SECOND AND THIRD ORDER ROTATABLE DESIGNS WITH MINIMUM NUMBER OF LEVELS AND SOME FURTHER METHODS FOR CONSTRUCTING THIRD ORDER DESIGNS

BY

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C O N T E N T S

INTRODUCTION:

A great deal of literature is available on response surface designs. Box and Hunter (1954, 1957) introduced the concept of rotatability and derived the necessary and sufficient conditions for a design to be rotatable of order d. They obtained the rotatable designs through geometrical configurations of regular and semi-regular solid figures. Next, Bose and Draper (1959) obtained an infinite series of second order rotatable designs in three dimensions by using a different technique. Gardiner et al (1959) provided some third order rotatable designs upto 6 factors. Draper (1960) again, constructed an infinite series of second order rotatable designs in four or more dimensions. Das (1961) obtained both second and third order designs as fractional replicate of factorial designs. More recently, Das and Narasimham (1962) and Das (1963) have given a different technique for the construction of such designs through B.I.B. designs both with blocks of equal and unequal sizes.

Though a number of authors have constructed rotatable designs, none has given due consideration to the number of levels of different factors. Obviously, whatever the design may be, if the number of levels is large, sometimes, it becomes difficult to operate the design in practical situations. Hence there is a need for obtaining designs with smaller number of levels.

The minimum number of levels with which a second order rotatable design can be obtained is three and only a limited number of such three levelled designs is available. Box and Behnken (1960 b) and also Das and Narasimham (1962) independently obtained three levelled second order rotatable designs through B.I.B. designs with $r = 3\lambda$.

Other series of second order designs obtained by Das and Narasimham (1962) require five levels if $r \neq 3\lambda$. Furthermore, Das (1963) has also given some other second order rotatable designs which require seven or even more levels.

Das (1961) attempted for the first time to convert designs of the central composite type in five levels to designs in three levels. The present investigation aims at supplying methods through which the existing designs can be converted to three levelled designs.

We have also presented in the present thesis, a new series of second order rotatable designs, both in five and three levels, obtainable through the B.I.B. design with the parameters $(v, k, r, b^{*}, \lambda)$.

Third order rotatable designs obtained by Das and Narasimham (1962) require a large number of levels of each of the factors. We have presented a method through which third order rotatable designs, both non-sequential and sequential can be obtained in five levels.

A series of third order rotatable designs in five levels has also been obtained through the doubly B.I.B. design $(v, k, r, b^{*}, \lambda, \mu)$.

Lastly, we have presented third order rotatable designs upto 10 factors obtainable through the B.I.B. designs with blocks of unequal sizes.

The notations and method of construction adopted by Das and Narasimham (1962) have been followed in this thesis.

2. Second order rotatable designs in 3 levels:

By extending the method given by Das (1961) for converting central composite designs to designs in 3 levels, it is possible, in general, to convert most of the second order designs, available in literature, to three levelled designs. The technique is the same viz., replacement of the unknown levels in some of the combinations so as to have only one unknown level followed by repetition of these combinations, may be different number of repetitions for different combinations, so as to satisfy the relation $\sum x_i^4 = 3\sum x_i^2 x_j^2$. The number of repetitions as also the type of combinations to be repeated depends on the method of construction of the designs which have to be converted to three levelled designs. However, the following theorem for obtaining the number of repetitions of particular types of combinations is true in the case of all second order designs which are obtainable by using B.I.B. designs. The sets (b, b, ..., b) or (b, 0, ..., 0) will also be taken as particular cases of balanced designs, λ being r in the former and zero in the latter.

Theorem 1:

If a second order rotatable design can be obtained through sets of unknown combinations obtainable from one or more B.I.B. designs with or without unequal block sizes, then this design can be converted to a 3 levelled design by first replacing the unknowns in all the blocks, by a single unknown and then repeating the blocks of sizes k_i, p_i times where $p_i = a_i^4/a^4$ (as obtained from the original design) and a_i is the unknown associated with the blocks of sizes k_i in the original design and a is the unknown associated to the largest blocks.

If p_i happens to be a fraction and is equal to q_i/q_i', then it has to be converted first to the fraction Q_i/Q by multiplying each of q_i and q_i' by an integer such that Q becomes the lowest common multiple of all q_i''s. Next, the blocks of sizes k_i are to be repeated Q_i times and those of the largest size k, Q times.

This theorem can easily be proved as indicated in the case of 3 levelled designs.

Let the unknown a be associated to the blocks of the larger size and a_1 to those of the other size. Let ca_1^4 be the contribution from the blocks of smaller size in $\sum x_i^4$ and /or $\sum x_i^2 x_j^2$. When the unknown a_1 in the blocks is replaced by a, their contribution will evidently be ca^4. Now, in order to make the latter contribution, ca^4 equal to the former ca_1^4, let us repeat the smaller blocks p times, after replacing the unknown a_1 by a, so that the contribution becomes pca^4. Now, if $pca^4 = ca_1^4$ i.e., $p = a_1^4/a^4$, the contributions from both will be the same.

If p happens to be a fraction, that is, $p = s/q$ say, it can always be made integral by repeating the blocks of the larger size q times or any of its multiple.

The case where there are more than two unknowns can also be proved on the same lines.

We have obtained below, the values of a_1^4/a^4 for various types of designs arising from the different methods of construction. The cases 1, 2 and 3 which are already known from the work of Das (1961), Raghavarao (1963) and Tyagi (1964) have also been given for the sake of completion. They, however, approached the problem from a different angle. The present work which was started in August, 1963 was done independently. A paper containing some of the results presented in the thesis was read at the 17th annual conference of the Indian Society of Agricultural Statistics held in the first week of January, 1964.

Case 1: Through central composite designs:

Das (1961) obtained three levelled second order rotatable designs through five levelled central composite designs by taking the sets (a,a, ..., a) and (a, 0, ..., 0) instead of (a,a, ...,a) and (b, 0, ..., 0). If the number of points obtained from the set (a, a, ..., a) be n, the value of $b^4/a^4 = n$ in the original central composite design. Hence the set (a, 0, ..., 0) obtained from (b, 0, ..., 0) by replacing b by a is to be repeated n times.

Case 2: Through B.I.B. designs with $r < 3\lambda$:

For, the construction of second order rotatable designs, we have to introduce a set of type (b,0, ..., 0) if $r < 3\lambda$ in a B.I.B. design. Now, we take the set (a, 0, ..., 0) in place of (b, 0, ..., 0). With the set (b, 0, ... 0) we have from the relation $\sum x_i^4 = 3 \sum x_i^2 x_j^2$, the equtION

$$r.2^m a^4 + 2b^4 = 3\lambda . 2^m a^4 \qquad \text{.........} (1)$$

where m ($\leq$ k) is such that a fraction of 2^m combinations out of 2^k can be so taken that its identity group does not include any interaction with less than five factors.

