ROLE OF TRANSFORMATION FOR CONSTRUCTION OF ASYMMETRICAL AND SYMMETRICAL ROTATABLE DESIGNS

BY

MEHTA, J.S.

Dissertation submitted in fulfilment of the requirements for the award of Diploma in Agricultural and Animal Husbandry Statistics of the Institute of Agricultural Research Statistics (I.C.A.R.), NEW DELHI

1964.

ACKNOWLEDGEMENT.

I have great pleasure in expressing my deep sense of gratitude to Prof. M.N. Das, Professor of Statistics, Institute of Agricultural Research Statistics, for his valuable guidance, keen interest, valuable criticisms and constant encouragement throughout the course of the investigation and immense help in writing this thesis.

I am grateful to Dr. V.G. Panse, Statistical Advisor, Institute of Agricultural Research Statistics, for providing necessary facilities to carry out this work.

(J. S. Mehta)

CONTENTS.

I.	INTRODUCTION	1
II.	TRANSFORMATION OF THE DOSE VARIATES OF ROTATABLE DESIGNS	3
III.	CHOICE OF TRANSFORMATION	15
IV.	TRANSFORMATION OF FACTORIAL DESIGNS WHERE THE FACTORS ARE EACH AT TWO LEVELS	33
V.	SUMMARY	37
VI.	REFERENCES	38

*****

INTRODUCTION:-

Box and Hunter (1957) introduced rotatable designs for exploration of response surfaces. They constructed some second order designs by making use of the geometrical configurations. After them a number of authors, Box & Behnken (1958), Bose & Draper (1959) and Draper (1960) obtained those designs through different techniques. Subsequently, third order rotatable designs were developed by Gardiner, Grandage & Hader (1959), Draper (1960), Thaker (1960). Box & Behnken (1960) presented a class of second order rotatable designs which were derivable from those of the first order. An altogether new approach for the construction of these designs was introduced by Das (1961), Das & Narasimaham (1962) and Das (1963). The idea behind this approach is to get those designs through the "factorials" and the "incomplete block designs". However, the designs obtained thus far were symmetrical in as much as the factors involved in the designs all had the same number of levels. Ramachander (1963) directed successfully his endeavours to get some new series of response surface designs which are of asymmetrical type but they did not satisfy the criterion of Rotatability. No further attempt seems to have been made to obtain asymmetrical rotatable designs.

As the asymmetrical designs are always more flexible for applications, they may be more useful in certain situations. Consequently, need was felt to have response surface designs which satisfy the conditions of rotatability and are, at the same time, asymmetrical. In the present thesis, an attempt has been made to devise a technique which enables us to get asymmetrical rotatable designs from the already existing symmetrical rotatable designs, without changing the number of design points.

The method has been modified further to get three levelled designs from five levelled designs, this time, however, by adding some extra design points. The technique has been utilised to establish the inter-relationship between the central composite designs and the rotatable designs obtained through incomplete block designs.

The above method has been extended to the factorial experiments, where all factors are at two levels each and it has been possible to get fraction of 3^n designs (for n even) as the transformed design. General properties of such transformed designs are also considered and illustrated with the aid of a 2^6 factorial experiment.

TRANSFORMATION OF THE DOSE VARIATES OF ROTATABLE DESIGNS.

This chapter deals primarily with the general considerations which have got to be kept in mind while selecting linear transformations to be applied to the dose variate of a symmetrical rotatable design - the basis of our technique. The two conditions to be satisfied by the transformation coefficients for keeping the transformed design rotatable have been enunciated and the reason for such a choice is also dealt with in this chapter. An attempt has been made to provide some simplified guiding principals regarding the actual choice of transformation. Towards the end of this chapter suitable choice of transformations which introduce asymmetry in the transformed design, yet ensuring its rotatability criterion has also been considered. The actual obtaining of such designs is, however, postponed till the following chapter.

Method of construction:-

The method of construction of asymmetrical rotatable designs consists in converting the existing symmetrical rotatable designs to asymmetrical designs by applying suitable linear transformation to the dose variates of the symmetrical design. Let there be v factors in a rotatable design such that the variate x_i denotes the levels of the ith factor. Let further (x_{1u}, x_{2u}, ... , x_{vu}) denote the uth design point of a rotatable design. Out of this point we can get the point (x'_{1u}, x'_{2u}, ..., x'_{vu}) through the transformation

$$x'_{ju} = \sum_i b_{ji} x_{iu} \quad \text{for } i \,\&\, j = 1, 2, \dots v \qquad \text{———— I}$$

As x'_{ju} need not be equal to x_{iu}, the transformed levels of any factor may be different from those of its original levels. It is also not necessary that the number of levels of the different factors in the two designs should remain the same. The relation

$\sum_u x^2_{iu} = N\lambda_2$ or N by a proper choice of scale so that $\lambda_2 = 1$ for $i = 1, 2, \ldots, v$ ....(1),

is satisfied in the original symmetrical rotatable design. Consider the relation $\sum_u x'^2_{ju}$ in the transfored design. Using the transformation (T), we have

$$\sum_u x'^2_{ju} = \sum_u \left(\sum_i b_{ji} x_{iu}\right)^2 \qquad \text{for } i \,\&\, j = 1, 2, \ldots, v.$$

$$= \sum_u \left[\sum_i x^2_{iu} b^2_{ji} + \sum_{i \neq i'} b_{ji} b_{ji'} x_{iu} x_{i'u}\right]$$

$$= \left[\left(\sum_u x^2_{iu}\right)\left(\sum_i b^2_{ji}\right) + \left(\sum_{\substack{u \\ i \neq i'}} x_{iu} x_{i'u}\right)\left(\sum_{i \neq i'} b_{ji} b_{ji'}\right)\right]$$

$$= \left(\sum_u x^2_{iu}\right)\left(\sum_i b^2_{ji}\right) = N \sum_i b^2_{ji}$$

since, $\sum_{\substack{u \\ (i \neq i')}} x_{iu} x_{i'u} = 0.$ and $\sum_u x^2_{iu} = N.$

Now the relation

$$\sum_u x'^2_{ju} = N \quad \ldots\ldots\ldots\ldots \quad A',$$

will hold if

$$\sum_i b^2_{ji} = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad R_1$$

Consequently we choose the transformation coefficients b_{ji}s in such a manner that the relation R_1 is satisfied. This is always possible since, given a set of coefficients, all that is required is the division of these coefficients by a suitable factor so as to satisfy R_1.

Again consider the relations

$$\sum_u x_{iu}^4 = 3N\lambda_4 \quad i = 1, 2, \ldots, v. \quad \ldots\ldots (2) \quad \&$$

$$\sum_{\substack{u \\ (i\neq i')}} x_{iu}^2 x_{i'u}^2 = N\lambda_4 \quad i \neq i' = 1, 2, \ldots, v. \ \ldots\ldots (3)$$

which are satisfied in the original design. In the transformed design

$$\sum_u x'^4_{ju} = \sum_u \left(\sum_i b_{ji} x_{iu}\right)^4$$

$$= \sum_u \left[\sum_i b_{ji}^4 x_{iu}^4 + 4 \sum_{i\neq i'} b_{ji} b_{ji'}^3 x_{iu} x_{i'u}^3 + 3 \sum_{i\neq i'} b_{ji}^2 b_{ji'}^2 x_{iu}^2 x_{i'u}^2 \right]$$

$$= \left(\sum_u x_{iu}^4\right) \left(\sum_i b_{ji}^4\right) + 4\left(\sum_{i\neq i'} b_{ji} b_{ji'}^3\right)\left(\sum_{\substack{u \\ (i\neq i')}} x_{iu} x_{i'u}^3\right) + 3\left(\sum_{i\neq i'} b_{ji}^2 b_{ji'}^2\right)\left(\sum_{\substack{u \\ (i\neq i')}} x_{iu}^2 x_{i'u}^2\right)$$

$$= 3N\lambda_4 \left[\sum_i b_{ji}^4 + \sum_{i\neq i'} b_{ji}^2 b_{ji'}^2\right]$$

$$= 3N\lambda_4 \left(\sum_i b_{ji}^2\right)^2 = 3N\lambda_4 \ \ldots\ldots\ldots B'.$$

by R_1 and since $\sum_{\substack{u \\ (i\neq i')}} x_{iu} x_{i'u}^3 = 0$.

