable design can be obtained by including at least one central point. The designs points for 4,7,10,16 factors are respectively,

(i) $a(v = 4,\ k = 2,\ \ r = 3,\ b=6,\ \lambda = 1) \times 2^2$

(ii) $a(v = 7,\ k = 3,\ \ r = 3,\ b=7\ \ \lambda = 1) \times 2^3$

(iii) $a(v = 10, k = 4,\ \ r = 6,\ b=15,\ \lambda = 2) \times 2^4$

(iv) $a(v = 16, k = 6,\ \ r = 6,\ b=16,\ \lambda = 2) \times (1/2 \text{ repl. } 2^6)$.

The number of non central points in these second order rotatable designs with 4,7,10,16 factors obtainable through them will respectively be (i) 6×2^2, (ii) 7×2^3, (iii) 15×2^4, (iv) 16×2^5. The design for seven factors has been presented in Appendix I. This design has also been obtained by Box and Behnken (1960) through the first order design.

If the relation $r = 3\lambda$ does not hold in any BIB design, we can always get a second order rotatable design through it by taking some more combinations involving one more unknown level 'b' and then multiplying these with requisite number of associate combinations. The combinations to be taken are either the v combinations,

b 0 0

0 b 0 ...0

0 0 b ...0

. . . . . .

. . . . . .

0 0 0 ... b

obtained from the combination (b 0 ....0) by permuting over the different factors or the combination (b b . . . b) according as $r < 3\lambda$ or $r > 3\lambda$.

The same letter 'b' has been used in two different contexts. It is used once to denote an unknown level 'b' and also to denote the total number of blocks 'b' be in the incomplete block design (v, r, b, k, λ). There is no possibility of these two notations being confused.

We have so far used two types of combinations viz. one involving the unknown 'a' and the other involving 'b' The combinations obtained through BIB designs will hereafter be called the a-combinations, while the v combinations obtained from (b 0 ... 0) by permutation will be called combinations of the type (b 0 . . . 0). The design points obtained by combinations of type (b 0 ...0) and (b b ...b) after multiplication with requisite associate combinations will hereafter be denoted as (b 0 ... 0) x 2^1 and (b b ... b) x 2^v or (b b ....b) x (suitable fractional repl. 2^v) respectively.

In the above design Σx_i^4 and $\Sigma x_i^2 x_j^2$ will be functions of a and b. From the relation $\Sigma x_i^4 = 3 \Sigma x_i^2 x_j^2$, we shall get an equation connecting a and b. This equation will always give a positive solution of b^2/a^2, provided that extra sets are suitably chosen taking into account the fact whether $r < 3\lambda$ or $r > 3\lambda$. For determining the unknowns a and b, we have one more equation (viz.) $\sum x_i^2 = N$ where N is the total number of points including central points.

(i) For example, in the design v = 8, k = 2, r = 7, b = 28, $\lambda = 1$, $r > 3\lambda$ and hence the combination (b b . . . b) has to be taken together with the 28 a-combinations, given by the BIBD, in order to get a second order rotatable design in 8 factors. The design points will be (i) a(v = 8, k = 2, r = 7, b = 28, λ = 1) x 2^2, (ii) (b b . . . b) x (1/4 repl. 2^8). In this design we have,

$$\sum x_i^4 = 28a^4 + 64b^4.$$
$$\sum x_i^2 x_j^2 = 4a^4 + 64b^4.$$

Hence, relation D gives the equation

$$28a^4 + 64b^4 = 3(4a^4 + 64b^4)$$
$$b^4/a^4 = 1/8.$$

The above equation together with $\sum x_i^2 = 28a^2 + 64b^2 = N$, will completely determine the two unknowns a and b. The number of points in this design will be 176. No central points are necessary in this design though they may be added if otherwise necessary.

(ii) Again considering the BIBD (v = 8, k = 4, r = 7, b = 14, λ = 3) another second order rotatable design in 8 factors can be obtained by taking a further combination of the type (b 0 . . . 0) as in this case $r < 3\lambda$.) The design points will be (i) $a(v = 8,\ k = 4,\ r = 7,\ b = 14, \lambda = 3) \times 2^4$ (ii) $(b\ 0\ldots0) \times 2^1$.

The equation from relation D comes out in this case,

$$112a^4 + 2b^4 = 3(48a^4)$$

whence $b^2/a^2 = 4$.

As all the points are equidistant, at least one central point will be necessary. The number of non-central points in the design is 224 + 16 = 240.

The above two exmaples show that by properly choosing the corresponding BIB design, the number of design points can be reduced. A list of second order rotatable designs together with the additional type of combination when necessary, to be taken for the construction of such designs upto 16 factors, has been presented in Appendix II together with relevant details. Designs for larger number of factors can, however, be obtained in the same lines. It will be seen that all these designs have either 3 or 5 levels according as extra combinations with b are taken or not.

It can be easily seen that all the central composite second orders rotatable designs come as particular cases of the method indicated through BIB designs. For in this case we can take the design points $a(v,\ r = 1,\ b = v,\ k = 1, \lambda = 0) \times 2^1$ along with the combination of type $(b\ b \ldots b) \times$ (suitable fractional repl. 2^v) to get a second order rotatable design in 'v' factors.

Construction of Second order rotatable designs through balanced incomplete block designs with unequal block sizes.

In the previous chapter we have construted second order rotatable designs by choosing treatment combinations of unknown levels with the help of balanced incomplete block designs. In this chapter we shall discuss a method of obtaining second order rotatable designs with the help of balanced incomplete block designs with unequal block sizes. These designs in general do not lead to a second order rotatable design, but through a particular class of them described below, such designs, can always be obtained.

A balanced incomplete block design with unequal block sizes $(v, b, r, k_1, k_2, \ldots k_m, \lambda)$ is defined as an arrangement of v treatments in b blocks of sizes $k_1, k_2 \ldots k_m$ such that each treatment occurs exactly in r blocks and any pair of treatments occur in exactly λ blocks. If b_i be the number of blocks of size k_i, we have for these designs

$$vr = \sum_{i=1}^{m} b_i k_i.$$

$$\lambda v(v-1) = \sum_{i=1}^{m} b_i k_i . (k_i - 1).$$

For the purpose of constructing second order rotatable designs we take a particular class of these designs in which the replication of every treatment is a constant r_i in the set of b_i blocks each of size k_i for all i. So for this class of designs we must further have $vr_i = b_i k_i$, $\sum r_i = r$.

By taking a BIBD (v, b, r, k, λ) and omiting any treatment (say t_v) wherever it occurs we get a balanced incomplete block design with unequal block sizes (v, b, r, k, $k-1$, λ) with $r_1 = r - \lambda$, $r_2 = \lambda$, $k_1 = k$, $k_2 = k-1$.

Through these balanced incomplete block designs with unequal block sizes (v, b, r, k_1, k_2, ...k_m, λ) second order rotatable designs can be obtained as follows. Let us write the balanced incomplete block design with uneuqal block sizes (v, b, r, k_1, k_2, . . . k_m, λ) as a $b \times v$ matrix the elements of which are zero and unity. As before we write the unknown level 'a' wherever unity occurs in the set of b_1 blocks of size k_1, similarly 'b' for unity in the set of b_2 blocks of size k_2 and so on. The design points are generated as before by multiplying each of the sets with 2^{k_1} or a fraction of 2^{k_1} associate combinations with levels 1 and -1 of k_1 factors, depending on k_1. This procedure gives a number of design points in which the relations A and B are satisfied. Relation (C) gives (m-1) equations where m is the number of block sizes. If $r = 3\lambda$ relation D is automatically satisfied. According as $r \lessgtr 3\lambda$ we have to add as before the points $(x\ 0....0) \times 2^1$ or $(x\ x\ ...x) \times$ suitable fraction of 2^v and solve for the unknowns from the relations C and D to give a second order rotatable design.