From equation (i) it follows that

$$b^4/a^4 = (3\lambda - r)\ 2^{n-1} \quad \ldots\ldots\ldots \quad (ii)$$

Thus, the set (a, 0, ..., 0) is to be repeated p times where p =

$$p = b^4/a^4 = (3\lambda - r)\ 2^{n-1}.$$

Case 2: Through B.I.B. designs with $r > 3\lambda$:

When $r > 3\lambda$, Das and Narasimhan (1962) took the set (b, b, ..., b) in addition to the a - combinations obtained from B.I.B. design and multiplied it with 2^v (or a suitable fraction $2^{v'}$) associate combinations to obtain the design points in addition to those obtainable from the B.I.B. design.

From, the relation $\sum x_i^4 = 3 \sum x_i^2 x_j^2$ it follows that

$$r.\ 2^n a^4 + 2^{v'} b^4 = 3\left[\lambda.\ 2^n a^4 + 2^{v'} b^4\right]$$

$$\text{i.e.,}\quad a^4/b^4 = 2.2^{v'} / (r-3\lambda).2^n \quad \ldots\ldots\ldots \quad (iii)$$

where $n \leq k$, $v' \leq v$.

Thus, for converting this five levelled design to three levelled design we will take the set (a, a, ..., a) in place of the set (b, b, ..., b) and repeat the a- combinations of B.I.B. design p times where

$$p = a^4/b^4 = 2.2^{v'} / (r - 3\lambda)\ 2^n.$$

If a^4/b^4 is fractional, i.e., of the form s/q, then the $(a, a, \ldots, a)$ set is to be repeated q times and the s - combinations of the B.I.B. design s times.

Case 4: Through B.I.B. designs with unequal block sizes:

(i) Through B.I.B. designs with three block sizes obtained through B.I.B. designs of the series $v = S^2$, $b^* = S^2 + S$, $r = S + 1$, $k = S$, $\lambda = 1$, where S is a prime or a prime power.

From the above series of B.I.B. designs Das (1963) obtained a new series of B.I.B. designs with unequal block sizes and $r = S\lambda$.

Das (1963) took three different unknown levels a_1, a_2 and a_3 for associating with blocks of sizes $k_1 = S$, $k_2 = 2$ and $k_3 = S(S-1)$ respectively. For such designs $\sum x_i^2 x_j^2$ will be constant if

$$2^{k_3'} \cdot a_3^4 = 2^2 \cdot a_2^4 = 2^{k_1'} \cdot a_1^4$$

where $2^{k'_3}$ and $2^{k'_1}$ are suitable fractions of 2^{k_3} and 2^{k_1} respectively such that the identity groups of interactions necessary for obtaining them do not contain any interaction with less than 5 factors.

Now, to convert this 7 levelled design to three levelled design, we will use the theorem 1.

We replace all the three unknowns into only one unknown say a_1. Then the blocks of size $k_2 = 2$ are to be repeated

$p_1 = a_2^4/a_3^4 = 2^{k'_3 - 2}$ times and the blocks of size k_1 are to

be repeated $p_2 = a_1^4/a_3^4 = 2^{k'_3 - k'_1}$, times so as to give a three levelled design.

Example: If we take $s = 3$, we get $v = 9$, $b^* = 18$, $k_1 = 3$, $k_2 = 2$, $k_3 = 6$, $r = 6$ and $\lambda = 2$; $r = 3\lambda$.

Then, $p_2 = 2^5/2^3 = 4$ and $p_1 = 2^5/2^2 = 8$.

Thus, we get a three levelled design in 576 non-central points.

(ii) Through B.I.B. designs with two block sizes obtained through P.B.I.B. designs with two associate classes:

From a P.B.I.B. design with parameters $(v, b^*, r_1, k, \lambda_1, \lambda_2, n_1, n_2)$ $\lambda_1 > \lambda_2$, we can get $vn_2/2$ blocks each of size 2 by putting together in a block two treatments i & j such that the two treatments are second associates. The b^* blocks of the P.B.I.B. design together with all the $vn_2/2$ blocks of size 2 each repeated $(\lambda_1 - \lambda_2)$ times will form a B.I.B. design with blocks of sizes k and 2, $\lambda = \lambda_1$ and $r = r_1 + n_2(\lambda_1 - \lambda_2)$.

Das (1963) obtained second order rotatable designs by associating a_1 with the blocks of P.B.I.B. design and a_2

with the $vn_2/2$ blocks. For simplicity, we would consider first the case when $r = 3\lambda$. $\sum x_i^2 x_j^2$ will be constant if

$$2^2 . a_2^4 + 2^{k'} . \lambda_2 a_1^4 = 2^{k'} . \lambda_1 a_1^4$$

$$\text{i.e., } a_2^4/a_1^4 = 2^{(k'-2)} . (\lambda_1 - \lambda_2).$$

To convert this design to three levelled design we use the theorem 1 as follows:

Firstly, we replace the two unknown levels into only one level say a_1. Secondly, we repeat the $vn_2/2$ blocks

$p_1 = a_2^4/a_1^4 = 2^{(k'-2)} . (\lambda_1 - \lambda_2)$ times to get a second order rotatable design in 3 levels.

If, however, $r \neq 3\lambda$, the sets $(a_1, 0, \ldots, 0)$ or $(a_1, a_1, \ldots, a_1)$ has to be taken as indicated earlier in cases 2 and 3.

<u>Example</u>: Let us take the P.B.I.B. design with $v = 12$, $r_1 = 4$, $k = 6$, $b^* = 8$, $\lambda_1 = 2$, $\lambda_2 = 0$; $n_1 = 10$, $n_2 = 1$

1	2	3	4	5	6
7	8	10	10	5	6
1	2	9	10	11	12
7	8	3	4	11	12
10	5	12	1	8	3
4	5	12	7	2	9
4	11	6	1	8	9
10	11	6	7	2	3

Using the above design together with the following blocks:

1	7
2	8
3	9
4	10
5	11
6	12

it is possible to get a second order rotatable design in three levels by associating a_1 to both types of blocks and repeating the second set of blocks of size 2, p_1 times where

$$p_1 = a_2^4/a_1^4 = 2^{(k'-2)} \cdot (\lambda_1 - \lambda_2) = 16.$$

(iii) Through B.I.B. designs with two block sizes obtained from P.B.I.B. designs with more than two associate classes:

From a P.B.I.B. design with parameters (v, b^o, r_1, k; λ_1, λ_2, ..., λ_t; n_1, n_2, ..., n_t) where $\lambda_1 > \lambda_2 > ... > \lambda_t$, we can get $vn_\ell/2$ blocks ($\ell = 2, 3, ..., t$) each of size 2 by including in a block all possible pairs of treatments which are mutually ℓ th associates. Then, the b^o blocks of the P.B.I.B. design together with all $vn_\ell/2$ blocks of size 2, each repeated ($\lambda_1 - \lambda_\ell$) times, will form a B.I.B. design with parameters

v, $b^o + \sum_\ell (vn_\ell/2)(\lambda_1 - \lambda_\ell)$; $r_1 + \sum n_\ell(\lambda_1 - \lambda_\ell)$; k, 2 ; λ.