Also, we have

$$\sum_{\substack{u \\ (j\neq j')}} x_{ju}^2 x_{j'u}^2 = \sum_u \left[\left(\sum_i b_{ji} x_{iu}\right)^2 \left(\sum_i b_{j'i} x_{iu}\right)^2 \right]$$

$$= \sum_u \left[\left(\sum_i b_{ji}^2 x_{iu}^2 + \sum_{i\neq i'} b_{ji} b_{ji'} x_{iu} x_{i'u}\right) \times \left(\sum_i b_{j'i}^2 x_{iu}^2 + \sum_{i\neq i'} b_{j'i} b_{j'i'} x_{iu} x_{i'u}\right) \right]$$

$$= \sum_u \left[\sum_i b_{ji}^2 b_{j'i}^2 x_{iu}^4 + 2 \sum_{i\neq i'} b_{ji} b_{j'i} b_{ji'} b_{j'i'} x_{iu}^2 x_{i'u}^2 + \sum_{i\neq i'} b_{ji}^2 b_{j'i'}^2 x_{iu}^2 x_{i'u}^2 \right]$$

$$= N\lambda_4 \left[3 \sum_i b_{j'i}^2 b_{ji}^2 + 2 \sum_{i\neq i'} b_{ji} b_{ji'} b_{j'i} b_{j'i'} + \sum_{i\neq i'} b_{ji}^2 b_{j'i'}^2 \right]$$

Let the transformation coefficients be be such that

$$\sum_{\substack{i \\ (j\neq j')}} b_{ji} b_{j'i} = 0 \quad \text{for} \quad i = 1, 2, \ldots, v. \quad \ldots\ldots \; R_2$$

This on squaring gives

$$\sum_i b_{ji}^2 b_{j'i}^2 + \sum_{i\neq i'} b_{ji} b_{j'i} b_{ji'} b_{j'i'} = 0 \quad \ldots\ldots \; (4).$$

Making use of (4) we have

$$\sum_{\substack{u \\ (j\neq j')}} x'^2_{ju} x'^2_{j'u} = N\lambda_4 \left[\sum_{\substack{i \\ (j\neq j')}} b_{ji}^2 b_{j'i}^2 + \sum_{\substack{i\neq i' \\ (j\neq j')}} b_{ji}^2 b_{j'i'}^2 \right]$$

$$= N\lambda_4 \left(\sum_i b_{ji}^2\right) \left(\sum_i b_{j'i}^2\right) = N\lambda_4 \quad \ldots . \; 6'$$

because of R_1.

Thus we note that R_2 ensures the constancy of $\sum_u x'^2_{iu} x'^2_{i'u}$. Next we may consider the relation given below.

$$\sum_{\substack{u \\ (j\neq j')}} x'_{ju} x'_{j'u} = \sum_u \left[\left(\sum_i b_{ji} x_{iu} \right) \quad \left(\sum_i b_{j'i} x_{iu} \right) \right]$$

$$= \sum_u \left[\sum_i b_{ji} b_{j'i} x^2_{iu} + \sum_{i\neq i'} b_{ji} b_{j'i'} x_{iu} x_{i'u} \right]$$

$$= \left(\sum_u x^2_{iu} \right) \left(\sum_i b_{ji} b_{j'i} \right) + \left(\sum_{\substack{i\neq i' \\ (j\neq j')}} b_{ji} b_{j'i'} \right) \left(\sum_{\substack{u \\ (i\neq i')}} x_{iu} x_{i'u} \right)$$

$$= 0 \qquad \ldots\ldots\ldots\ldots\ldots D'$$

because first term on right hand side is zero owing to R_2 *and second term is so by the very definition of rotatable* design.

Now we know that for a second order rotatable design the conditions to be satisfied by the design points are

$$\sum_u x^2_{iu} = N \lambda_2 \text{ or } N \text{ by proper choice of scale} \qquad \ldots\ldots\ldots\ldots\ldots A$$

$$\sum_u x^4_{iu} = 3N \lambda_4 \qquad \ldots\ldots\ldots\ldots\ldots B$$

$$\sum_u x^2_{iu} x^2_{i'u} = N \lambda_4 \qquad \ldots\ldots\ldots\ldots\ldots C$$

$$\sum_u x_{iu} = 0, \quad \sum_{\substack{u \\ (i\neq i')}} x_{iu} x_{i'u} = 0, \quad \sum_{\substack{u \\ (i\neq i')}} x_{iu} x^2_{i'u} = 0$$

$$\sum_u x^3_{iu} = 0, \quad \sum_{\substack{u \\ (i\neq i')}} x_{iu} x^3_{i'u} = 0, \ldots$$

$$\sum_{\substack{u \\ (i\neq i'\neq i''\neq i''')}} x_{iu} x_{i'u} x_{i''u} x_{i'''u} = 0 \qquad \ldots\ldots\ldots\ldots\ldots D$$

all these relations being true for $i = 1, 2, \ldots, v$. Lastly

$$\frac{\lambda_4}{\lambda_2^2} > \frac{v}{v+2} \qquad \ldots\ldots\ldots\ldots\ldots E$$

is to be satisfied. Now relations A, B, C, D are the relations A', B', C', D' which the transformed design points satisfy and as such the transformed design achieves the criterion of rotatability. Thus we find that relations R_1 and R_2 are the key relations for deciding the choice of the coefficients of the transformations and provide us with broad outlines of the method towards the choice of transformations. Suitable choices of transformations which lead us to asymmetrical rotatable designs can be discussed now.

Before taking up the particular cases for illustration purposes, it will be worthwhile mentioning another point of no less an importance. Let a second degree curve be fitted to the data available, for the dose-response relationship

$$y_u = a_0 + \sum_{i=1}^{v} a_i x_{iu} + \sum_{i \le j = 1}^{v} a_{ij} x_{iu} x_{ju}$$

where y_u is the response corresponding to the uth experimental design point and the coefficients a_0, a_i, a_{ij} are to be estimated by the least squares technique. Now, the variance of the estimated response thus obtained will be a function of the variances of the estimates of the coefficients a_0, a_i, a_{ij} etc which can be easily got while solving out the coefficients through the least squares method. Again, the variances of the estimates of these coefficients will themselves be functions of $\sum_u x_{iu}^2$ (or $N\lambda_2$) and $\sum_u x_{iu}^4$, $\sum_u x_{iu}^2 x_{i'u}^2$ (or λ_4). Since the relations A, B, C, D remain invariant under the transformations satisfying the conditions R_1 and R_2, we may conclude that the variance of the response estimated from the transformed design remains unaltered.