With this method the second order rotatable design for $v = 6, 8,$ and 10 factors are obtained with fewer number of points than when they are obtained through BIB designs. These designs contain 44, 144, 196 points respectively whereas those obtained through BIB designs contain 92, 176, 240 points respectively.

By trial and error two new designs one for five factors and another for 9 factors are also obtained on the same lines with slight modifications. These designs contain 44 and 194 points respectively where as those obtained through BIB designs contain 56 and 224 points respectively.

Method of construction of these designs has been indicated below:

<u>Design in 6 factors</u>:- We take the balanced incomplete block design with unequal block sizes ($v = 6$, $r_1 = 2$, $r_2 = 1$, $k_1 = 3$, $k_2 = 2$, $b = 7$, $\lambda = 1$) obtained by omitting any one treatment from the BIB design ($v = 7$, $r = 3$, $k = 3$, $b = 7$, $\lambda = 1$). We attach the unknown levels 'a' and 'b' with blocks of sizes 3 and 2 respectively and generate the design points by multiplying with the suitable number of associate combinations.

		Combinations of unknown levels.	Number of design points
	1 2 4	a a 0 a 0 0	8
	2 3 5	0 a a 0 a 0	8
a	3 4 6	0 0 a a 0 a	8
	1 5 6	a 0 0 0 a a	8
	1 3	b 0 b 0 0 0	4
b	2 6	0 b 0 0 0 b	4
	4 5	0 0 0 b b 0	4
		Total number of design points	44

We have for this design $\sum x_i^2 = 16a^2 + 4b^2$, $\sum x_i^4 = 16a^4 + 4b^4$, $\sum x_i^2 x_j^2 = 8a^4 + 4b^4$, therefore $\sum x_i^4 = 3 \sum x_i^2 x_j^2$

$s^2 = b^4/a^4 = 2$. 'a' is obtained from the relation $\sum x_i^2 = 16a^2 + 4b^2 = N$. No central point is necessary. ~~unless otherwise~~. Hence we get a second order rotatable design for 6 factors in 44 points.

Design in 8 factors:-

We take the BIB design with unequal block sizes ($v = 8, r_1 = 3, r_2 = 1, k_1 = 3, k_2 = 2, b = 12, \lambda = 1$) which is obtained by omitting any treatment from the BIB design ($v = 9, k = 3, r = 4, b = 12, \lambda = 1$). We attach the unknown levels 'a' and b with blocks of sizes 3 and 2 respectively. As $r > 3\lambda$ in this case we further add combinations of type $(c\ c \ldots c) \times 1/4$ repl. 2^8.

Here we have

$$\sum x_i^2 = 24a^2 + 4b^2 + 64c^2.$$
$$\sum x_i^4 = 24a^4 + 4b^4 + 64c^4.$$
$$\sum x_i^2 x_j^2 = 8a^4 + 64c^4 = 4b^4 + 64c^4.$$

therefore $s^2 = b^4/a^4 = 2.$, and $t^2 = a^4/c^4 = 16.$

'a' is obtained from $\sum x_i^2 = N$.

No central point is necessary ~~unless otherwise~~. So we get a a design for 8 factors in 144 points.

Design in 10 factors:-

We take the BIB design of unequal block sizes ($v = 10, r_1 = 3, r_2 = 2, k_1 = 5, k_2 = 4, b = 11, \lambda = 2$). which is obtained by omiting any treatment from the BIB design ($v = 11, r = 5, k = 5, b = 11, \lambda = 2$). We attach the unknown levels 'a' and 'b' with blocks of sizes 5 and 4 respectively. As $r < 3\lambda$ we further add combinations of type $(c\ 0\ 0 \ldots 0) \times 2^1$.

We have

$$\sum x_i^2 = 48a^2 + 32b^2 + 2c^2.$$
$$\sum x_i^4 = 48a^4 + 32b^4 + 2c^4.$$
$$\sum x_i^2 x_j^2 = 16a^4 + 16b^4 = 32a^4.$$

Therefore, $s^2 = b^4/a^4 = 1$ and $t^2 = c^4/a^4 = 8$.

a^2 is obtained from $\sum x_i^2 = N$.

No central point is necessary. ~~unless otherwise.~~ Thus we get a design for 10 factors in 196 points.

In the case of 5 and 9 factors, suitable BIB designs with unequal block sizes of the type we require and which can give reasonably small number of design points are not available. But by trial and error designs with fewer number of points have been obtained proceeding on same lines with slight modifications.

Design in 5 factors:-

We take the following incomplete block design with unequal block sizes (not balanced) and attach the unknown levels a and b as indicated below and obtain the design points by multiplying each of the blocks with the requisite associate combinations.

	Combinations of unknown levels.	Number of design points.
1 2 4	a a 0 a 0	8
2 3 5	0 a a 0 a	8
3 4	0 0 a a 0	4
3 4	0 0 a a 0	4
1 5	a 0 0 0 a	4
1 5	a 0 0 0 a	4

4 5	0 0 0 b b	4
1 3	b 0 b 0 0	4
2	0 b 0 0 0	2
2	0 b 0 0 0	2
	Total number of design points	44

We have

$$\sum x_i^2 = 16a^2 + 4b^2$$
$$\sum x_i^4 = 16a^4 + 4b^4$$
$$\sum x_i^2\, x_j^2 = 8a^4 = 4b^4$$

therefore $s^2 = b^4/a^4 = 2.$

a^2 is obtained from $\sum x_i^2 = N.$

No central point is necessary. ~~unless otherwise~~. So we get a design for five factors in 44 points.

9 factors:-

We take the following incomplete block design with unequal block sizes (not balanced) and attach the unknown level 'a' as indicated below and obtain the design points by multiplying each of the blocks with the requisite associate combinations

1 2 3 5 8

2 3 4 6 9

5 6 7 9 1

3 4 5 7

2 6 7 8

1 4 8 9 attach the unknown level 'a' with each of

4 5 6 8 the blocks.

7 8 9 3

1 2 4 7

1 3 6

1 3 6

2 5 9

2 5 9

Thus this generates 176 design points. To this further add the 18 design points of type $(b\ 0 \ldots\ 0) \times 2^{1}$.

We have

$$\Sigma x_i^2 = 80a^2 + 2b^2$$

$$\Sigma x_i^4 = 80z^4 + 2b^4$$

$$\Sigma x_i^2 x_j^2 = 32a^4$$

Relation D gives $s^2 = b^4/a^4 = 8$.

a^2 is obtained from $\Sigma x_i^2 = N$.

No central point is necessary. unless otherwise. Thus, we get a design for 9 factors in 194 points.

CHAPTER IV

Construction of third order rotatable designs through doubly balanced incomplete designs

The a-combinations chosen through BIB designs for the construction of second order rotatable designs donot usually satisfy the relations $c_1(ii)$ together with D, $D_1(i)$ and $D_1(ii)$ and E. If the BIBD happens to be a doubly balanced i.e. in addition to pairs of treatments occuring a constant number of times, λ, the triplets of treatments also occur a constant number of times, μ in the blocks (Calvin 1954), the relation $C_1(ii)$ is also satisfied. For satisfying the other relations D, $D_1(i)$ and $D_1(ii)$, we have to introduce combinations involving the fresh unknowns which can be evaluated by solving the equations obtained through D, $D_1(i)$ and $D_1(ii)$.

For example, each of the following designs are doubly balanced.