Das (1963) obtained second order rotatable designs by associating as many unknown levels as there are associate classes excluding the first one with which the unknown a_1 is associated, the blocks in the ℓ th set being associated with a_ℓ ($\ell = 2, 3, ..., t$). $\sum x_i^2 x_j^2$ will be constant if

$$2^2 . a_\ell^4 + 2^{k'} . \lambda_\ell a_1^4 = 2^{k'} . \lambda_1 a_1^4$$

$$\text{i.e., } a_\ell^4/a_1^4 = 2^{(k' - 2)} . (\lambda_1 - \lambda_\ell).$$

For $v > 5$, the set (a, a, ..., a) has to be taken and thus the design can be obtained in $[2(\ell + 1) + 1]$ levels where $\ell = 1, 2, ..., t$.

By theorem 1, we can reduce the number of levels of such designs from $[2(l+1)+1]$ to 3.

Let us associate the same unknown level a for all associate classes including the first one. Then $vn_l/2$ blocks are to be repeated p_l times where

$$p_l = a_l^4/a_1^4 = (\lambda_1 - \lambda_l).\ 2^{(k'-2)}, \quad l = 2, 3, \ldots, t.$$

For $v > 5$, the set $(a_1, a_1, \ldots, a_1)$ has to be taken and we can get a second order rotatable design in 3 levels by proceeding as described in case 3.

<u>Example:</u> Let the P.B.I.B. design be $v = b^* = 8$, $k = 4$, $r_1 = 4$, $\lambda_1 = 3$, $\lambda_2 = 2$, $\lambda_3 = 1$, $\lambda_4 = 0$; $n_1 = 2$, $n_2 = 2$, $n_3 = 2$ and $n_4 = 1$.

Thus $vn_2/2 = 8$ blocks are to be repeated p_2 times, $vn_3/2 = 8$ blocks, p_3 times and $vn_4/2 = 4$ blocks, p_4 times where

$$p_2 = a_2^4/a_1^4 = 2^{(k'-2)}.\ (\lambda_1 - \lambda_2) = 4,$$
$$p_3 = a_3^4/a_1^4 = 2^{(k'-2)}.\ (\lambda_1 - \lambda_3) = 8$$
$$\text{and } p_4 = a_4^4/a_1^4 = 2^{(k'-2)}.\ (\lambda_1 - \lambda_4) = 12.$$

Here, for the combined design $k = 4$, $\lambda = 3$, $r = 12$, $r > 3\lambda$; consequently by taking the set $(a_1, a_1, \ldots, a_1)$ and repeating the above design p_1 times where

$$p_1 = 2.2^{v'}/(r - 3\lambda).\ 2^m = 2.2^6/4.2^4 = 2,$$

we get a second order rotatable design in three levels.

3. <u>Series of second order rotatable designs through B.I.B. designs with blocks of unequal sizes.</u>

When there is a B.I.B. design with parameters (v, k, r, b^{*}, λ), we can always get a series of B.I.B. designs for (v - x) treatments, $0 \leq x \leq v - 2$, with unequal block sizes k_i, i = 1, 2, ..., such that the parameters r and λ remain unchanged while the parameter b^{*} may change for $x \geq k$.

By associating an unknown a to each of these blocks we can get 2^{k_1} (or a suitable fraction $2^{k'_1}$) associate combinations from the blocks of the largest size k_1 where $k'_1 \leq k_1 \leq k$. If k_i 's, i = 2, 3, ..., be the other block sizes, then by repeating the blocks of sizes k_i, p_i times where $p_i = 2^{k'_1 - k'_i}$, i = 2, 3, ..., and $2^{k'_i}$ is a suitable fraction of 2^{k_i}, i = 1, 2, ..., as required from the general principles of construction, we shall be getting as many design points from this design as are obtainable from the B.I.B. design for v treatments provided $x < k$ and $k_1 = k$. If for some $x < k$, there exists a $k_1 < k$, then the design points obtained through such a design will be less than those obtained for v treatments. For $x \geq k$, the number of design points can always be made less by omitting the treatments all occuring in a block.

From a three levelled second order rotatable design for v factors, obtained through the B.I.B. design with $r = 3\lambda$, a series of three levelled second order rotatable designs can be obtained for (v - x) factors also, $0 \leq x \leq v - 2$, with the same or smaller number of design points. These designs for

$(v - x)$ factors are usually with the minimum number of design points, known so far, provided for a given v, x is so taken that for $(v - x)$ factors there does not exist any B.I.B. design with equal block sizes and with $r = 3\lambda$. Thus, if such a design with $r = 3\lambda$ exists for $(v - x')$ factors, then x can take value upto $(x' - 1)$.

From the B.I.B. design (16, 6, 6, 16, 2) which gives a three levelled design in 512 points, we can get three levelled second order designs for 15, 14, 13, 12 and 11 factors also with 512 points by omitting respectively 1, 2, 3, 4 and 5 treatments. The numbers of points in these designs are minimum, known so far, excepting the design in 11 factors for which the number of points will be 352 when it is obtained by using the a - combinations of the B.I.B. design (11, 5, 5, 11, 2) alongwith the set (a, 0, ..., 0) repeated 8 times.

Though we have a B.I.B. design with equal block sizes for $v = 10$ and $r = 3\lambda$, this design can also be obtained by omitting from the symmetrical B.I.B. design (16, 6, 6, 16, 2), 6 treatments occuring in a block. Thus from the design (16, 6, 6, 16, 2) we can continue to omit the treatments till the number of treatments is 8 and by using these designs we can also get three levelled second order rotatable designs with 240 points for 10, 9 and 8 factors.

We cannot use the design (16, 6, 6, 16, 2) for obtaining 7 - factor rotatable design as we have a B.I.B. design with equal block sizes and with $r = 3\lambda$ for 7 factors viz., the design

(7, 3, 3, 7, 1). By using this design we can get three levelled second order rotatable designs for 6 and 5 factors with 36 points, for 4 and 3 factors with 24 points and for 2 factors with 20 points.

From a five levelled second order rotatable design for v factors obtained through the B.I.B. design with $r \neq 3\lambda$, we can get similarly, a series of five levelled second order designs for $(v - x)$ factors with the same sets of combinations.

We can also get a series of three levelled second order rotatable designs for $(v - x)$ factors, from a three levelled second order rotatable design for v factors, obtained through B.I.B. designs with $r \neq 3\lambda$, through the same sets of combinations viz., $(a, 0, \ldots, 0)$ or $(a, a, \ldots, a)$ (vide cases 2 and 3 of section 2). For $r < 3\lambda$, the number of repetitions of the set $(a, 0, \ldots, 0)$ remains the same for each of the $(v - x)$ factors, $0 \leq x \leq v - 2$ except for the few cases when $k_1 < k$. For $r > 3\lambda$, the number of repetitions of the different sets remain the same if $k_1 = k$ and if the design points obtained through the set $(a, a, \ldots, a)$ remain the same. If $k_1 < k$, designs can still be found from the general principles discussed in cases 2 and 3 of section 2.