Let the transformed design be expressed in the matrix notation as given below:-

$$\begin{bmatrix} x'_{11} & x'_{21} & x'_{31} & \cdots & x'_{v1} \\ x'_{12} & x'_{22} & x'_{32} & \cdots & x'_{v2} \\ x'_{13} & x'_{23} & x'_{33} & \cdots & x'_{v3} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ x'_{1N} & x'_{2N} & x'_{3N} & \cdots & x'_{vN} \end{bmatrix} = \begin{bmatrix} x_{11} & x_{21} & x_{31} & \cdots & x_{v1} \\ x_{12} & x_{22} & x_{32} & \cdots & x_{v2} \\ x_{13} & x_{23} & x_{33} & \cdots & x_{v3} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ x_{1N} & x_{2N} & x_{3N} & \cdots & x_{vN} \end{bmatrix} \begin{bmatrix} b_{11} & b_{21} & b_{31} & \cdots & b_{v1} \\ b_{12} & b_{22} & b_{32} & \cdots & b_{v2} \\ b_{13} & b_{23} & b_{33} & \cdots & b_{v3} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ b_{1v} & b_{2v} & b_{3v} & \cdots & b_{vv} \end{bmatrix}$$

(NxV) (Nxv) (vxv)

Transformed Design — Original Design — Transformation Matrix (of transformation coeffs).

where N is the total number of design points. If x_{ij} denotes the jth point under consideration of the ith factor then the transformed design point can be written down as

$$\begin{aligned} x'_{11} &= x_{11}b_{11} + x_{21}b_{12} + x_{31}b_{13} + \cdots + x_{v1}b_{1v} \\ x'_{21} &= x_{11}b_{21} + x_{21}b_{22} + x_{31}b_{23} + \cdots + x_{v1}b_{2v} \\ x'_{31} &= x_{11}b_{31} + x_{21}b_{32} + x_{31}b_{33} + \cdots + x_{v1}b_{3v} \\ &\vdots \\ x'_{v1} &= x_{11}b_{v1} + x_{21}b_{v2} + x_{31}b_{v3} + \cdots + x_{v1}b_{vv} \end{aligned}$$

Similarly we can write down the rest of the transformed design points. The transformation matrix of the transformation coefficients is a v x v matrix where v is the number of factors involved in the design.

Another point which has to be kept in mind while deciding upon the transformations will be dealt with here. Let us consider a central composite design in four factors so that each factor is at five levels 0, $\pm$ a and $\pm$ b and there are 24 non-central design points which are given below:-

$$\begin{bmatrix} a & a & a & a \\ a & a & a & -a \\ a & a & -a & a \\ a & -a & a & a \\ -a & a & a & a \\ a & a & -a & -a \\ a & -a & a & -a \\ -a & a & a & -a \\ a & -a & -a & a \\ -a & a & -a & a \\ -a & -a & a & a \\ a & -a & -a & -a \\ -a & a & -a & -a \\ -a & -a & a & -a \\ -a & -a & -a & a \\ -a & -a & -a & -a \\ b & 0 & 0 & 0 \\ -b & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & -b & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & -b & 0 \\ 0 & 0 & 0 & b \\ 0 & 0 & 0 & -b \end{bmatrix}$$

D_1

Relation $\sum_u x_{iu}^4 = 3 \sum_u x_{iu}^2 x_{i'u}^2$, which is to be satisfied by these design points, requires the relation b = 2a to hold. Let us take the following most general form of a 4 x 4 transformation matrix.

$$\begin{bmatrix} b_{11} & b_{21} & b_{31} & b_{41} \\ b_{12} & b_{22} & b_{32} & b_{42} \\ b_{13} & b_{23} & b_{33} & b_{43} \\ b_{14} & b_{24} & b_{34} & b_{44} \end{bmatrix} = M$$

where the elements b_{ij} satisfy the relations R_1 and R_2. That the levels of the factors in the transformed design depend upon the choice of these elements will be amply demonstrated by taking the following two cases of transformation matrices.

Case I:-

Here we choose the elements such that the transformation matrix is

$$\begin{bmatrix} c & -f & -e & -d \\ f & c & d & -e \\ e & -d & c & f \\ d & e & -f & c \end{bmatrix}$$

and the values of c, d, e, f will be determined by the relation

$$c^2 + d^2 + e^2 + f^2 = 1 \qquad \ldots\ldots\ldots\ldots(5)$$

since by a proper choice of the signs the solutions of (5) can be made to satisfy R_1 and R_2 simultaneously. In the transformed design, obtained as a result of applying the transformation given above on the design matrix D_1, we have the following 24 new levels of the first factor and the same levels for the other factors as well

$\pm$ (ea + fa + ca + da); $\pm$ (ea + fa + ca + da); $\pm$ (ea + fa − ca + da)

$\pm$ (ea − fa + ca + da); $\pm$ (−ea + fa + ca + da); $\pm$ (ea + fa − ca − da)

$\pm$ (ea − fa + ca − da); $\pm$ (−ea + fa + ca − da); $\pm$ eb; $\pm$ fb; $\pm$ cb; $\pm$ db.

then the elements c, d, e, f, are selected without any further restriction on them except that.

$$c^2 + d^2 + e^2 + f^2 = 1,$$

which follows from relation R_1, the number of distinct levels of each of the factors as obtained through the transformations is likely to be large. So ways and means have to be devised so as to reduce the number of levels and for this the following sub-cases are considered.

<u>Sub-case (i)</u>

Let e = 0 f ≠ d ≠ c ≠ 0 so that now (5) reduces to

$$f^2 + c^2 + d^2 = 1$$

and some of the 24 levels obtained above will be equal. As a matter of fact now the levels are

$\pm$ (fa + ca + da); $\pm$(fa + ca − da); $\pm$ (fa − ca + da); $\pm$ (−fa + ca + da)

$\pm$ fb; $\pm$ cb; $\pm$ db; 0,

some of which may again be equal for particular choices of f, c, d subject to the above restrictions. Thus through the restriction e = 0, the number of levels could be reduced considerably from a maximum of 24 to 15.

<u>Sub-case (ii)</u>

Let e = f = 0 and c ≠ d ≠ 0 so that (5) reduces to $c^2 + d^2 = 1$ and the transformed factors have for their levels.

$\pm$ (ca + da); $\pm$ (ca - da); $\pm$ cb; $\pm$ db; 0

which , under the assumption that all levels are distinct, are nine in number.

Sub-case (iii)

Let e = 0 and c = d = f = $1/3^{\frac{1}{2}}$ which satisfy relation (5). In the transformed design the factors have levels as

$0;\ \pm\ 3^{\frac{1}{2}}a;\ \pm\ a/3^{\frac{1}{2}};\ \pm\ b/3^{\frac{1}{2}}$

i.e. just seven levels each.

Sub-case (iv)

Let e = f = 0 and c = d = $1/2^{\frac{1}{2}}$. In the transformed design each of the factor involved has its levels 0, $\pm$ $2^{\frac{1}{2}}a$ which are only three in number.

It will be seen that all the above transformations indicated in the different sub-cases have kept the transformed designs symmetrical rotatable though the number of levels of the factors has changed depending upon the choice of the elements in the transformation matrix. The first two sub-cases are illustrations of the fact that by taking a smaller number of non-zero and distinct elements the number of levels in the transformed design can be reduced. The number can be reduced further if, in addition to taking a smaller number of non-zero elements they are made equal; a fact brought forth in sub-cases (iii) and (iv). Thus we may say that

(i) As far as possible, in keeping with the basic requirements of the elements so as to satisfy R_1 and R_2, minimum number of non-zero elements in the transformation matrix should be taken since this ensures smaller number of distinct levels of each of the factors in the transformed design.

(ii) Further, this advantage can be enhanced by taking these elements equal.

Case II

In this section we shall attempt to show how asymmetry can be introduced.