(i) $v = 3,\ k = 2,\ r = 2,\ b = 3,\ \lambda = 1,\ \mu = 0.$

(ii) $v = 4,\ k = 3,\ r = 3,\ b = 4,\ \lambda = 2,\ \mu = 1.$

(iii) $v = 5,\ k = 3,\ r = 6,\ b = 10,\ \lambda = 3,\ \mu = 1.$

(iv) $v = 6,\ k = 3,\ r = 10,\ b = 20,\ \lambda = 4,\ \mu = 1.$

(v) $v = 8,\ k = 4,\ r = 7,\ b = 14,\ \lambda = 3,\ \mu = 1.$

(vi) $v = 9,\ k = 3,\ r = 28,\ b = 84,\ \lambda = 7,\ \mu = 1.$

(vii) $v = 10,\ k = 4,\ r = 13,\ b = 30,\ \lambda = 4,\ \mu = 1.$

(viii) $v = 11,\ k = 5,\ r = 15,\ b = 33,\ \lambda = 6,\ \mu = 2.$

(ix) $v = 12,\ k = 6,\ r = 11,\ b = 22,\ \lambda = 5,\ \mu = 2.$

With the help of each of these designs which will supply us the a-combinations as described earlier for second order rotatable designs, third order designs, both sequential and non-sequential, can be obtained by taking one or more of the combinations of the type (b 0 0....0), (c c 0 0....0), (d d.....d) involving fresh unknown levels b, c, d and multiplying them with the associate combinations as earlier. The combination (c c 0 0....0) will give c_2^v combinations when permuted over all the v factors and these v(v-1)/2 combinations will hereafter be called combinations of type (c c 0 0....0). The design points obtained from the combinations of type (c c 0 0....0) after multiplying each one of them with the 2^2 associate combinations will be denoted as (c c 0....0)x 2^2. The other two types of combinations have been described earlier.

As an example, we can get a third order non-sequential rotatable design in 8 factors with the help of the following design points:

(i) 672 points from a-(9, 3, 28, 84, 7, 1) x 2^3

(ii) 256 " " (b b....b) x ½ repl.2^9

(iii) 256 " " (c c....c) x ½ repl.2^9

(iv) 18 " " (d 0....0) x 2

The equations for solving the unknowns come out as:

From D: $(28 \times 8)a^4 + 256(b^4 + c^4) + 2d^4$

$$= (21 \times 8)a^4 + 3\times256(b^4 + c^4)$$

From D_1(i): $(28\times8)a^6 + 256(b^6 + c^6) + 2d^6$

$$= (35 \times 8)a^6 + 5\times256(b^6 + d^6)$$

From D_1(ii): $(7\times8)a^6 + 256(b^6 + c^6)$

$$= 3\times8a^6 + 3\times256(b^6 + c^6)$$

Solving these equations we get

$$b^2/a^2 = 0.392768$$

$$c^2/a^2 = 0.122376$$

$$d^2/a^2 = 3.914868$$

The value of a can be obtained from

$\sum x_i^2 = N$. This design contains 1202 points.

Sequential third order designs can be obtained with the help of the same types of combinations viz. a- combinations through the doubly B.I.B designs together with one or more of the types of combinations (b 0 0....0), (c c 0,....0) and (d d.... d). For example, we can get a sequential third order rotatable design in 8 factors with the help of the following design points:

Block I : (i) 128 points of (d d ....d) x (1/2 repl. 2^8)

(ii) 16 points of (e 0 ....0) x 2^1.

Block II: (iii) 224 points of a(v = 8, k = 4, r=7, v=14, λ=3, u=1

μ= 1) x 2^4

(iv) 112 points of (c c 0 ....0) x 2^2.

The design equations will lead to

From

$$(D)\ 112a^4 + 28c^4 + 128d^4 + 2e^4 = 144a^4 + (3x128)d^4 + 12c^4.$$

$$D_1\ (i)\ 112a^6 + 28c^6 + 128d^6 + 2e^6 = 240a^6 + (5x128)d^6 + 20c^6.$$

$$D_1(ii)\ 48a^6 + 128d^6 + 4c^6 = 48a^6 + (3x128)d^6.$$

There is one more equation to make each block a second order rotatable design.

This equation gives

$$2e^4 + 128d^4 = (3x128)d^4.$$

Putting $a^2/d^2 = s$, $c^2/d^2 = u$, $e^2/d^2 = v$.

The equations become,

$$8u^2 + v^2 = 16s^2 + 128$$

$$4u^3 + v^3 = 64s^3 + 256$$

$$4u^3 = 2x128.$$

$$v^2 = 128.$$

Whence $u = 4$, $v = \sqrt{128}$ and $s = \sqrt{8}$.

The value of d cam be obtained from $\sum x_i^2 = N$.

The number of central points to be added to the two blocks will be determined from

$$\Sigma_1 x_i^2 / \Sigma_2 x_i^2 = (144 + n_{10})/(336 + n_{20})$$

Where $\sum_1 x_i^2$ is summed over the points in the first block and $\sum_2 x_i^2$ is summed over the points in the second block. As $\sum_1 x_i^2$ and $\sum_2 x_i^2$ are functions of the unknown levels which have been solved out, n_{10}, n_{20} the number of central points to be added to first and second block respectively, can be obtained from the above relation.

Actually $\sum_1 x_i^2 = 128d^2 + 2e^2$.

$\sum_2 x_i^2 = 112a^2 + 28c^2$.

Therefore substituting for s, u and v obtained earlier, n_{10}, n_{20} can be obtained from

$$(64+v)/(56s+14u) = (144 + n_{10})/(336 + n_{20})$$

Thus we get a sequential third order rotatable design for 8 factors in 480 non-central points.

2: Third order rotatable designs through complementary BIBD.

If a BIB design (which is not doubly balanced) together with its complementary BIB design, repeated once if necessary, be taken to generate the a-combinations as before, we can get points through these a-combinations which will satisfy C_1(ii), as μ will be a constant in the combined complementary BIBD, together with all the other relations (Sprott, D.A., A.M.S., 1955), excepting D, D_1(i) and D_1(ii), E. For satisfying these relations we have to take one or more of the types of combinations (b 0 ....0), (c c 0 ....0) and (d d ....d) involving fresh unknowns.

For example, a non-sequential third order rotatable design in 10 factors can be obtained with the following points:

(i) (18 x 32) points of $a(v = 10, k = 5, r=9, b=18, \lambda=4) \times 2^5$.

(ii) (18x 32) points of a(complementary BIBD of above BIBD

$v = 10, k = 5, r=9, b=18, \lambda=4) \times 2^5$

(iii) 20 points of $(b\ 0 \ldots 0) \times 2^1$.

(iv) 20 points of $(c\ 0 \ldots 0) \times 2^1$.

(v) 180 points of $(d\ d\ 0 \ldots 0) \times 2^2$.

Here $\mu = 3$ in the combined complementary BIB designs.

The relations D, D_1(i), D_1(ii) give the equations,

$$(18 \times 32)a^4 + 2b^4 + 2c^4 + 36d^4 = (24 \times 32)a^4 + 12d^4.$$

$$(18 \times 32)a^6 + 2b^6 + 2c^6 + 36d^6 = (40 \times 32)a^6 + 20d^6.$$

$$(8 \times 32)a^6 + 4d^6 = (9 \times 32)a^6.$$

Putting $b^2/a^2 = s$, $c^2/a^2 = t$, $d^2/a^2 = u$,

We get $u = 2$, $s^2 + t^2 = 48$, $s^3 + t^3 = 288$.

Solving, we get $s = 6.494805$

$t = 2.411955$

Thus, we get a non-sequential third order rotatable design in $1152 + 20 + 20 + 180 = 1372$ points.

Sequential third order designs can also be constructed with the help of the complementary BIB designs together with other three types of combinations involving fresh unknowns.

For example, with the following points we can get a sequential third order design for 7 factors.

<u>Block I</u>: (i) 112 points of $a(v=7, k=4, r=4, b=7, \lambda=2) \times 2^4$

(ii) 14 points of $(b\ 0 \ldots 0) \times 2^1$.