4. Third order rotatable designs in 5 levels:

It appears that the minimum number of levels of each of the factors with which a third order rotatable design could be obtained is five, though the absolute minimum can be four also. But only few designs with 5 levels of each of the factors are available in literature. The designs given by Das and Narasimham (1962) for 7 and 15 factors are in five levels. The other designs given by them require larger number of levels ; some require as many as 13 levels. Moreover, apart from the practical considerations, as the number of levels increases it becomes difficult to solve the algebric equations. We thus, attempted to obtain third order designs in 5 levels.

In the cases of second order rotatable designs, the existing designs could be converted to three levelled designs by repeating certain sets of combinations. In the case of third order rotatable designs, this technique, in general, does not seem to be applicable, through the repetition of particular types of sets is necessary here also. The method of obtaining such designs is, in general, to take at least two suitably chosen sets from the following sets of combinations:

(i) $(a, a, \ldots, a)$; (ii) $(b, b, \ldots, b)$;

(iii) $(a, 0, \ldots, 0)$; (iv) $(b, 0, \ldots, 0)$;

(v) $(b, b, 0, \ldots, 0)$; (vi) $(a, a, 0, \ldots, 0)$

and (vii) the a - combinations obtainable from B.I.B. designs either doubly balanced or complementary designs;

and repeat them p, q, r etc., times. Next, through the relations of the third order designs we shall be getting equations in terms of p, q, r etc., and the positive integral solutions of p, q, r etc., will give us the number of repetitions, necessary for the respective sets.

When $\lambda = 3\mu$ in a doubly B.I.B. design with parameters $(v, k, r, b^{*}, \lambda, \mu)$, third order designs with five levels can, in general, be obtained by taking the sets (a, 0, ..., 0) and (b, 0, ..., 0), repeated q and r times respectively, together with the a - combinations of the doubly B.I.B. design which are to be repeated p times.

For $\lambda \neq 3\mu$, no general method for the choice of such sets seems possible.

When a doubly B.I.B. design is not available, we take a B.I.B. design alongwith its complementary design so that the combined design becomes a doubly B.I.B. design.

Example: A non-sequential third order rotatable design in 4 factors can be obtained with the following points:

(i) 4 x 6 p points of a - (4, 2, 3, 6, 1, 0) x 2^2,

(ii) 8q points of (a, 0, 0, 0) x 2,

(iii) 8r points of (b, 0, 0, 0) x 2

and

(iv) 16m points of (a, a, a, a) x 2^4.

The three relations of the design give the following equations:

$$16\,ma^4 + 2\,rb^4 + 12\,pa^4 + 2\,qa^4 = 3\left[16\,ma^4 + 4\,pa^4\right]$$

$$16\,ma^6 + 2\,rb^6 + 12\,pa^6 + 2\,qa^6 = 5\left[16\,ma^6 + 4\,pa^6\right]$$

and $$16\,ma^6 + 4\,pa^6 = 3\left[16\,ma^6\right].$$

The third equation gives $4\,p = 16 \times 2\,m$ which is satisfied when $m = 1$ and $p = 8$. Putting these values in the first two equations and letting $b^2/a^2 = s$, we get

$$s^2 = (16 - q)/r\,, \qquad s^3 = (64 - q)/r.$$

These equations are satisfied if $q = 0$ and $r = 1$, giving $s^2 = 16$ and $s^3 = 64$ which implies that $s = 4$. Thus, we get a five levelled third order rotatable design in 216 points.

From the five levelled non-sequential third order rotatable designs, obtained as described earlier, we could obtain sequential designs by dividing the sets into two blocks such that each block is a second order rotatable design. No general technique for such sub-division of sets seems possible.

In certain designs, we can divide the design points of the sets into two blocks in several ways. For example, let us consider the non-sequential third order rotatable design for 3 factors obtained through the following design points:

(i) 12 p points of $a - (3, 2, 2, 3, 1, 0) \times 2^2$

(ii) 6 q points of $(a, 0, 0) \times 2$

(iii) 6 r points of $(b, 0, 0) \times 2$

and (iv) 8 n points of $(a, a, a) \times 2^3$,

where $p = 8$, $q = 16$, $r = 1$ and $n = 2$. Total number of design points is thus 214.

We can get a sequential third order design by taking the two blocks as

B_1: 16 points of $(a, a, a) \times 2^3$

6 points of $(b, 0, 0) \times 2$

B_2: 96 points of $a - (3, 2, 2, 3, 1, 0) \times 2^2$

96 points of $(a, 0, 0) \times 2$.

An alternative to above is to take the two blocks as:

B_1: 16 points of $(a, a, a) \times 2^3$

6 points of $(b, 0, 0) \times 2$

48 points of $a - (3, 2, 2, 3, 1, 0) \times 2^2$

48 points of $(a, 0, 0) \times 2$

B_2: 48 points of $a - (3, 2, 2, 3, 1, 0) \times 2^2$

48 points of $(a, 0, 0) \times 2$.

Another alternative is to take the two blocks as:

B_1: 16 points of $(a, a, a) \times 2^3$

6 points of $(b, 0, 0) \times 2$

72 points of $a - (3, 2, 2, 3, 1, 0) \times 2^2$

72 points of $(a, 0, 0) \times 2$

B_2: 24 points of $a - (3, 2, 2, 3, 1, 0) \times 2^2$

24 points of $(a, 0, 0) \times 2$.

One more alternative is:

B_1: 16 points of $(a, a, a) \times 2^3$

6 points of $(b, 0, 0) \times 2$

84 points of $a - (3, 2, 2, 3, 1, 0) \times 2^2$

84 points of $(a, 0, 0) \times 2$

B_2: 12 points of $a - (3, 2, 2, 3, 1, 0) \times 2^2$

12 points of $(a, 0, 0) \times 2$.

Lists of non-sequential and sequential third order rotatable designs in five levels upto 15 factors have been presented in the appendices I and II respectively.

5. Series of five levelled third order rotatable design through doubly B.I.B. designs with blocks of unequal sizes:

From a doubly B.I.B. design with the parameters $(v, k, r, b^*, \lambda, \mu)$, we can always get a series of doubly B.I.B. designs with $(v - x)$, $0 \leq x \leq v - 3$, treatments by omitting any x treatments. In the resulting design the values of b^*, r, λ and μ remain the same. For $x \geq k$, the value of b^* may also decrease.

By associating an unknown a to each of these blocks we can get 2^k (or a suitable fraction $2^{k'}$) associate combinations from the blocks of the largest size k. If k_i's, $i = 2, 3, \ldots,$ be the other block sizes then by repeating the blocks of sizes k_i, $2^{k' - k'_i}$ times $(k'_i \leq k_i)$, we shall begetting as many design points to be included in a third order rotatable design for $(v - x)$ factors as are obtainable through the doubly B.I.B. design with v treatments provided $x < k$. If, however, the largest block size is k_1 $(< k)$ then those blocks are to be repeated $2^{k' - k'_1}$ times where $k'_1 \leq k_1$. For $x \geq k$ the number of design points can always be made smaller by omitting treatments all occurring in a block.