If we choose our transformation matrix as

$$\begin{bmatrix} e & -f & 0 & 0 \\ f & e & 0 & 0 \\ 0 & 0 & c & d \\ 0 & 0 & d & -c \end{bmatrix}$$

and apply it on D_1, the first two factors will have levels $\pm$ (ea + fa); $\pm$ (ea - fa); $\pm$ eb; $\pm$ fb; 0, while the last two factors have levels $\pm$ (ca + da); $\pm$ (ca - da); $\pm$ cb; $\pm$ db; 0, in the transformed design. The number of levels of the first two factors need not be the same as of the last two factors. For example let $e = f = 1/2^{\frac{1}{2}}$ and $c = 1/5^{\frac{1}{2}}$ and $d = 2/5^{\frac{1}{2}}$. Consequently the first two factors have 0; $\pm\, 2^{\frac{1}{2}}a$ as their levels while the levels of the last two factors are 0, $\pm\, a/5^{\frac{1}{2}}$; $\pm\, 2a/5^{\frac{1}{2}}$ and $\pm\, 3a/5^{\frac{1}{2}}$. Thus the transformed design is an incomplete factorial of $3^2 \times 7^2$ and is an asymmetrical rotatable design. This case is enough to suggest that judicious choice of elements can help us to introduce asymmetry in the rotatable designs.

Details regarding the actual choice of these transformations are intended to be discussed in the next chapter.

CHOICE OF TRANSFORMATION:-

In the present chapter an attempt has been made to discuss in detail the choice of transformations for obtaining rotatable designs. Second order rotatable designs obtained through this method of transformation of doses can be both asymmetrical and symmetrical. Several transformations are discussed for getting such designs from the already existing symmetrical rotatable designs which may either be central composite designs or those obtained through incomplete block designs. Discussion has been restricted to only such transformations which enable us to keep the number of levels of the factors in the transformed design reasonably small. This, however, has been done without any loss of generality of the form of the transformations.

I. Transformations - factors taken two at a time.

1) Central composite designs.

Let us consider the transformation matrix

$$\begin{bmatrix} D_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & D_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & D_3 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & D_{(v-x)/2} & 0 \\ 0 & 0 & 0 & \cdots & 0 & I_x \end{bmatrix} = Mt$$

where D_i is a 2 x 2 matrix having for its general form

$$\begin{bmatrix} 1/(1+y_i^2)^{\frac{1}{2}} & y_i/(1+y_i^2)^{\frac{1}{2}} \\ y_i/(1+y_i^2)^{\frac{1}{2}} & -1/(1+y_i^2)^{\frac{1}{2}} \end{bmatrix}$$

so that D_1, D_2, ... etc are the sub-matrices obtained by assigning particular real values to "y_i", and I_x is the unit matrix of order x. Again, v is the total number of factors and x is the number of those factors which can be kept unaltered with respect to the number as well as the magnitudes of their levels by not subjecting them to any transformation. It may be noted that x will be odd or even (zero included) according as v is odd or even.

Consider now a central composite design obtained with the help of the two sets (i) (aa ... a) and (ii) (b00 ... 0) through the two operations of rotation and multiplication (Das 1961) so that each factor has the five levels, 0; $\pm$ a; $\pm$ b. Let us now take a particular case of Mt in which the sub-matrices are such that

$$D_1 = D_2 = D_3 = \ldots = D_{(v-x)/2} = D_1 \text{ (say)}$$

where D_1 is the 2 x 2 matrix

$$\begin{bmatrix} 1/2^{\frac{1}{2}} & 1/2^{\frac{1}{2}} \\ 1/2^{\frac{1}{2}} & -1/2^{\frac{1}{2}} \end{bmatrix}$$

being obtained by giving to y_1 the value unity. When this transformation matrix is operated on the design points obtained above, the resulting design will be symmetrical rotatable and of the form $5^{(v-x)} \times 5^{x}$. It may be seen that although all the factors have the same number of levels yet the magnitudes of the levels of the two sets of factors will differ - the last x factors have the five levels 0; $\pm$ a; $\pm$ b while each of the (v-x) factors has its levels as 0; $\pm\ 2^{\frac{1}{2}}a$; $\pm\ b/2^{\frac{1}{2}}$. A point which does not escape our notice is that if the relation b = 2a is satisfied in the central composite design to start with, then in the transformed design the levels $2^{\frac{1}{2}}a$ and $b/2^{\frac{1}{2}}$ of the (v-x) factors become equal. So to say, now x factors which were left unaltered, retain their five levels 0; $\pm$ a and $\pm$ b, whereas (v-x) factors are such that each is at three levels 0; $\pm$ $2^{\frac{1}{2}}a$. Now the transformed design is an asymmetrical rotatable design and is an incomplete factorial of $3^{(v-x)} \times 5^{x}$. However, for v even, when x = 0 we have the transformed design as 3^{v} which is symmetrical rotatable design. This clearly brings out the advantage we can gain in reducing the number of levels of factors by using relation b = 2a in the context of the transformation considered. Though this relation is not true for all values of v, and even when it holds much will depend upon the transformation matrix taken; as will be shown now, yet in a subsequent section of this chapter it will be shown that it is always possible to satisfy the relation b = 2a in any central composite design by adding a number of design points. The choice of transformation matrix considered in this section seems to be the best in so far as we are able to reduce the number of levels of the factors in the transformed design when relation b = 2a is satisfied.

The next best choice of the transformation matrix is when in Mt we take

$$D_1 = D_2 = \dots = D_{(v-x)/2} = D_2 \text{ (say)}$$

where D_2 is a 2 x 2 matrix of the form

$$\begin{bmatrix} 1/10^{\frac{1}{2}} & 3/10^{\frac{1}{2}} \\ 3/10^{\frac{1}{2}} & -1/10^{\frac{1}{2}} \end{bmatrix}$$

being obtained by putting $y_1 = 3$. When the dose variates of the central composite design are subjected to this transformation matrix then the transformed design is asymmetrical rotatable and is an incomplete factorial of $5^x \times 9^{(v-x)}$. First x-factors have five levels $0;\ \pm a;\ \pm b$ and the $(v-x)$ factors have each nine levels in $0; \pm 2a/10^{1/2};\ \pm 4a/10^{1/2};\ \pm b/10^{1/2};\ \pm 3b/10^{1/2}$. If we have the relation $b = 2a$ satisfied then the nine levels will not be all distinct and, as a matter of fact, now they are only seven in number viz $0;\ 2a/10^{1/2}; \pm 4a/10^{1/2};\ \pm 3b/10^{1/2}$. However, if y_i be given the value of 2, it can be shown that no such reduction in number of levels will be possible despite $b = 2a$ being satisfied. Similarly by giving to y_i other values it is possible to have any number of transformation matrices and consequently any number of asymmetrical rotatable designs. But one thing which may be noted is that for transformations when factors are taken two at a time, in the transformed design, the maximum number of levels for any factor is nine while the corresponding minimum is three depending upon the values which are assigned to 'y_i'.