<u>Block II</u>: (iii) 112 points of a(complementary BIBD of above BIBD

$v=7, k=3, r=3, b=7, \lambda=1) \times 2^3$

each of these 56 points in (iii) being repeated once more.

Here $\mu = 1$ in the combined complementary BIB designs. The complementary BIBD in (iii) has to be repeated once more as each a-combination from the first BIBD gives 16 design points on multiplication with the associate combinations and each a-combination from the second complementary BIBD gives only 8 design points on multiplication with the associate combinations. Hence unless all the points obtained from the a-combinations of the complementary design be repeated once, $\sum x_i^2 \, x_j^2 \, x_k^2$ will not be a constant for all i, j, k.

In the case of the above design points condition D_1(ii) is satisfied as $\lambda = 3\mu$ and relation that each block is a second order design is satisfied as $r = 3\lambda$ in block II.

Hence relations D and D_1(i) gives the equations,

D : $112a^4 + 2b^4 = (3 \times 48)a^4$.

D_1(i): $112a^6 + 2b^6 = (5 \times 48)a^6$

Putting $b^2/a^2 = t$,

these equations become

$$t^2 = 16,$$

$$t^3 = 64.$$

Therefore, $t = 4$.

'a' can be obtained from the equation $\sum x_i^2 = N$. Number of central points to be added to the two blocks is given by the relation,

$$48a^2/(64a^2 + 2b^2) = (112 + n_{10})/(126 + n_{20})$$

i.e. $48/(64 + 2t) = (112 + n_{10})/(126 + n_{20})$.
Thus we get a sequential third order rotatable design for 7 factors in 238 non central points, with same central points to be added.

Appendices III and IV present respectively non-sequential and sequential third order rotatable designs upto 15 factors obtained by utilising doubly balanced incomplete block designs or by a BIB design together with its complementary BIB design.

In this chapter a general method of construction of third order rotatable designs both sequential and non-sequential, for any number of factors through doubly balanced incomplete block designs and complementary BIB designs has been described. In particular, with this method we could get sequential third order rotatable designs for 6 and 7 factors in 260 and 238 non central points respectively. But by trial and error designs for 6 and 7 factors in fewer number of design points have been obtained on the same lines with slight modifications. These are presented below. These designs seem to contain reasonably small number of design points. For estimating the 84 and 120 coefficients in the third order surfaces of 6 and 7 factors, we get sequential third order rotatable designs in 180 and 182 non central points respectively.

<u>Sequential third order rotatable design in 7 factors:</u>

We take the following 182 non central points into two blocks for the design.

Block I : (i) 56 points of $b(v = 7, k=3, r=3, b=7, \lambda=1) \times 2^3$.

Block II: (ii) 112 points of a(complementary of the above BIBD $v = 7, k=4, r=4, b=7, \lambda=2) \times 2^4$.

(iii) 14 points of $(c\ 0 \ldots 0) \times 2^1$.

With these points we have,

$$\Sigma x_i^2 = 64a^2 + 24b^2 + 2c^2.$$
$$\Sigma x_i^4 = 64a^4 + 24b^4 + 2c^4.$$
$$\Sigma x_i^6 = 64a^6 + 24b^6 + 2c^6.$$
$$\Sigma x_i^2 x_j^2 = 32a^4 + 8b^4.$$
$$\Sigma x_i^2 x_j^4 = 32a^6 + 9b^6.$$
$$\Sigma x_i^2 x_j^2 x_k^2 = 16a^6 = 8b^6.$$

Relations C_1(ii), D, D1(i), D_1(ii) give C_1(ii) in order

$$s^3 = b^6/a^6 = 2.$$

(D) $64a^4 + 24b^4 + 2c^4 = 96a^4 + 24b^4.$

D_1(i) $64a^6 + 24b^6 + 2c^6 = (5 \times 32)a^6 + 40b^6$

Relation D_1(ii) is automatically satisfied.

from (D) we get $t^2 = c^4/a^4 = 16.$

from D_1(i) we get $t^3 = c^6/a^6 = 64.$

Block I obviously forms second order rotatable design, with atleast one central point.

The number of central points n_{10}, n_{20} to be added to the two blocks can be determined by

$$\Sigma_1 x_i^2 / \Sigma_2 x_i^2 = (56+n_{10})/(126 + n_{20}) = 24b^2/(64a^2 + 2c^2).$$

<u>Sequential third order rotatable design in 6 factors:-</u>

We take the following 180 non central points into two blocks for the design.

Block I: b $\left[\begin{array}{ccc} 1 & 2 & 4 \\ 2 & 3 & 5 \\ 3 & 4 & 6 \\ 5 & 6 & 1 \\ & 4 & 5 \\ & 4 & 5 \\ & 6 & 2 \\ & 6 & 2 \\ & 1 & 3 \\ & 1 & 3 \end{array}\right]$ (b)

Block II: a $\left[\begin{array}{cccc} 1 & 2 & 3 & 6 \\ 1 & 3 & 4 & 5 \\ 2 & 4 & 5 & 6 \\ & 3 & 5 & 6 \\ & 3 & 5 & 6 \\ & 1 & 4 & 6 \\ & 1 & 4 & 6 \\ & 1 & 2 & 5 \\ & 1 & 2 & 5 \\ & 2 & 3 & 4 \\ & 2 & 3 & 4 \end{array}\right]$

and $(c\ 0 \ldots 0) \times 2^1$.

The treatment combinations of the unknown levels are generated as before from the incomplete block designs (not balanced) and the design points are obtained by multiplying each of them with the requisite associate combinations.

$$\sum x_i^2 = 64a^2 + 24b^2 + 2c^2$$
$$\sum x_i^4 = 64a^4 + 24b^4 + 2c^4$$
$$\sum x_i^6 = 64a^6 + 24b^6 + 2c^6$$
$$\sum x_i^2 x_j^2 = 32a^4 + 8b^4$$
$$\sum x_i^2 x_j^4 = 32a^6 + 8b^6$$
$$\sum x_i^2 x_j^2 x_k^2 = 16a^6 = 8b^6$$

Relations C_1(ii), D, D_1(i), D_1(ii) give

$$s^3 = b^6/a^6 = 2.$$
$$t = c^2/a^2 = 4.$$

Block I obviously forms a second order rotatable design, with atleast one central point.

The number of central points n_{10}, n_{20} to be added to the two blocks can be determined by

$$\Sigma_1 x_i^2 / \Sigma_2 x_i^2 = (56+n_{10})/(124+n_{20}) = 24b^2/(64a^2+2c^2)$$

CHAPTER V

Multifactorial designs suitable for studying simultaneously (i) the main effects and first order interactions of the different factors and (ii) the second degree response surface:

Rotatable designs are advocated by Box and Hunter for the exploration of response surfaces in multifactorial experiments. The most commonly used designs are the second order rotatable designs for the exploration of the second degree surfaces. It may happen that an experimenter is interested in studying the second order surface along with main effects and first order interactions of the factors involved in a multifactorial experiment at the same time.

All the second order rotatable designs available donot lend themselves to serve for such an experimentation. With them all the main effects and first order interactions are not estimable and even if some of them are estimable, their precisions will not be the same. We will show later that all the second order rotatable designs constructed through BIB designs as indicated in the previous chapters, will lend themselves very efficiently for such double-purpose experimentation.

Let us denote the set of design points of any second order rotatable design by (R.). We add further a set of a few more design points (A), suitably chosen, to the experiment. The part (R) can be used to exploit the response surface efficiently, while the design with all the (R + A) points, if necessary, without a set of certain design points (B) from R can be used to estimate the main effects and first order interactions of the factors involved, with the same precision. The sets of points (A) and (B) are suitably chosen such that a two way tables of frequencies for any two factors give either equal or proportional frequencies for the various combinations of levels in the experiment. This ensures that all the main effects and first order interactions are estimable with equal precision.