From a five levelled third order rotatable design for v factors obtained through doubly B.I.B. design with $\lambda = 3\mu$, we can get, through the same sets, a series of five levelled third order designs for $(v - x)$ factors, $0 \leq x \leq v - 3$, with the same numbers of repetitions of the different sets as required for the design in v factors and the value of b^2/a^2 will remain the same. As such the number of design points in the design with $(v - x)$ factors, $0 \leq x \leq v - 3$, will be less than that of the design with v factors.

From a five levelled third order rotatable design for v factors, obtained through doubly B.I.B. design with $\lambda \neq 3\mu$, a series of five levelled third order rotatable design can likewise be obtained for $(v - x)$ factors, $0 \leq x \leq v - 3$, also.

An interesting application of this method of obtaining third order rotatable designs in five levels is that for 8 factors if we use the sets (b, 0, ..., 0) and those a-combinations obtainable from the doubly B.I.B. design (8, 4, 7, 14, 3, 1) where $\lambda = 3\mu$, we get only a rotatable arrangement and not a design. But if we omit one treatment from the doubly B.I.B. design and generate a third order rotatable design with the same two types of sets as described above, we shall get a five levelled third order rotatable design in 7 factors with just 238 points. Likewise, by omitting, in turn, 2, 3, 4 and 5 treatments from the same doubly B.I.B. design with 8 factors, it will be possible to get five levelled third order rotatable designs in 6, 5, 4 and 3 factors with 236, 234, 232 and 214 points respectively.

Similarly, we can use the rotatable arrangement for 16 factors obtainable through the a-combinations of the doubly B.I.B. design (16, 6, 15, 30, 7, 3) for obtaining third order third order design for all numbers of factors ranging from 15 to 3.

In general, by omitting x treatments from (16, 8, 15, 30, 7, 3) and taking the sets (b, b, 0, ...,0) and (b, 0, ..., 0) we can get rotatable designs for $(16 - x)$ factors, $0 \leq x \leq 13$ when the first and second sets are each taken once and the third set is taken q times where $q = 2x$. Thus, for 15, 14, 13, 12, 11, 10, 9 and 8 factors, the values of q will be 2, 4, 6, 8, 10, 12, 14 and 16 respectively and the number of design points respectively will be : 4320, 4316, 4308, 4296, 4280, 4260, 4236 and 4080. These numbers are considerably smaller than those obtainable by using other methods.

6. Third order rotatable designs through B.I.B. designs with blocks of unequal sizes:-

Several methods are available for constructing third order rotatable designs. In many cases these designs require too many design points. Recently, Das (1962) obtained second order rotatable designs through B.I.B. designs with blocks of unequal sizes which suggested that third order rotatable designs with reasonably smaller number of points may be obtained through this approach. Hence an attempt was made to construct third order rotatable designs, both non-sequential and sequential, through B.I.B. designs with blocks of unequal sizes. We are presenting below designs upto 10 factors.

(i) A design for 4 factors: By omitting one treatment from the doubly B.I.B. design (5, 3, 6, 10, 3, 1) we get two separate doubly B.I.B. designs as (4, 3, 3, 4, 2, 1) and (4, 2, 3, 6, 1, 0). Let us associate the unknown a with the doubly B.I.B. designs (4, 3, 3, 4, 2, 1) and (4, 2, 3, 6, 1, 0) and repeat the second doubly B.I.B. design 2 times. We get $\mu = 1$ for the combined design. Then, by taking further, two sets (b, 0, 0, 0) and (c, 0, 0, 0), we get from the relations of a third order design, the three equations as:

$$48 a^4 + 2 (b^4 + c^4) = 3 (24 a^4),$$

$$48 a^6 + 2 (b^6 + c^6) = 5 (24 a^6)$$

and $$24 a^6 = 3 (8 a^6).$$

Putting $b^2/a^2 = s$ and $c^2/a^2 = t$, we get from the first two equations

$s^2 + t^2 = 12$ and

$s^3 + t^3 = 36$,

giving $s = 3.247$ and $t = 1.205$. We thus get a design in 4 factors with 96 points.

A sequential third order design can also be obtained with the same number of design points if we take the two blocks as:

B_1: $a - (4, 2, 3, 6, 1, 0) \times 2^2$
repeated once more

B_2: $a - (4, 3, 3, 4, 2, 1) \times 2^3$
$(b, 0, 0, 0) \times 2$
$(c, 0, 0, 0) \times 2.$

The solutions of s and t remain the same for this design.

(ii) <u>A design for 5 factors</u>: By associating the unknown a with the doubly B.I.B. designs (5, 4, 4, 3, 3, 2) and (5, 2, 4, 10, 1, 0) and repeating the second doubly B.I.B. design, 4 times and taking further the sets (b, 0, 0, 0, 0), (c, 0, 0, 0,0) and (d, d, 0, 0, 0) we get the equations as:

$$128\, a^4 + 16\, d^4 + 2\,(b^4 + c^4) = 3\,(64\, a^4 + 4\, d^4),$$
$$128\, a^6 + 16\, d^6 + 2\,(b^6 + c^6) = 3\,(64\, a^6 + 4\, d^6)$$
and
$$64\, a^6 + 4\, d^6 = 3\,(32\, a^6).$$

Putting $s = b^2/a^2$, $t = c^2/a^2$ and $u = d^2/a^2$, we get from the third equation $u^3 = 8$, i.e., $u = 2$.

The first two equations give

$s^2 + t^2 = 24$ and

$s^3 + t^3 = 112$, which are satisfied for $s = 4.809$ and $t = 0.933$. We, thus, get a third order rotatable design with 300 points in 5 factors.

A sequential design with 332 points can also be obtained if we take the two blocks as:

B_1: $a - (5,\ 2,\ 4,\ 10,\ 1,\ 0) \times 2^2$

$(b,\ b,\ b,\ b,\ b) \times 2^5$

B_2: $a - (5,\ 4,\ 4,\ 5,\ 3,\ 2) \times 2^4$

$a - (5,\ 2,\ 4,\ 10,\ 1,\ 0) \times 2^2$ repeated 3 times

$(c,\ c,\ 0,\ 0,\ 0) \times 2^2$

$(d,\ 0,\ 0,\ 0,\ 0) \times 2$

$(e,\ 0,\ 0,\ 0,\ 0) \times 2$

For this design the solutions will be:

$s = a^2/b^2 = 4.000$; $t = c^2/b^2 = 8.032$;

$u = d^2/b^2 = 19.938$ and $v = e^2/b^2 = 7.183$.

(iii) <u>A design for 6 factors</u>: If we omit one treatment each from the B.I.B. designs (7, 3, 3, 7, 1) and (7, 4, 4, 7, 2) we get the resulting design in three block sizes viz., 2, 3 and 4. Then by repeating the blocks of size 2, 4 times and blocks of size 3, 2 times and associating the same unknown a with all the blocks we can get a third order rotatable design in 6 factors by taking further the set (b, 0, ..., 0). This design, in five levels, happens to be the same as given in section 5.