The transformation matrices considered so far are such that the sub-matrices D_i for $i = 1, 2, \ldots (v-x)/2$ in the diagnal of the transformation matrix Mt are all either D_1s or D_2s where D_1 and D_2 are 2x2 matrices as defined earlier. With the help of these transformation matrices asymmetrical rotatable designs can be obtained in which only two groupings of factors, each group characterised by different levels (in number as well as magnitude), are possible. The number of factor groupings in the transformed design may be taken as a measure of the degree of asymmetry which a transformation is capable of introducing. Let Mt be a transformation matrix where all the sub-matrices D_is are not the same e.g. $D_1, D_2, D_3, D_4, \ldots$ being taken respectively as

$$\begin{bmatrix} 1/2^{\frac{1}{2}} & 1/2^{\frac{1}{2}} \\ 1/2^{\frac{1}{2}} & -1/2^{\frac{1}{2}} \end{bmatrix}, \begin{bmatrix} 1/5^{\frac{1}{2}} & 2/5^{\frac{1}{2}} \\ 2/5^{\frac{1}{2}} & -1/5^{\frac{1}{2}} \end{bmatrix}, \begin{bmatrix} 1/10^{\frac{1}{2}} & 3/10^{\frac{1}{2}} \\ 3/10^{\frac{1}{2}} & -1/10^{\frac{1}{2}} \end{bmatrix}, \begin{bmatrix} 1/17^{\frac{1}{2}} & 4/17^{\frac{1}{2}} \\ 4/17^{\frac{1}{2}} & -1/17^{\frac{1}{2}} \end{bmatrix}, \ldots$$

obtained by giving to y_1 the values of 1, 2, 3, 4, ... and let it be operated on the central composite design we shall get a transformed design which is asymmetrical rotatable and is an incomplete factorial of $3^{x_1} \times 5^{x_2} \times 7^{x_3} \times 9^{x_4}$ where $x_1 + x_2 + x_3 + x_4 = v$. If v is even then all the xs are even whereas if v is odd, one of them must be odd. We find here that four factor groupings are possible such that the number of levels which each group of factors possesses is a number lying between the minimum, 3 and the maximum of 9 levels associated with this type of transformations. Consequently, by choosing D_is as different it is possible to increase the degree of asymmetry in the transformed rotatable designs.

For illustration, an example of central composite design in four factors is taken up. Each factor has five levels 0; $\pm$ a; $\pm$ b and the relation b = 2a holds. Let us consider the transformation matrix

$$Mt = \begin{bmatrix} D_1 & 0 \\ 0 & I_2 \end{bmatrix}$$

where D_1 is a 2 x 2 matrix

$$\begin{bmatrix} 1/2^{\frac{1}{2}} & 1/2^{\frac{1}{2}} \\ 1/2^{\frac{1}{2}} & -1/2^{\frac{1}{2}} \end{bmatrix}$$

and I_2 is a unit matrix of order 2. When Mt is operated on the central composite design, we have the transformed rotatable design whose points are given below:-

	F_1	F_2	F_3	F_4	F'_1	F'_2	F'_3	F'_4
	a	a	a	a	$2^{\frac{1}{2}}a$	0	a	a
	a	a	a	-a	$2^{\frac{1}{2}}a$	0	a	-a
	a	a	-a	a	$2^{\frac{1}{2}}a$	0	-a	a
	a	-a	a	a	0	$2^{\frac{1}{2}}a$	a	a
	-a	a	a	a	0	$-2^{\frac{1}{2}}a$	a	a
	a	a	-a	-a	$2^{\frac{1}{2}}a$	0	-a	-a
	a	-a	a	-a	0	$2^{\frac{1}{2}}a$	a	-a
	-a	a	a	-a	0	$-2^{\frac{1}{2}}a$	a	-a
	a	-a	-a	a	0	$2^{\frac{1}{2}}a$	-a	a
	-a	a	-a	a	0	$-2^{\frac{1}{2}}a$	-a	a
	-a	-a	a	a	$-2^{\frac{1}{2}}a$	0	a	+a
	a	-a	-a	-a	0	$2^{\frac{1}{2}}a$	-a	-a
	-a	a	-a	-a	0	$-2^{\frac{1}{2}}a$	-a	-a
	-a	-a	a	-a	$-2^{\frac{1}{2}}a$	0	a	-a
	-a	-a	-a	a	$-2^{\frac{1}{2}}a$	0	-a	a
	-a	-a	-a	-a	$-2^{\frac{1}{2}}a$	0	-a	-a
	b	0	0	0	$b/2^{\frac{1}{2}}$	$b/2^{\frac{1}{2}}$	0	0
	-b	0	0	0	$-b/2^{\frac{1}{2}}$	$-b/2^{\frac{1}{2}}$	0	0
	0	b	0	0	$b/2^{\frac{1}{2}}$	$-b/2^{\frac{1}{2}}$	0	0
	0	-b	0	0	$-b/2^{\frac{1}{2}}$	$b/2^{\frac{1}{2}}$	0	0
	0	0	b	0	0	0	b	0
	0	0	-b	0	0	0	-b	0
	0	0	0	b	0	0	0	b
	0	0	0	-b	0	0	0	-b
Levels	5	5	5	5	3	3	5	5

Design points of the original central composite design.

Design points of the transformed asymmetrical rotatable design.

Thus we see that the transformed design is an incomplete factorial of $3^2 \times 5^2$ where F_1', F_2' factors in the transformed design have each three levels viz $0; \pm 2^{\frac{1}{2}}a$, while F_3', F_4' have levels in $0; \pm a; \pm b$.

ii) Designs obtained through incomplete block designs:-

Consider now the effect of Mt on the symmetrical rotatable designs obtained through the incomplete block designs (Das & Narasimaham 1962). Let there be v factors each at five levels $0; \pm a; \pm b$, where relation $b = 2a$ exists. If in Mt all D_is are the same as D_1, where D_1 is the 2×2 matrix

$$\begin{bmatrix} 1/2^{\frac{1}{2}} & 1/2^{\frac{1}{2}} \\ 1/2^{\frac{1}{2}} & -1/2^{\frac{1}{2}} \end{bmatrix},$$

then the transformed asymmetrical rotatable design is an incomplete factorial of $5^x \times 5^{(v-x)}$ where x factors remain unaltered and have levels $0; \pm a; \pm b$ and $(v-x)$ factors have each five levels viz $0; \pm 2^{\frac{1}{2}}a; \pm a/2^{\frac{1}{2}}$. On the contrary if the relation $b = 2a$ does not hold then the transformed design will be an incomplete factorial of $5^x \times 7^{(v-x)}$ where now $(v-x)$ factors have seven levels viz $0; \pm 2^{\frac{1}{2}}a; \pm a/2^{\frac{1}{2}}; \pm b/2^{\frac{1}{2}}$. For any other transformation matrix the number of levels will be larger.

Thus the salient feature of the difference between central composite designs and the rotatable designs obtained through incomplete block designs is that for the same transformation matrix, the former gives designs with smaller number of levels. The best we can achieve through the latter type is to get transformed rotatable designs in which factors have the same number of levels as the factors in the original design.

II. Transformations:- factors taken three at a time.

1) Central composite designs.

The next case which we now take up is the transformations where three factors are taken at a time to form one group of transformations. Though it is possible to put forth the most general form of such transformations but it is considered better to take up the particular cases which will give us the transformed designs where factors have reasonably small number of levels. Accordingly we go back to the transformation matrix Mt and choose D_is so that $D_1 = D_2 = \dots = D_{(v-x)/3} = D$ (say) where D, in this case, can be taken, in its general form, as the matrix

$$\begin{bmatrix} 1/(2+y_1^2)^{\frac{1}{2}} & y_1/(1+y_1^2)^{\frac{1}{2}} & 1/\{1+y_1^2+(2+y_1^2)^2\}^{\frac{1}{2}} \\ y_1/(2+y_1^2)^{\frac{1}{2}} & -1/(1+y_1^2)^{\frac{1}{2}} & y_1/\{1+y_1^2+(2+y_1^2)^2\}^{\frac{1}{2}} \\ 1/(2+y_1^2)^{\frac{1}{2}} & 0 & -(1+y_1^2)/\{1+y_1^2+(2+y_1^2)^2\}^{\frac{1}{2}} \end{bmatrix}$$

where y_1 can be assigned different real values. But instead of taking the general matrix D we shall consider a particular case of D (say) D_1 viz

$$\begin{bmatrix} 1/3^{\frac{1}{2}} & -1/2^{\frac{1}{2}} & 1/6^{\frac{1}{2}} \\ 1/3^{\frac{1}{2}} & -1/2^{\frac{1}{2}} & 1/6^{\frac{1}{2}} \\ 1/3^{\frac{1}{2}} & 0 & -2/6^{\frac{1}{2}} \end{bmatrix}$$

as through this choice the number of levels of the factors can be kept small.