The second order rotatable design constructed through BIB designs are particularly flexible for such a double purpose experimentation. By simply adding a few more central points to such second order rotatable designs, designs useful for the two purposes can be obtained. These designs are interesting in the sense that even after adding such further central points, these designs still remain to be rotatable designs and further, with a higher efficiency for exploring the response surface. Thus, in fact, we are able to construct second order rotatable designs of type (R + A) for studying the multifactor dependence, and a part of the design can also be used for estimating the main effects and interactions with equal precisions. Construction of such designs is indicated below.

Case (i):- When a second order rotatable design (R) for v factors is obtained from a BIB design (v, r, b, k, λ) with $r = 3\lambda$, and n_0 central points, then by adding $(A) = n(r^2/\lambda - b) - n_0$ more central points (so many different observations being made at the central point), where n is the number of associate combinations used for multiplication to generate the design points, we always get a design of the type required for the double-purpose experimentation. The whole design (R + A) consisting of all the design points $a(v, r, b, k, \lambda) \times 2^3$ and $n(r^2/\lambda - b)$ central points can be used as the second order rotatable design. The main effects and first order interactions are also estimable mutually independently with all these design points (R + A) for all the two way tables, for any two factors are orthogonal in the frequencies of the various combinations of the levels in the design as seen from the table below:

		Factor F_1			
		$-a$	a	0	
Factor F_2	$-a$	$\lambda n/4$	$\lambda n/4$	$(r-\lambda)n/2$	
	a	$\lambda n/4$	$\lambda n/4$	$(r-\lambda)n/2$	
	0	$(r-\lambda)n/2$	$(r-\lambda)n/2$	$(b-2r+\lambda)n+n(r^2/\lambda-b)$	
					bn

$n(r^2/\lambda - b)$ is always positive in a BIBD and is an integer in this case as $r = 3\lambda$.

<u>Case (ii)</u> :- When a second order rotatable designs (R) for v factors is obtained from a BIBD (v, r, b, k, λ) along with 2v design points $(b\ 0 \ldots 0) \times 2^1$ and n_0 central points, then also we have to add $(A) = [n(r^2/\lambda - b) - n_0]$ central points for the purpose.

In this the (R), (A) are

$(R) = a(v, r, b, k, \lambda) \times n + (b\ 0 \ldots 0) \times 2^1 + n_0$ central points

$(A) = [n(r^2/\lambda - b) - n_0]$ central points.

For exploiting the response surface the whole design (R + A) can be used efficiently. For estimating the main effects and first order interactions the design points a(v, r, b, k, λ)n and $n(r^2/\lambda - b)$ central points can be used (i.e.) from whole design (R + A), the set of points (B) $\equiv$ (b 0 ....0) x 2^1 are omited for the time being to estimate various first order interations and main effects. From these points the main effects and first order interactions can be estimated mutually independently as is shown earlier. The only condition is that $n(r^2/\lambda - b)$ should be an integer.

Case (iii): When a second order rotatable design (R) for v factors is obtained from a BIBD (v, r, b, k, λ) along with the set of points (b b ....b) x suitable fractional repl. of 2^v and n_0 central points, we need not add any more further design points. The whole design (R) can be used for the exploration of the response surface and the part consisting of the design points (b b ....b) x suitable fractional repl. of 2^v can be used to study the main effects and in fact all the interactions upto order three, with equal precisions.

Thus, we have seen when the second order rotatable designs are constructed with the help of BIB designs, by adding a few more central points to these designs, we can get designs in which the response surface and the main effects and first order interactions can be studied simultaneously. Though we have made the study of constructing such designs only from those second order rotatable designs constructed with the help of BIB designs, the method of getting such designs from any second order rotatable design is identically the same, as is indicated in the beginning of the chapter. But the only difficulty may be that after adding a few more design points the whole design may not still remain to be rotatable as is the case with those obtained through BIB designs. In any case they can be used in parts to study the response surface along with main effects and first order interactions.

A list of second order rotatable designs of the type (R + A) upto 7 factors, from which main effects and first order interactions can be recovered with equal precisions, has been presented below:

No. of factors. 'v'	Second order Rotatable design of type (R+A) to explore the response surface.	Design points to be used from (R+A) for the study of main effects and first order interactions.	Total number of design points in the second order rotatable design of type (R + A)
3.	a(v=3,k=2,r=2, b=3, λ=1) x 2^2. (b 0 0 ..0) x 2^1	a(v=3,k=2,r=2,b=3, λ=1) x 2^2 4 central points	18+4 = 22
4.	a(v=4,k=2,r=3,b=6, λ=1) x 2^2 12 central points	a(v=4,k=2,r=3,b=6, λ=1) x 2^2 12 central points	24+12 =36
5.	a(v=5,k=2,r=4,b=10 λ=1) x 2^2 (b b ..b)x1/2 repl.2^5	(b b...b)x½ repl.2^5	56+0 =56
6.	a(v=6,k=3,r=5,b=10, λ=2) x 2^3. (b 0 ...0) x2^1 20 central points	a(v=6,k=3,r=5,b=10, λ=2) x 2^3 20 central points	92+20=112.

7.	a(v=7,k-3,r=3,b=7 λ=1) x 2^3	a(v=7,k=3,r=3,b=7, λ=1) x 2^3	56 + 16 = 72.
	16 central points	16 central points.	

APPENDIX I

A second order rotatable design in seven factors:-

The a-combinations are generated by writing the BIB design $v = 7$, $k = 3$, $r = 3$, $b = 7$, $\lambda = 1$ as a bxv matrix with elements 'a' and '0'.

BIBD	a-combinations.
1 2 4	a a 0 a 0 0 0
2 3 5	0 a a 0 a 0 0
3 4 6	0 0 a a 0 a 0
4 5 7	0 0 0 a a 0 a
5 6 1	a 0 0 0 a a 0
6 7 2	0 a 0 0 0 a a
7 1 3	a 0 a 0 0 0 a

As $p = k = 3$, the associate design is the 2^3 - design with levels 1 and -1, with which the three a's of each a-combination have to be multiplied so as to give design points.

The 56- non central points of the second order rotatable design in 7-factors thus obtained are presented in the next page. The value of 'a' has to be obtained from the relation:

$24a^2 = N$ where N denotes the total number of design points which include at least one central point together with the 56 points.

Points in the design are

a a 0 a 0 0 0

a a 0 -a 0 0 0

a -a 0 a 0 0 0

a -a 0 -a 0 0 0

-a a 0 a 0 0 0

-a a 0 -a 0 0 0

-a -a 0 a 0 0 0

-a -a 0 -a 0 0 0

0 a a 0 a 0 0

0 a a 0 -a 0 0

0 a -a 0 a 0 0

0 a -a 0 -a 0 0

0 -a a 0 a 0 0

0 -a a 0 -a 0 0

0 -a -a 0 a 0 0

0 -a -a 0 -a 0 0

0 0 a a 0 a 0

0 0 a a 0 -a 0

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0 0 0 0 0 0 0

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APPENDIX II

List of second order rotatable designs:-

Number of factors 'v'	Types of combinations with the associate design, to be used for multiplication.	Number of points eadh type of combinations of unknown levels.	Solutions of the unknowns in terms of a^2
(1)	(2)	(3)	(4)
3.	a(v=3, k=2, r=2, b=3, λ=1) x 2^2	12	$b^2/a^2 = \sqrt{2}$
	(b 0 0) x 2^1	6	
4(i)*	a(v=4, k=2, r=3, b=6, λ=1) x 2^2	24	
4(ii)	a(v=4, k=3, r=3, b=4, λ=2) 2^3	32	$b^2/a^2 = 2\sqrt{3}$.
	(b 0 0 0) x 2^1	8	
5.	a(v=5, k=2, r=4, b=10, λ=1) x 2^2	40	$b^2/a^2 = 1/2\sqrt{2}$.
	(b b ...b) x ½ repl.2^5	16	
6(i)	a(v=6, k=2, r=5, b=15 λ=1) x 2^2	60	$b^2/a^2 = 1/2\sqrt{2}$.
	(b b ..b)x ½ repl.2^6	32	
6(ii)	a(v=6, k=3, r=5, b=10, λ=2)x2^3	80	$b^2/a^2 = 2$.
	(b 0 ...0) x 2^1	12	