(iv) <u>A design for 7 factors</u>: By associating the unknown a with the B.I.B. designs (7, 3, 3, 7, 1) and (7, 4, 4, 7, 2) we get a third order rotatable design if we repeat the first B.I.B. design, 2 times and take the set (b, 0, ..., 0). This design is also presented in section 5.

(v) A design for 8 factors: If we omit one treatment from the B.I.B. design (9, 3, 4, 12, 1), we get the resulting design in two block sizes viz., 3 and 2. Then, if we take their complementary designs we shall be getting two more block sizes viz., 5 and 6. Then, by associating the same unknown a with all those blocks and repeating the blocks of size 2, 16 times, blocks of size 3, 8 times and blocks of size 5, 2 times we get a third order rotatable design if we take further the sets (b, 0, ..., 0), (c, c, 0, ..., 0), (d, d, 0, ..., 0) and (e, 0, ..., 0). The three equations will be:

$$4 \times 192\, a^4 + 2(b^4 + e^4) + 28(c^4 + d^4) = 3\left[2 \times 192\, a^4 + 4(c^4 + d^4)\right],$$
$$4 \times 192\, a^6 + 2(b^6 + e^6) + 28(c^6 + d^6) = 5\left[2 \times 192\, a^6 + 4(c^6 + d^6)\right]$$
and
$$2 \times 192\, a^6 + 4(c^6 + d^6) = 3\,(192\, a^6).$$

Putting $s = b^2/a^2$, $t = c^2/a^2$, $u = d^2/a^2$ and $v = e^2/a^2$, we get from the above equations:

$$v^2 + s^2 = 192 - 8\,(t^2 + u^2),$$
$$v^3 + s^3 = 3 \times 192 - 4\,(t^3 + u^3) \text{ and}$$
$$t^3 + u^3 = 48.$$

They are satisfied for

$s = 3.425$; $t = 3.196$;

$u = 2.480$ and $v = 7.006$.

We, thus get a third order rotatable design in 1792 points.

A sequential design with 2032 points can also be obtained if we take in B_1, the blocks of sizes 2 and 3 repeated 15 and 8 times respectively, alongwith the set (d, d, ..., d) and in B_2, the blocks of sizes 5 and 6, the blocks of size 3 repeated 2 times, alongwith the sets (b, 0, ..., 0), (c, c, 0, ..., 0), (e,e, 0, ..., 0), (1, 1, 0, ..., 0) and (w, 0, ..., 0). The three relations of the design will be satisfied for:

$$s = a^2/d^2 = 2.000 \; ; \; t = b^2/d^2 = 7.388 \; ; \; u = c^2/d^2 = 6.525 \; ;$$
$$v = e^2/d^2 = 4.735 \; ; \; m = 1^2/d^2 = 4.000 \text{ and } n = w^2/d^2 = 18.827.$$

(vi) A design for 9 factors: By associating the unknown a with the B.I.B. designs (9, 3, 10, 18, 5) and (9, 4, 8, 18, 9) and repeating the second design 2 times, we get a third order rotatable design by taking further the sets (b, 0, ..., 0), (d, 0, ..., 0) and (c, c, 0, ..., 0). Here $\mu = 3$ for the combined design. This design with 1332 design points was given by Das and Narasimham (1962).

(vii) A design for 10 factors: If we omit one treatment from the doubly B.I.B. design (11, 5, 15, 33, 6, 2) we get the two B.I.B. designs as (10, 5, 9, 18, 4) and (10, 4, 6, 15, 2). Let us associate the unknown a with the B.I.B. designs (10, 5, 9, 18, 4) and (10, 4, 6, 15, 2) and repeat the second B.I.B. design 2 times. We get $\mu = 2$ for the combined design. Then, by taking further the sets (b, 0, ..., 0) and (c, 0, ..., 0) we get a third order design in 1096 points. The three equations will be:

$$15 \times 32 \ a^4 + 2(b^4 + c^4) = 3(6 \times 32 \ a^4),$$

$$15 \times 32 \ a^6 + 2(b^6 + c^6) = 5(6 \times 32 \ a^6)$$

and $$6 \times 32 \ a^6 = 3(2 \times 32 \ a^6).$$

By putting $s = b^2/a^2$ and $t = c^2/a^2$, we get the first two equations as:

$$s^2 + t^2 = 48$$

and $s^3 + t^3 = 240$, which are satisfied for $s = 5.444$ and $t = 4.286$.

A sequential design can also be obtained with the same number of design points by taking the two blocks as:

B_1: $a-(10, 4, 6, 15, 2) \times 2^4$

repeated once more

B_2: $a-(10, 5, 9, 18, 4) \times 2^5$

$(b, 0, \ldots, 0) \times 2$

$(c, 0, \ldots, 0) \times 2.$

We can get a series of 7 levelled third order rotatable designs by omitting x treatments, $0 \leq x \leq v - 3$, from a doubly B.I.B. design with the parameters $(v, k, r, b^*, \lambda, \mu)$ and with $\lambda = 3\mu$ provided a third order rotatable design can be obtained through the a-combinations of the doubly B.I.B. design $(v, k, r, b^*, \lambda, \mu)$, $\lambda = 3\mu$ and the sets $(b, 0, \ldots, 0)$ and $(c, 0, \ldots, 0)$. The solutions of $s = b^2/a^2$ and $t = c^2/a^2$ will remain the same for all the $(v - x)$ treatments, $0 \leq x \leq v - 3$. The repetitions of blocks of different sizes, obtained by omitting x treatments from the doubly B.I.B. design, will be made as discussed in the section 3.

For example, there exists a 7 levelled third order rotatable design for 11 factors obtainable through the c-combinations of the doubly B.I.B. design (11, 5, 15, 33, 6, 2), $\lambda = 3\mu$ and the sets (b, 0, ..., 0) and (c, 0, ..., 0) with 1100 design points. Now, by omitting in turn 1, 2, 3, 4 and 5 treatments from the doubly B.I.B. design (11, 5, 15, 33, 6, 2), we can get third order rotatable designs for 10, 9, 8, 7 and 6 factors respectively with 1096, 1092, 1088, 1084 and 1048 design points. Similarly, the 7 levelled third order rotatable design for 5 factors, obtainable through the c-combinations of the doubly B.I.B. design (5, 3, 6, 10, 3, 1), $\lambda = 3\mu$ and the sets (b, 0, ..., 0) and (c, 0, ..., 0), with 100 design points, may be used to give third order rotatable designs for 4 and 3 factors with 96 and 92 design points respectively.

SUMMARY:

A method for obtaining second order rotatable designs in three levels has been described. Through this method the existing second order rotatable designs can be converted to designs with three levels of each of the factors. A number of examples have been given to illustrate this method.

The investigation has been extended to obtain third order rotatable designs, both non-sequential and sequential, with five levels of each of the factors. Two tables giving third order rotatable designs in five levels have been given alongwith a detailed example for 4 factors.