Let us consider the effect of this transformation matrix on the central composite design in v factors each of which is at five levels 0; $\pm$ a; $\pm$ b. Let x be the number of factors which have been predicided not to be subjected to any transformation. It may, however, be noted here that if v is a multiple of three then x can take one of the values 0, 3, 6, 9, ... so that x (v-3), whereas if v is not a multiple of three, x can take one of the values from either of the two sets of values 1, 4, 7, 10, 13, ... & 2, 5, 8, 11, ... according as v is of the form 3n + 1 or 3n + 2 for all positive integral values of n. The transformed design will be asymmetrical rotatable, being an incomplete factorial of $7^{(v-x)/3} \times 3^{(v-x)/3} \times 5^{(v-x)/3} \times 5^{x}$ provided relation b = 2a holds. We find that (v-x) factors have been grouped into three sets such that the factors in the first set have for their seven levels 0; $\pm a/3^{\frac{1}{2}}$; $\pm 3^{\frac{1}{2}}a$; $\pm b/3^{\frac{1}{2}}$ while the factors in the remaining two sets have for them three and five levels 0; $\pm 2^{\frac{1}{2}}a$ and 0; $\pm 2a/6^{\frac{1}{2}}$; $\pm 4a/6^{\frac{1}{2}}$ respectively. There is, however, the set of x factors which has remained unaltered with respect to its levels. If, on the other hand, relation b = 2a is not satisfied then the transformed asymmetrical rotatable design will be an incomplete factorial of $7^{(v-x)/3} \times 5^{(v-x)/3} \times 9^{(v-x)/3} \times 5^{x}$. So that now (v-x)/3 factors forming one group have each five levels viz 0; $\pm 2^{\frac{1}{2}}a$; $\pm b/2^{\frac{1}{2}}$. The levels of another group of (v-x)/3 factors which in the last case had five levels, have increased to nine viz 0; $\pm 2a/6^{\frac{1}{2}}$; $\pm 4a/6^{\frac{1}{2}}$; $\pm b/6^{\frac{1}{2}}$; $\pm 2b/6^{\frac{1}{2}}$.

Similarly by assigning to y certain other particular values we can have alternative transformation matrices and consequently more of the asymmetrical rotatable designs but the number of levels of the factors will be more for anyother transformation matrix than for the one discussed above. The maximum number of levels of the factors, in fact, could go upto 15 in case D is of the form used above. One point about this type of transformations worth mentioning is that the transformed design is invariably asymmetrical unlike the type of transformations when factors are taken two at a time, when the transformed designs can be symmetrical also.

ii) Designs obtained through incomplete block designs:

As in the case of transformations involving two factors at a time, it will be seen that in the present case also the transformed designs obtained through rotatable designs which in turn have been constructed by employing incomplete block design technique, have larger number of levels for the factors than when the same have been obtained through the central composite design. Consider the transformation matrix Mt where all the D_is are the same as D, which is a 3 x 3 matrix

$$\begin{bmatrix} 1/3^{\frac{1}{2}} & 1/2^{\frac{1}{2}} & 1/6^{\frac{1}{2}} \\ 1/3^{\frac{1}{2}} & -1/2^{\frac{1}{2}} & 1/6^{\frac{1}{2}} \\ 1/3^{\frac{1}{2}} & 0 & -2/6^{\frac{1}{2}} \end{bmatrix},$$

and operate it upon the rotatable design in v factors obtained through the incomplete block design. Let x factors remain unaltered where x should take one of the values in the three sets of values defined in the last section, which will depend upon whether or not v is a multiple of three. In case the relation b = 2 a holds then the transformed rotatable design is an incomplete factorial of $7^{(v-x)/3} \times 5^{(v-x)/3} \times 9^{(v-x)/3} \times 5^{x}$. The transformed factors have been classified into three equal groups characterised by seven, five and nine levels viz $0; \pm a/3^{\frac{1}{2}}; \pm 2a/3^{\frac{1}{2}}; \pm 3a/3^{\frac{1}{2}}$; $0; \pm a/2^{\frac{1}{2}}; 2a/2^{\frac{1}{2}}$, and $0; \pm a/6^{\frac{1}{2}}; \pm 2a/6^{\frac{1}{2}}; \pm 3a/6^{\frac{1}{2}}; \pm 4a/6^{\frac{1}{2}}$ respectively. If, however, relation b = 2a does not hold, the transformed asymmetrical rotatable design is an incomplete factorial of $9^{(v-x)/3} \times 7^{(v-x)/3} \times 13^{(v-x)/3} \times 5^{x}$ showing clearly the increase in the number of levels of all the factors as compared to the case when relation b = 2a is satisfied. Any other transformation matrix, when factors are taken three at a time, will give rise to a transformed design having larger number of levels of the factors, thus giving us an indication of the limitations of the transformations in getting transformed rotatable designs having factors with smaller number of levels, through those generated with the help of incomplete block design.

In order to increase the degree of asymmetry in the transformed designs, transformation matrix which involve sub-matrices of orders 2 & 3 simultaneously can be used.

III. Transformations:- factors taken four at a time.

The case when sub-matrices in the transformation matrix are of order four, can be discussed similarly with respect to the central composite designs and the rotatable designs obtained through the incomplete block designs. The transformation matrix involving sub-matrix $\underline{D}$ of order four,

$$\begin{bmatrix} 1/2 & 1/2 & 1/2 & -1/2 \\ 1/2 & 1/2 & -1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 & 1/2 \\ -1/2 & 1/2 & 1/2 & 1/2 \end{bmatrix}$$

when operated on the rotatable design in four factors obtained through SIBD with parameters $v = 4$ $k = 3$ $b = 4$ $r = 3$ $\lambda = 2$, however, gives us an interesting case for study of the transformed design. In the original design each factor has five levels and there are 40 points - none of these being central. The relation $b^2 = 12^{\frac{1}{2}} a^2$ holds. The transformed design is such that it is an incomplete factorial of 6^4 where each factor has six levels $\pm a/2$; $\pm 3a/2$; $\pm b/2$, which are all non-zero. The design so obtained is unique because this is the only one got thus far with the use of transformations where the increase in the number of levels of the factors is only by unity so as to give us an even-levelled rotatable design.

If in the above rotatable design obtained through incomplete block design we repeat b-points 12 times then the relation b = a is established. Now the total number of design points is 128 which is large as compared to the 40 points in the original design and each factor is at three levels 0; ± a. If now the same transformation matrix is operated on these 128 design points we shall get the transformed design as an incomplete factorial of 4^4. Thus we have got another even levelled design in 4 factors.

When the transformation matrix is operated on the central composite design in four factors where relation b = 2a holds, we will find that the transformed design has remained unaltered in all respects. For example, the transformed design is again a central composite design having the same levels for its factors in number as well as in magnitude, as in the original central composite design.

Suitable combination of sub-matrices discussed in I, II & III, taken simultaneously can be used to increase the degree of asymmetry in the transformed design.

IV. <u>Reduction in the number of levels in the transformed designs.</u>

In the preceding sections it was observed that when the relation b = 2a holds for the central composite designs it was possible to choose transformation matrices which will give rotatable designs with some or all of the transformed factors having reduced number of levels than in the original design. In the present section it is intended to give a technique which will enable us to achieve that relation viz b = 2a in such central composite designs where it does not ordinarily hold, so that reduction in the number of levels in the transformed design can be brought about in the case of all such designs.