(1)	(2)	(3)	(4)
7.*	a(v=7,k=3,r=3,b=7, λ=1) x 2^3	$\overline{56}$	
8(i)	a(v=8,k=2,r=7,b=28, λ=1) x 2^2	112	$b^2/a^2=1/2\sqrt{2}$.
	(b b...b)x$\frac{1}{4}$ repl.2^8	$\overline{64}$	
8(ii)*	a(v=8,k=4,r=7,b=14, λ=3) x 2^4.	224	$b^2/a^2 = 4$
	(b 0 ...0) x 2^1	$\overline{16}$	
9.	a(v=9,k=3,r=4,b=12, λ=1) x 2^3	$\overline{96}$	$b^2/a^2=1/4\sqrt{2}$.
	(b b ..b)x $\frac{1}{4}$ repl.2^9	128	
10.*	a(v=10,k=4,r=6,b=15, λ=2) x 2^4	240	
11.	a(v=11,k=5,r=5,b=11, λ=2) x 1/2 repl.2^5	176	$b^2/a^2=2\sqrt{2}$.
	(b 0 ...0) x 2^1.	22	
12(i)	a(v=12,k=2,r=11,b= 66 λ=1) x 2^2	$\overline{264}$	$b^2/a^2=1/4\sqrt{2}$.
	(b b ..b)x$\frac{1}{8}$ repl.2^{12}	512	

(1)	(2)	(3)	(4)
12(ii)	$\bar{a}(v=12, k=6, r=11, b=22, \lambda=5) \times 1/2$ repl.2^6	704	$b^2/a^2 = 8$
	$(b\ 0 \ldots\ldots 0) \times 2^1)$	24	
13(i)	$a(v=13, k=4, r=4, b=13, \lambda=1) \times 2^4$	208	$b^2/a^2 = \sqrt{3}/16$
	$(\bar{b}\ b \ldots b) \times \frac{1}{8}$ repl.2^{13}	1024	
13(ii)	$a(v=13, k=3, r=6, b=26, \lambda=1) \times 2^3$	208	$b^2/a^2 = \sqrt{3}/16$.
	$(\bar{b}\ b \ldots b) \times \frac{1}{8}$ repl.2^{13}	1024	
14.	$a(v=14, k=2, r=13, b=91, \lambda=1) \times 2^2$	364	$b^2/a^2 = \sqrt{5}/16$.
	$(b\ b \ldots b) \times 1/16$ repl.2^{14}	1024	
15.	$a(v=15, k=7, r=7, b=15, \lambda=3) \times \frac{1}{8}$ repl.2^7	960	$b^2/a^2 = 8$
	$(b\ 0 \ldots 0) \times 2^1$	30	
16*	$\bar{a}(v=16, k=6, r=6, b=16, \lambda=2) \times \frac{1}{8}$ repl.2^6	512	

N.B. * denotes at least one central point is to be added.

APPENDIX III

List of nonsequential third order rotatable designs.

(1)	(2)	(3)	(4)
3.	$a(v=3, k=2, r=2, b=3,$	12	$b^2/a^2 = 2.109000$
	$\lambda=1, \mu=0) \times 2^2$		
	$(b\ 0\ 0\ ) \times 2^1$	6	$c^2/a^2 = 0.852600$
	$(c\ 0\ 0\ ) \times 2^1$	6	$d^2/a^2 = 0.629960$
	$(d\ d\ d\ ) \times 2^3$	8	
4.	$a(v=4, k=3, r=3, b=4,$	32	$a^2/d^2 = 0.793701$
	$\lambda=2, \mu=1) \times 2^3$		
	$(b\ 0\ 0\ 0) \times 2^1$	8	$b^2/d^2 = 2.577472.$
	$(c\ 0\ 0\ 0) \times 2^1$	8	$c^2/d^2 = 0.957168.$
	$(d\ d\ 0\ 0) \times 2^2$	24	
5(i)	$a(v=5, k=3, r=6, b=10,$	80	
	$\lambda=3, \mu=1) \times 2^3$		
	$(b\ 0\ \ldots 0) \times 2^1$	10	$b^2/a^2 = 3.247410$
	$(c\ 0\ \ldots 0) \times 2^1$	10	$c^2/a^2 = 1.205956$
5^*(ii)	$a(v=5, k=4, r=4, b=5,$	80	$a^2/d^2 = 0.436790.$
	$\lambda=3, \mu=2) \times 2^4$		
	$(b\ 0\ \ldots 0) \times 2^1$	10	$b^2/d^2 = 1.975158$
	$(c\ 0\ \ldots 0) \times 2^1$	10	$c^2/d^2 = 0.856008$
	$(d\ d\ 0 \ldots 0) \times 2^2$	40	$e^2/d^2 = 1.000000$
	$(e\ 0 \ldots 0) \times 2^1$	10	

(1)	(2)	(3)	(4)
6^*(i)	$a(v=6, k=3, r=10, b=20, \lambda=4, \mu=1) \times 2^3$	160	$a^2/d^2 = 2.519842$
	$(b\ 0 \ldots 0) \times 2^1$	12	$b^2/d^2 = 7.226732$
	$(c\ 0 \ldots 0) \times 2^1$	12	$c^2/d^2 = 3.683908$
	$(d\ d \ldots d) \times 2^6$	64	$e^2/d^2 = 7.000000$
	$(e\ 0 \ldots 0) \times 2^1$	12	
6^*(ii)	$a(v=6, k=4, r=10, b=15, \lambda=6, \mu=3) \times 2^4$	240	$a^2/d^2 = 0.436790$
	$(b\ 0 \ldots 0) \times 2^1$	12	$b^2/d^2 = 2.015918$
	$(c\ 0 \ldots 0) \times 2^1$	12	$c^2/d^2 = 1.465050$
	$(d\ d\ 0 \ldots 0) \times 2^2$	60	$e^2/d^2 = 1.000000$
	$(e\ 0 \ldots 0) \times 2^1$	12	$x^2/d^2 = 1.000000$
	$(x\ 0 \ldots 0) \times 2^1$	12	
6^*(iii)	$a(v=6, k=5, r=5, b=6, \lambda=4, \mu=3) \times 2^5$	192	
	$(b\ 0 \ldots 0) \times 2^1$	12	$b^2/a^2 = 3.657940$
	$(c\ 0 \ldots 0) \times 2^1$	12	$c^2/a^2 = 1.272580$
	$(d\ d\ 0 \ldots 0) \times 2^2$	60	$d^2/a^2 = 3.195920$
	$(e\ e\ 0 \ldots 0) \times 2^2$	60	$e^2/a^2 = 1.945794$
	$(w\ 0 \ldots 0) \times 2^1$	12	$w^2/a^2 = 4.000000$
	$(x\ 0 \ldots 0) \times 2^1$	12	$x^2/a^2 = 5.000000$