A series of five levelled second order rotatable designs, obtainable through the omission of treatments in a B.I.B. design has been presented. A more useful series of three levelled second order rotatable designs has been obtained through the omission of treatments. Particularly, the series obtained through the B.I.B. designs with $r = 3\lambda$ seems to be of paramount importance. This series helps us in getting three levelled designs with smaller number of design points. A number of such useful designs have been given in section 3.

We have also obtained a series of five levelled third order rotatable designs by omitting some treatments in a doubly B.I.B. design. The utility is that the numbers of design points in these third order rotatable designs are reduced to large extent. A use of rotatable arrangements to obtain third order rotatable designs is quite interesting.

Third order rotatable designs upto 10 factors, both non-sequential and sequential, have been obtained through the B.I.B.

designs with blocks of unequal sizes. A series of seven levelled third order rotatable designs obtained through the omission of treatments in a doubly B.I.B. design has also been given. This series also, helps in reducing the number of design points to a large extent.

<u>APPENDIX I.</u>

List of non-sequential third order rotatable designs in five levels.

No. of factors (v)	Types of combinations with the associate design to be used for multiplication.	No. of repetitions of each of the combinations	Total number of points	Solutions
Col. (1)	Col. (2)	Col. (3)	Col. (4)	Col. (5)
3	c-(3, 2,2,3,1,0) x 2^2	8		
	(a, 0, 0) x 2	16		
	(b, 0, 0) x 2	1		
	(a, a, a) x 2^3	2	214	$b^2/a^2 = 4$
3	c-(3,2,2,3,1,0) x 2^2	1		
	(a, 0, 0) x 2	10		
	(b, b, b) x 2^3	16	200	$a^2/b^2 = 4$
4	c-(4,3,3,4,2,1) x 2^3	2		
	(a, 0, 0, 0) x 2	8		
	(b, 0, 0, 0) x 2	1		
	(a, a, 0, 0) x 2^2	4	232	$b^2/a^2 = 4$
4	c-(4,2,3,6,1,0) x 2^2	1		
	(a, 0, 0, 0) x 2	8		
	(b, b, b, b) x 2^4	8	216	$a^2/b^2 = 4$
4	c-(4,2,3,6,1,0) x 2^2	8		
	(b, 0, 0, 0) x 2	1		
	(a, a, a, a) x 2^4	1	216	$b^2/a^2 = 4$

Col. (1)	Col. (2)	Col. (3)	Col. (4)	Col. (5)
5	a-(5,2,4,10,1,0) $\times 2^3$	1		
	(a, 0, 0, 0, 0) $\times 2$	8		
	(b, b, b, b, b) $\times 2^3$	4	228	$a^2/b^2 = 4$
5	a-(5,3,6,10,3,1) $\times 2^3$	2		
	(a, 0, 0, 0, 0) $\times 2$	8		
	(b, 0, 0, 0, 0) $\times 2$	1	250	$b^2/a^2 = 4$
6	a-(6,2,5,15,1,0) $\times 2^3$	1		
	(a, 0, ..., 0) $\times 2$	4		
	(b, b, ..., b) $\times 2^6$	2	236	$a^2/b^2 = 4$
6	a-(6,3,10,20,4,1) $\times 2^3$	16		
	(b, 0, ..., 0) $\times 2$	12		
	(a, a, ..., a) $\times 2^6$	1	2768	$b^2/a^2 = 4$
7	a(7,4,4,7,2) $\times 2^4$ Complementary B.I.B.D. a-(7,3,3,7,1) $\times 2^3$ repeated once more } $\mu = 1$	1		
	(b, 0, ..., 0) $\times 2$	1	238	$b^2/a^2 = 4$
7	a-(7,2,6,21,1,0) $\times 2^2$	1		
	(a, 0, ..., 0) $\times 2$	2		
	(b, b, ..., b) $\times \frac{1}{2}(2^7)$	2	260	$a^2/b^2 = 4$
8	a-(8,4,7,14,3,1) $\times 2^4$	1		
	(b, 0, ..., 0) $\times 2$	1		
	(b, b, 0, ..., 0) $\times 2^2$	1		
	(a, a, ..., a) $\times \frac{1}{2}(2^8)$	1	480	$b^2/a^2 = 4$

Col.(1)	Col. (2)	Col.(3)	Col. (4)	Col. (5)
8	a-(8,4,7,14,3,1) x 2^4	23		
	(a, 0, ..., 0) x 2	100		
	(b, 0, ..., 0) x 2	12	7392	$b^2/a^2 = 5$
9	a-(9,5,10,18,5) x 2^5 complementary B.I.B.D. a-(9,4,8,18,3) x 2^4 repeated once more $\Big\}$ $\mu = 3$	8		
	(b, 0, ..., 0) x 2	88		
	(b, b, 0, ..., 0) x 2^2	1	10044	$b^2/a^2 = 6$
10	a-(10,3,9,18,4) x 2^5 complementary B.I.B.D. a-(10,5,9,18,4) x 2^5 $\Big\}$ $\mu = 3$	8		
	(b, 0, ..., 0) x 2	28		
	(b, b, 0, ..., 0) x 2^2	1	10116	$b^2/a^2 = 4$
11	a-(11,5,15,33,6,2) x 2^5	15		
	(a, 0, ..., 0) x 2	144		
	(b, 0, ..., 0) x 2	16	19360	$b^2/a^2 = 6$
12	a-(12,6,11,22,5) x 2^6 complementary B.I.B.D. a-(12,6,11,22,5) x 2^6 $\Big\}$ $\mu = 4$	2		
	(b, 0, ..., 0) x 2	16		
	(b, b, 0, ..., 0) x 2^2	1	6280	$b^2/a^2 = 4$
13	a-(13,6,12,26,5) x 2^6 complementary B.I.B.D. a-(13,7,14,26,7) x $\frac{1}{2}(2^7)$ $\Big\}$ $\mu = 5$	4		
	(b, 0, ..., 0) x 2	26		
	(b, b, 0, ..., 0) x 2^2	3	14924	$b^2/a^2 = 4$

Col.(1)	Col. (2)		Col. (3)	Col. (4)	Col. (5)
14	a-(14,7,13,26,6) x ½ (2^7) complementary B.I.B.D. a-(14,7,13,26,6) x ½ (2^7)	$\mu = 5$	4		
	(b, 0, ..., 0) x 2		20		
	(b, b, 0, ..., 0) x 2^2		8	14964	$b^2/a^2 = 6$
15.	a-(15,7,7,15,8) x 2^7 complementary B.I.B.D. a-(15,6,6,15,4) x ½ (2^8)	$\mu = 8$	1		
	(b, 0, ..., 0) x 2		2		
	(b, b, 0, ..., 0) x 2^2		1	4320	$b^2/a^2 = 4$

APPENDIX XI

List of sequential third order rotatable designs in five levels.