The central composite design is obtained with the help of sets (i) (aa ... a) and (ii) (b00 ... 0) by the two operations of rotation and multiplication. The relation between b and a is determined by the equation

$$\sum_u x_{iu}^4 = 3 \sum_u x_{iu}^2 x_{i'u}^2 \qquad \ldots\ldots\ E'$$

The technique consists in replacing b by 2a in the combinations (b00 ... 0) and repeating them p times, where p is determined from the equation E'. The value of p gives the number of times the sets (b00 ... 0) have to be repeated so as to satisfy the relation b = 2a. This technique holds for all number of factors greater than three.

For central composite designs in 4 and 5 factors the relation b = 2a holds and it has been shown that transformation matrix can be had which shall give transformed designs in which at least some of the factors have three levels. Let us next consider the case for six factor central composite design having 44 non-central points and where relation $b^4 = 32\ a^4$ holds. Repeat b combinations p times where p is determined from the equation

$$32\ a^4 + 2p\ (16\ a^4) = 3 \times 32\ a^4$$

$$p = 2.$$

Thus by repeating the sets (b00 ... 0) twice the relation b = 2a is satisfied, so that the modified central composite design in six factors is still five levelled but has now 32 + 2 x 12 = 56 non-central points. As a result of the operation of the transformation matrix

$$\begin{bmatrix} D_1 & 0 & 0 \\ 0 & D_2 & 0 \\ 0 & 0 & D_3 \end{bmatrix}$$

where $D_1 = D_2 = D_3 = D$ (say) is 2 x 2 matrix

$$\begin{bmatrix} 1/2^{\frac{1}{2}} & 1/2^{\frac{1}{2}} \\ 1/2^{\frac{1}{2}} & -1/2^{\frac{1}{2}} \end{bmatrix}$$

on the design points of the modified central composite design thus obtained, we will get a transformed rotatable design which is an incomplete factorial of 3^6 where each factor has 3 levels 0; $\pm 2^{\frac{1}{2}}a$. Similarly the cases of other factors can also be dealt with. We have given below a table depicting the difference in the number of design points between the central composite design and its modified form from which three levelled designs can be obtained by suitable choice of transformation matrix.

No. of factors	No. of non-central points in the central composite design	No. of non-central points in the modified central composite design where b = 2a
4	24	24
5	26	26
6	44	56
7	78	120
8	80	128
9	146	272
10	148	288

V. Inter-relationship between central composite designs and the rotatable designs obtained through incomplete block designs.

Consider a central composite design in ten factors having 148 points of which 128 points of the type (aa ... a) have been obtained by confounding the three independent interactions ABCDE, ACGIJ and EFGHI. As a matter of fact seven interactions in all will be confounded; ABCDFGHI, BDFHJ, ACEFHJ and BDEGIJ being the remaining four and are the generalised interactions, none of which is of order less than four. If the design points obtained above are subjected to the transformation matrix

$$\begin{bmatrix} D_1 & 0 & 0 & 0 & 0 \\ 0 & D_1 & 0 & 0 & 0 \\ 0 & 0 & D_1 & 0 & 0 \\ 0 & 0 & 0 & D_1 & 0 \\ 0 & 0 & 0 & 0 & D_1 \end{bmatrix}$$

where the sub-matrix D_1 is

$$\begin{bmatrix} 1/2^{\frac{1}{2}} & 1/2^{\frac{1}{2}} \\ 1/2^{\frac{1}{2}} & -1/2^{\frac{1}{2}} \end{bmatrix}$$

then the transformed design points will be the same as given by *Das (1963) through the following incomplete block design with block sizes 5 and 2;*

1	2	3	4	5
6	7	8	9	5
1	2	8	9	10
6	7	3	4	10
1	7	3	9	5
6	2	8	4	5
1	7	8	4	10
6	2	3	9	10

Block Size 5

1	6
3	8
4	9
5	10

Block Size 2.

where 1, 2, 3, ... 10 stand for factors A, B, C, ... J respectively. Blocks of size 5 are utilized to get 128 design points of the type (aa ... a) while blocks of size 2 are used to generate points of type (b00 ... 0).

Here in the above example we have applied transformation when factors are taken two at a time. In fact (AF), (BG), (CH), (DI) and (EJ) is the grouping in which way the transformation matrix has been selected.

Similarly we can take up a central composite design in 12 factors having 280 design points. Of these 280 design points, 256 of the type (aa ... a) have been obtained by confounding 4 independent interactions ABCDEF, ABCDGHIJ, ABCEGIK, ABDEHIL. There will in all be 15 interactions confounded, the remaining being generalised interactions, none of them will be of order less than four. If now the transformation matrix

$$\begin{bmatrix} D_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & D_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & D_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & D_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & D_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & D_1 \end{bmatrix}$$

where D_1 is 2 x 2 matrix

$$\begin{bmatrix} 1/2^{\frac{1}{2}} & 1/2^{\frac{1}{2}} \\ 1/2^{\frac{1}{2}} & -1/2^{\frac{1}{2}} \end{bmatrix}$$

and the grouping of the factors is (AG), (BH), (CI), (DJ), (EK), (FL), is operated on the design points obtained above we will get the transformed design whose points will be the same as given by Das (1963) through the incomplete block design with block sizes 6 and 2

1	2	3	4	5	6
7	8	9	10	5	6
1	2	9	10	11	12
7	8	3	4	11	12
10	5	12	1	8	3
4	5	12	7	2	9
4	11	6	1	8	9
10	11	6	7	2	3

Block Size 6

1	7
2	8
3	9
4	10
5	11
6	12

Block Size 2

Both the cases discussed above lead us to think on the lines that with the help of the transformations it is possible to pass from a central composite design to the rotatable designs constructed through the technique of incomplete block design. Though it may be seen that a judicious choice of interactions to be confounded to get central composite design points, is needed.

Again, in the case of 4 factor central composite design we are taking the complete replication of 2^4 points of the type (aa ... a). Consequently if this design is operated upon by the transformation matrix

$$\begin{bmatrix} D_1 & 0 \\ 0 & D_1 \end{bmatrix}$$

where D_i is as defined earlier, the transformed design points can be obtained from the BIBD with the parameters $v = 4$ $k = 2$ $b^* = 6$ $r = 3$ & $\lambda = 1$. So here we have balanced incomplete block design with equal block size. In the central composite design in six factors we need confound only one interaction which we may take ABCDEF to get 2^5 design points of the type (aa ... a). On transforming those design points with the help of the transformation matrix

$$\begin{bmatrix} D_1 & 0 & 0 \\ 0 & D_1 & 0 \\ 0 & 0 & D_1 \end{bmatrix}$$

we will get a rotatable design, the points of which can be obtained through the BIBD.

1	2	3		1	4
1	5	6		2	5
4	2	6		3	6
4	3	5			

Block Size 3 Block Size 2

of unequal block size 3 and 2. This BIBD design has also been given by Das (1963).

It seems that this method of transformation to show inter-relationship between central composite designs and the rotatable designs obtained through incomplete block design will hold for all even factor cases. The only exception appears to be that of 8 factors. [illegible]

Transformation of factorial designs where the factors are each at two levels.

In this chapter transformations - when factors are taken two at a time have been applied on the design points of a factorial experiment of the form 2^n and the resultant design in which all the factors are at three levels each, has been obtained thus giving us, for a transformed design, a fraction of 3^n in 2^n combinations.