(1)	(2)	(3)	(4)
7(i)	a(v=7, k=4, r=4, b=7,		
	λ=2,) x 2^4		
	2.a(complementary		
	BIBD v=7, k=3, r=3,	224	b^2/a^2 = 4.000000
	b=7, λ=1, μ=1) x 2^3		
	(b 0 ....0) x 2^1	14	
7(ii)	a(v=7, k=3, r=15, b=35,	280	a^2/d^2 = 2.000000
	λ=5, μ=1) x 2^3		
	(b 0 ....0) x 2^1	14	b^2/d^2 = 7.542256
	(c 0 ....0) x 2^1	14	c^2/d^2 = 2.667280
	(dd .....d) x $\frac{1}{2}$ repl. 2^7	64	
8*	a(v=8, k=4, r=7, b=14,	224	a^2/d^2 = 1.000000
	λ=3, μ=1) x 2^4		
	(b 0....0) x 2^1	16	b^2/d^2 = 4.000000
	(cc0....0) x 2^2	112	c^2/d^2 = 4.000000
	(d d....d) x $\frac{1}{2}$ repl. 2^8	128	
9(i)	a(v=9, k=3, r=28, b=84,	672	
	λ=7, μ=1) x 2^3		
	(b b ....b) x $\frac{1}{2}$ repl. 2^9	256	b^2/a^2 = 0.392768
	(c c ....c) x $\frac{1}{2}$ repl. 2^9	256	c^2/a^2 = 0.122376
	(d 0 ....0) x 2^1	18	d^2/a^2 = 3.914868

Col. (1)	Col. (2)	Col. (3)	Col. (4)
9(ii)	a-(9, 5, 10, 18, 5) x 2^5	576	
	Complementary B. I. B. D		
	a-(9, 4, 8, 18, 3) x 2^4	576	$b^2/a^2 = 5.944129$
	repeated once more.		
	(b 0....0) x 2	18	$c^2/a^2 = 4.546079$
	(c 0....0) x 2	18	$d^2/a^2 = 2.000000$
	(d d 0...0) x 2^2	144	
10(i)	a-(10, 4, 12, 30, 4, 1) x 2^4	480	
	(b b.....b) x ½ repl. 2^{10}	512	$b^2/a^2 = 0.248096$
	(c c.....c) x ½ repl. 2^{10}	512	$c^2/a^2 = 0.064468$
	(d 0....0) x 2	20	$d^2/a^2 = 2.371260$
	(d 0....0) x 2	20	
	(d 0....0) x 2	20	
	(d 0....0) x 2	20	
	(d 0....0) x 2	20	
	(d 0....0) x 2	20	
0(ii)	a-(10, 5, 9, 18, 4) x 2^5	576	
	Complementary B. I. B. D		
	a-(10, 5, 9, 18, 4) x 2^5	576	$b^2/a^2 = 6.494805$
	(b 0....0) x 2	20	$c^2/a^2 = 2.411955$
	(c 0....0) x 2	20	$d^2/a^2 = 2.000000$
	(d d 0...0) x 2^2	180	

(1)	(2)	(3)	(4)
11(i)	a(v=11, k=5, r=15, b=33, λ=6, μ=2) x 2^5	1056	
	(b 0 ...0) x 2^1	22	b^2/a^2 = 5.443720
	(c 0 ...0) x 2^1	22	c^2/a^2 = 4.285562
11^*(ii)	a(v=11, k=6, r=6, b=11, λ=3) x 2^6		
	2.a(complementary BIBD v=11, k=5, r=5, b=11, λ = 2, μ=2) x 2^5	1408	b^2/a^2 = 0.572357
	(b b...b)x½ repl.2^{11}	512	c^2/a^2 = 3.647317.
	(c c 0...0) x 2^2	220	d^2/a^2 = 1.954600
	(d d 0...0) x 2^2	220	e^2/a^2 = 2.000000
	(e e 0...0) x 2^2	220	
11^*(iii)	a(v=11, k=3, r=45, b=165, λ=9, μ=1) x 2^3	1320	
	(b b...b)x½ repl.2^{11}	512	b^2/a^2 = 0.333192
	(c c...c)x½ repl.2^{11}	512	c^2/a^2 = 0.199449
	(d d...d)x½ repl.2^{11}	512	d^2/a^2 = 0.125000
	(e 0...0) x 2^1	22	e^2/a^2 = 3.634241
12^*(i)	a(v=12, k=6, r=11, b=22, λ=5, μ=2) x 2^6	1408	
	(b b 0...0) x 2^2	264	b^2/a^2 = 3.161774
	(c c 0...0) x 2^2	264	c^2/a^2 = 2.000797
	(d d ....0) x½ repl.2^{12}	1024	d^2/a^2 = 0.396850
	(e e 0...0) x 2^2	264	e^2/a^2 = 2.000000

(1)	(2)	(3)	(4)
12(ii)	a(v=12,k=6,r=11,b=22, λ=5) x 2^6		
	a(complementary BIBD. v=12,k=6,r=11,b=22, λ=5,μ=4) x 2^6	2816	b^2/a^2 = 8.000000
	(b 0...0) x 2^1	24	d^2/a^2 = 2.983100
	(d d 0..0) x 2^2	264	e^2/a^2 = 1.761050
	(e e 0...0) x 2^2	264	
12(iii)	a(v=12,k=3,r=55,b=220, λ=10,μ=1) x 2^3	1760	
	(b b...b)x½ repl.2^{12}	1024	b^2/a^2 = 0.282368.
	(c c...c)x½ repl.2^{12}	1024	c^2/a^2 = 0.169032
	(d 0 ...0) x 2^1	24	d^2/a^2 =3.301927
13*(i)	a(v=13,k=6,r=12,b=26, λ=5) x 2^6		
	a(complementary BIBD v=13,k=7,r=14,b=26, λ =7,μ=5)x½ repl.2^7	3328	
	(b 0...0) x 2^1	26	b^2/a^2 = 6.802642
	(c 0...0) x 2^1	26	c^2/a^2 = 4.660909
	(d d 0..0) x 2^2	312	d^2/a^2 = 3.610148.
	(e e 0...0) x 2^2	312	e^2/a^2 = 0.983286

(1)	(2)	(3)	(4)
13(ii)	a(v=13,k=4,r=44,b=143,	2288	
	λ = 11, μ =2) x 2^4		
	(b b 0...0) x 2^2	312	a^2/d^2 = 3.030288
	(c c 0...0) x 2^2	312	b^2/d^2 = 7.578788
	(d d.....d)x½ repl.2^{13}	2048	c^2/d^2 = 3.180174.
14*	a(v=14,k=7,r=13,b=26,		
	λ =6)x½ repl.2^7		
	a(complementary BIBD		
	v=14,k=7,r=13,b=26,	3328	b^2/a^2 = 6.782826
	λ = 6, μ =5)x½ repl.2^7		
	(b 0...0) x 2^1	28	c^2/a^2 = 1.998312.
	(c 0...0) x 2^1	28	d^2/a^2 = 3.629538
	(d d 0..0) x 2^2	364	e^2/a^2 = 0.571362
	(e e 0..0) x 2^2	364	
15	a(v=15,k=7,r=7,b=15,		
	λ =3) x 2^7		
	a(complementary BIBD		
	v=15,k=8,r=8,b=15,	3840	b^2/a^2 = 4.000000
	λ =4, μ =3)x½ repl.2^8		
	(b 0...0) x 2^1	30	c^2/a^2 = 4.000000
	(c 0...0) x 2^1	30	d^2/a^2 = 4.000000
	(d d 0...) x 2^2	420	

N.B: * denotes that these designs have infinite number of solutions, of which only one has been given.

APPENDIX IV

List of sequential third order rotatable designs.