No. of factors (v) Col. (1)	Types of combinations with the associate design to be used for multiplication. Col. (2)	No. of repetitions of each of the combinations Col. (3)	Total number of points for different blocks Col. (4)	Solutions Col. (5)
3*	B_1: $(a,a,a) \times 2^3$	2		
	$(b,0,0) \times 2$	1	22	$b^2/a^2 = 4$
	B_2: $a-(3,2,2,3,1,0) \times 2^2$	3		
	$(a,0,0) \times 2$	16	192	
3*	B_1: $(a,0,0) \times 2$	1		
	$(b,b,b) \times 2^3$	2	22	$a^2/b^2 = 4$
	B_2: $a-(3,2,2,3,1,0) \times 2^2$	1		
	$(a,0,0) \times 2$	9		
	$(b,b,b) \times 2^3$	14	178	
4*	B_1: $(a,a,0,0) \times 2^2$	4	96	
	B_2: $(a,0,0,0) \times 2$	8		$b^2/a^2 = 4$
	$(b,0,0,0) \times 2$	1		
	$a-(4,3,3,4,2,1) \times 2^3$	2	136	
4*	B_1: $a-(4,2,3,6,1,0) \times 2^2$	1	24	
	B_2: $(a,0,0,0) \times 2$	8		$a^2/b^2 = 4$
	$(b,b,b,b) \times 2^4$	8	192	
4*	B_1: $a-(4,2,3,6,1,0) \times 2^2$	8	192	
	B_2: $(b,0,0,0) \times 2$	1		$b^2/a^2 = 4$
	$(a,a,a,a) \times 2^4$	1	24	

* Designs with asterisk have got several alternatives of which only one is given.

Col. (1)	Col. (2)	Col. (3)	Col. (4)	Col. (5)
5ᵃ	B_1: $(b,b,b,b,b) \times 2^5$	1		
	$(a,0,0,0,0) \times 2$	2	52	$a^2/b^2 = 4$
	B_2: $(b,b,b,b,b) \times 2^5$	3		
	$(a,0,0,0,0) \times 2$	4		
	$a-(5,2,4,10,1,0) \times 2^2$	1	176	
5	B_1: $a-(5,3,4,10,3,1) \times 2^3$	1		
	$(a,0,0,0,0) \times 2$	12	200	$b^2/a^2 = 4$
	B_2: $a-(5,3,6,10,3,1) \times 2^3$	3		
	$(a,0,0,0,0) \times 2$	4		
	$(b,0,0,0,0) \times 2$	2	300	
6	B_1: $(b,b, \ldots, b) \times 2^6$	1		
	$(a,0, \ldots,0) \times 2$	4	112	$a^2/b^2 = 4$
	B_2: $(b,b, \ldots, b) \times 2^6$	2		
	$a-(6,2,5,15,1,0) \times 2^2$	1	124	
6ᵇ	B_1: $(a,a, \ldots,a) \times 2^6$	1		
	$(b,0, \ldots,0) \times 2$	4	112	$b^2/a^2 = 4$
	B_2 $a-(6,8,10,20,4,1) \times 2^3$	16		
	$(b,0,\ldots,0) \times 2$	8	2656	
7	B_1: $(b,b,\ldots,b) \times \frac{1}{2}\ (2^7)$	1		
	$(a,0,\ldots,0) \times 2$	4	120	$a^2/b^2 = 4$
	B_2: $(b,b,\ldots,b) \times \frac{1}{2}\ (2^7)$	3		
	$a-(7,2,6,21,1,0) \times 2^2$	2	360	

Col. (1)	Col. (2)	Col. (3)	Col. (4)	Col. (5)
8	B_1: a-(8,4,7,14,3,1) x 2^4	1		
	(b, 0, ...0) x 2	1	240	$b^2/a^2 = 4$
	B_2: (a,a, ..., a) x $\frac{1}{2}$ (2^3)	1		
	(b,b,0, ...,0) x 2^2	1	240	
8*	B_1: a-(8,4,7,14,3,1) x 2^4	1		
	(a,0, ..., 0) x 2	16	480	$b^2/a^2 = 8$
	B_2: a-(8,4,7,14,3,1) x 2^4	24		
	(a,0, ..., 0) x 2	64		
	(b,0, ..., 0) x 2	12	6912	
9*	B_1: a-(9,4,8,18,3) x 2^4 repeated once more	1		
	(b,0, ..., 0) x 2	1	594	$b^2/a^2 = 4$
	B_2: a-(9,4,8,18,3) x 2^4 repeated once more complementary B.I.B.D. to above	7		
	a-(9,5,10,18,5) x 2^5	8		
	(b,0, ..., 0) x 2	37		
	(b,b,0, ...,0) x 2^2	1	9450	
10*	B_1: a-(10,5,9,18,4) x 2^5	1		
	(b,0, ..., 0) x 2	3	636	$b^2/a^2 = 4$
	B_2: a-(10,5,9,18,4) x 2^5 complementary B.I.B.D. to above	7		
	a-(10,5,9,18,4) x 2^5	8		
	(b,0, ..., 0) x 2	33		
	(b,b,0, ..., 0) x 2^2	1	9450	

Col. (1)	Col. (2)	Col. (3)	Col. (4)	Col. (5)
11*	B_1: c-(11,5,15,33,6,2) $\times 2^5$	1		
	(a,0, ..., 0) $\times$ 2	48	2112	$b^2/a^2 = 6$
	B_2: c-(11,5,15,33,6,2) $\times 2^5$	14		
	(a,0, ..., 0) $\times$ 2	96		
	(b,0, ..., 0) $\times$ 2	16	17248	
12*	B_1: c-(12,6,11,22,5) $\times 2^6$	2		
	(b,b,0, ..., 0) $\times 2^2$	1	3080	$b^2/a^2 = 4$
	B_2: Complementary B.I.B.D. to above c-(12,6,11,22,5) $\times 2^6$	2		
	(b,0, ..., 0) $\times$ 2	16	3200	
13*	B_1: c-(13,7,14,26,7) $\times \frac{1}{2}(2^7)$	4		
	(b,0, ..., 0) $\times$ 2	2		
	(b,b,0, ..., 0) $\times 2^2$	3	7644	$b^2/a^2 = 4$
	B_2: Complementary B.I.B.D. to above c-(13,6,12,26,5) $\times 2^6$	4		
	(b,0, ..., 0) $\times$ 2	24	7280	
14*	B_1: c-(14,7,13,26,6) $\times \frac{1}{2}(2^7)$	2		
	(b,0, ..., 0) $\times$ 2	20	3888	$b^2/a^2 = 4$
	B_2: c-(14,7,13,26,6) $\times \frac{1}{2}(2^7)$ complementary B.I.B.D. to above	2		
	c-(14,7,13,26,6) $\times \frac{1}{2}(2^7)$	4		
	(b,b,0, ..., 0) $\times 2^2$	3	11076	
15	B_1: c-(15,7,7,15,3) $\times 2^7$	1		
	(b,0,...,0) $\times$ 2	3	2160	$b^2/a^2 = 4$
	B_2: c-(15,7,7,15,3) $\times 2^7$ complementary B.I.B.D. to above	3		
	c-(15,8,8,15,4) $\times \frac{1}{2}(2^8)$	4		
	(b,b,0, ..., 0) $\times 2^2$	4	15120	

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