(1)	(2)	(3)		(4)
3.	a(v=3, k=2, r=2, b=3, λ=1, μ=0) x 2^2	12	B_1	b^2/a^2=1.41421
	(b 0 0) x 2^1	6	B_1	
	(c 0 0) x 2^1	6	B_2	c^2/a^2=1.92849
	(d d d) x 2^3	8	B_2	d^2/a^2=0.32390
	(w w w) x 2^3	8	B_2	w^2/a^2=0.60000
4.	(d d 0 0) x 2^2	24	B_1	
	a(v=4, k=3, r=3, b=4, λ=2, μ=1) x 2^3	32	B_2	b^2/a^2=3.247410
	(b 0 0 0) x 2^1	8	B_2	c^2/a^2=1.205956
	(c 0 0 0) x 2^1	8	B_2	d^2/a^2=1.259921
5.*	(c c 0 0 0) x 2^2	40	B_1	a^2/d^2=1.587401
	(d d d d d) x 2^5	32	B_1	
	a(v=5, k=4, r=4, b=5, λ=3, μ=2) x 2^4	80	B_2	b^2/d^2=6.415804
	(b 0 0 0 0) x 2^1	10	B_2	c^2/d^2=4.000000
	(e 0 0 0 0) x 2^1	10	B_2	e^2/d^2=3.103404
	(w 0 0 0 0) x 2^1	10	B_2	w^2/d^2=5.000000
	(x 0 0 0 0) x 2^1	10	B_2	x^2/d^2=5.000000

(1)	(2)	(3)	(4)
6.	(d d ....d) x 2^6	64 } B_1	$a^2/d^2 = 2.519842$
	(e 0 ....0) x 2^1	12	
	a(v=6,k=3,r=10,b=20		
	λ=4, μ=1) x 2^3	160 } B_2	$b^2/d^2 = 5.039684$
	(b 0 ...0) x 2^1	12	$c^2/d^2 = 5.039684$
	(c 0 ...0) x 2^1	12	$e^2/d^2 = 8.000000$
7.	2.a(v=7,k=3,r=3,b=7,		
	λ=1) x 2^3	112 } B_1	
	a(complementary BIBD		
	v=7,k=4,r=4,b=7,		
	λ=2, μ=1) x 2^4	112 } B_2	$b^2/a^2 = 4.000000$
	(b 0...0) x 2^1	14	
8.	(d d ....d)x½ repl.2^8	128 } B_1	$a^2/d^2 = 2x\ 2^{\frac{1}{2}}$
	(e 0 ....0) x 2^1	16	
	a(v=8,k=4,r=7,b=14,		
	λ =3, μ=1) x 2^4	224 } B_2	$c^2/d^2 = 4.000000$
	(c c 0...0) x 2^2	112	$e^2/d^2 = 8x2^{\frac{1}{2}}$

(1)	(2)	(3)	(4)
9^*(i)	(c c ...c) x½ repl. 2^9	256 } B_1	a^2/b^2 = 3.023716
	(d 0 ...0) x 2^1	18 } B_1	c^2/b^2 = 0.899121
	(e 0 ...0) x 2^1	18 } B_1	d^2/b^2 = 8.507872
	(w 0 ...0) x 2^1	18 } B_1	e^2/b^2 = 2.563504
	(x 0 ...0) x 2^1	18 } B_1	
	a(v=9, k=3, r=28, b=84, λ=7, μ=1) x 2^3	672 } B_2	w^2/b^2 = 8.000000
	(b b ...b) x½ repl. 2^9	256 } B_2	x^2/b^2 = 8.000000
9(ii)	a(v=9, k=4, r=8, b=18, λ=3) x 2^4	576 } B_1	b^2/a^2 = 6.196762
	(e 0...0) x 2^1	18 } B_1	
	a (complementary BIBD v=9, k=5, r=10, b=18, λ=5, μ=3) x 2^5	576 } B_2	c^2/a^2 = 1.264954
	(b 0 0....0) x 2^1	18 } B_2	d^2/a^2 = 2.000000
	(c 0 .....0) x 2^1	18 } B_2	e^2/a^2 = 4.000000
	(d d 0....0) x 2^2	144 } B_2	
10.	a(v=10, k=5, r=9, b=18, λ=4) x2^5	576 } B_1	
	(b 0....0) x 2^1	20 } B_1	b^2/a^2 = 6.494805
	(c 0....0) x 2^1	20 } B_1	c^2/a^2 = 2.411955
	a(complementary BIBD (v=10, k=5, r=9, b=18, λ=4, μ=3) x 2^5	576 } B_2	d^2/a^2 = 2.000000
	(d d 0....0) x 2^2	180 } B_2	

(1)	(2)	(3)		(4)
11(i)	(b b 0....0) x 2^2	220		
	(c c 0....0) x 2^2	220	B_1	b^2/a^2 = 3.225490
	(d d .....d)x½ repl.2^{11}	512		c^2/a^2 = 0.453387
	a(v=11,k=5,r=15,b=33,			
	λ =6, μ=2) x 2^5	1056	B_2	d^2/a^2 = 0.538609
	(e e 0...0) x 2^2	220		e^2/a^2 = 1.851640
11(ii)	2^2 a(v=11,k=5,r=5,b=11,	220		
	λ =2) x 2^5	704	B_1	
	(e e 0...0) x 2^2	220		e^2/a^2 = 1.511858
	a(complementary BIBD			
	v=11,k=6,r=6,b=11,			
	λ =3, μ=2) x 2^6	704		b^2/a^2 = 0.572350
	(b b ...b)x½repl.2^{11}	512	B_2	c^2/a^2 = 3.572516
	(c c0...0) x 2^2	220		d^2/a^2 = 2.464714
	(d d 0..0) x 2^2	220		
12.	(b b 0 ...0) x 2^2	264		b^2/a^2 = 2.806072
	(c c 0....0) x 2^2	264	B_1	c^2/a^2 = 1.485024
	(d d .....d)x½ repl.2^{12}	1024		d^2/a^2 = 0.396850
	a(v=12,k=6,r=11,b=22,			
	λ = 5, μ =2) x 2^6	1408	B_2	e^2/a^2 = 2.828427
	(e e 0...0) x 2^2	264		

(1)	(2)	(3)		(4)
13*	a(v=13, k=6, r=12, b=26,			
	λ =5) x 2^6	1664	B_1	
	(e e 0....0) x 2^2	312		e^2/a^2 = 2.309401
	a(complementary BIBD			
	v=13, k=7, r=14, b=26,			
	λ =7, μ =5)x½ repl.2^7	1664	B_2	b^2/a^2 = 0.343768
	(c c 0....0) x 2^2	312		c^2/a^2 = 3.253305
	(d d 0....0) x 2^2	312		d^2/a^2 = 2.511237
	(b b .....b)x½ repl.2^{13}	2048		w^2/a^2 = 3.000000
	(w w 0....0) x 2^2	312		
14*	a(v=14, k=7, r=13, b=26,			
	λ =6) x½ repl.2^7	1664	B_1	
	(e e 0...0) x 2^2	364		e^2/a^2 = 2.828427
	a(complementary BIBD			
	v=14, k=7, r=13, b=26,			
	λ =6, μ =5)x½ repl.2^7	1664	B_2	b^2/a^2 = 0.235260
	(b b ...b)x½ repl.2^{14}	4096		c^2/a^2 = 3.172259
	(c c ...0) x 2^2	364		d^2/a^2 = 2.296065
	(d d 0..0) x 2^2	364		w^2/a^2 = 2.000000
	(w w 0...0) x 2^2	364		

(1)	(2)	(3)		(4)
15.	a(v=15,k=7,r=7,b=15, λ=3) x 2^7	1920	B_1	
	(e e 0 ....0) x 2^2	420		e^2/a^2= 2.412091
	a(complementary BIBD v=15,k=8,r=8,b=15, λ=4,μ=3)x½ repl.2^8	1920	B_2	b^2/a^2= 0.164658
	(b b ....b)x ⅛ repl.2^{15}	4096		c^2/a^2= 3.832317
	(c c 0...0) x 2^2	420		d^2/a^2= 1.413329
	(d d 0...0) x 2^2	420		

NB: (i) B_1 denotes block number one while B_2 denotes block number two.

(ii) * denotes that these designs have infinite number of solutions of which only one has been given.

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