Let us consider a 2^n factorial experiment where n, the number of factors, is even and each factor is at two levels, $\pm a$. If A, B, C, D, ..., are the letters used to denote factors then (AB), (CD), (EF), ..., is the grouping which we shall adopt for the purposes of writing down the matrix of transformation coefficients when factors are taken two at a time. Let the transformation matrix be

$$\begin{bmatrix} D_1 & 0 & 0 & \cdots & 0 \\ 0 & D_1 & 0 & \cdots & 0 \\ 0 & 0 & D_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots\vdots\vdots & \vdots \\ 0 & 0 & 0 & \cdots & D_1 \end{bmatrix}$$

where the sub-matrices D_1s are of order two each and are of the form

$$\begin{bmatrix} 1/2^{\frac{1}{2}} & 1/2^{\frac{1}{2}} \\ 1/2^{\frac{1}{2}} & -1/2^{\frac{1}{2}} \end{bmatrix}$$

When this transformation matrix is operated on the 2^n factorial experiment taken above, following are the general properties exhibited in the transformed design. These are, however, apart from the fact that the transformed ~~design~~ factors are at three levels 0; $\pm 2^{\frac{1}{2}}a$. Before giving these general properties it is considered worthwhile to put down the nomenclature of the interactions in the transformed design. Firstly, factors denoted by A, B, C, D, E, ..., will be the factors A', B', C', D', E', ..., in the transformed design but we shall still denote the factors by

A, B, C, D, etc, as this will not cause any confusion. Secondly, since transformed factors are at three levels each, we shall use $A_1B_1C_2D_1$ to denote a four factor interaction in which A, B, D, have their linear components while the factor C has its quadratic component involved in it. Similarly, $A_1B_2C_2D_2$ for example, will be used to denote interaction of the linear component of factor A and quadratic components of each of the factors B, C, D.

In the transformed design, without blocking

i) The full number of $(2^n - 1)$ degrees of freedom are estimable

ii) The interaction with the largest number of factors which can be estimated is the one with $n/2$ factors in it.

iii) Information regarding interactions which involve factors that are coming as groups defined earlier, is completely lost. Consequently, interactions of order $\geqslant n/2$ are not estimable.

iv) Linear components of all main effects and interactions of all linear components can be estimated independent of one another.

v) Consider a $(x + y)$ factor interaction where x is the number of factors of which linear components are involved while y factors, which determine y groups of two factors each, have their quadratic components. Obviously $(x + y) \leqslant n/2$. Then this interaction will form aliases with interactions of the same order. Other members of the aliases group can be determined by taking interactions in which factors having linear components are the same x factors while the remaining y factors, having their quadratic components, will be obtained as a permutation of all the possible permutations of 2y factors subject to the restriction that no two of the y factors come as a group defined earlier. If, however, x factors also have their quadratic components involved in the $(x + y)$ factor interaction to start with, then interactions obtained by all possible permutations of 2(x + y) factors of (x + y) groups will all form one aliases group.

If we start with a 2^6 design with blocking and apply the same transformation matrix then the transformed design is again a fraction of 3^6 but now all (2^6 - 1) i.e. 63 degrees of freedom are not estimable. The number of estimable degrees of freedom will depend upon the interactions confounded due to blocking. It has been observed, however, that if two factor interactions, such that the factors form one group, are confounded in the original design then the maximum number of degrees of freedom can be estimated in the transformed design. The interactions confounded will be those which involve only quadratic components.

All the above mentioned properties can be easily verified from the case of 2^6 factorial design considered below:

Confounded affects in the original 2^6 design		ACE, ABD, BCF, DEF, ABEF, ACDF, BCDE.	ABE, BDF, ACD, CEF, ADEF, ABCF, BCDE.	ACE, BCF, BDE, ADF, ABEF, ABCD, CDEF.	ACF, BCE, BDF, ADE, ABCD, ABEF, CDEF	AB, CD, EF, ABCD, CDEF, ABEF, ABCDEF.
(1)	(2)	(3)	(4)	(5)	(6)	(7)
Affects:	degree of freedom					
$A_1, B_1, C_1, D_1, E_1, F_1$.	6	6	6	6	6	6
$A_2 \equiv B_2; C_2 \equiv D_2; E_2 \equiv F_2$.	3	3	3	3	3	0
$A_1C_1, A_1D_1, A_1E_1, A_1F_1,$ $B_1C_1, B_1D_1, B_1E_1, B_1F_1$ $C_1E_1, C_1F_1, D_1E_1, D_1F_1$.	12	12	12	12	12	12

(1)	(2)	(3)	(4)	(5)	(6)	(7)
$A_1C_2 \equiv A_1D_2$; $A_2C_1 \equiv B_2C_1$; $A_2D_1 \equiv B_2D_1$; $A_1E_2 \equiv A_1F_2$; $A_2E_1 \equiv B_2E_1$; $A_2F_1 \equiv B_2F_1$; $B_1C_2 \equiv B_1D_2$; $B_1E_2 \equiv B_1F_2$; $C_1E_2 \equiv C_1F_2$; $C_2E_1 \equiv D_2E_1$; $C_2F_1 \equiv D_2F_1$; $D_1E_2 \equiv D_1F_2$.	12	8 A_2C_1* C_1E_2* A_2D_1* D_1E_2*	6 A_1C_2* A_2E_1* C_1E_2* B_1C_2* A_2F_1* D_1E_2*	12	12	12
$A_2C_2 \equiv A_2D_2 \equiv B_2C_2 \equiv B_2D_2$; $A_2E_2 \equiv D_2E_2 \equiv A_2F_2 \equiv B_2F_2$; $C_2E_2 \equiv C_2F_2 \equiv B_2E_2 \equiv D_2F_2$.	3	2 A_2E_2*	3	*	*	*
$A_1C_1E_1$, $B_1D_1F_1$, $A_1C_1F_1$, $A_1D_1E_1$, $A_1D_1F_1$, $B_1D_1E_1$, $B_1C_1E_1$, $B_1C_1F_1$	8	*	*	*	*	8
$A_1C_2E_1 \equiv A_1D_2E_1$; $B_1D_2F_1 \equiv B_1C_2F_1$	2	*	*	2	2	2
$B_1D_2E_1 \equiv B_1C_2E_1$; $A_1C_2F_1 \equiv A_1D_2F_1$	2	*	*	2	2	2
$A_2C_1E_1 \equiv B_2C_1E_1$; $A_1C_1E_2 \equiv A_1C_1F_2$	2	2	*	2	2	2
$B_1D_1F_2 \equiv B_1D_1E_2$; $B_2D_1F_1 \equiv A_2D_1F_1$	2	2	*	2	2	2
$A_1D_1E_2 \equiv A_1D_1F_2$; $A_2D_1E_1 \equiv B_2D_1E_1$	2	2	*	2	2	2
$B_1C_1E_2 \equiv B_1C_1F_2$; $A_2C_1F_1 \equiv B_2C_1F_1$	2	2	*	2	2	2
$A_1C_2E_2 \equiv A_1C_2F_2 \equiv A_1D_2E_2 \equiv A_1D_2F_2$, $A_2C_1E_2 \equiv A_2C_1F_2 \equiv B_2C_1E_2 \equiv B_2C_1F_2$, $A_2C_2E_1 \equiv A_2D_2E_1 \equiv B_2D_2E_1 \equiv B_2C_2E_1$. 6	3	3	3	3	3	3
$B_2D_2F_1 \equiv A_2C_2F_1 \equiv A_2D_2F_1 \equiv B_2C_2F_1$, $B_2D_1F_2 \equiv A_2D_1E_2 \equiv A_2D_1F_2 \equiv B_2D_1E_2$, $B_1D_2F_2 \equiv B_1D_2E_2 \equiv B_1C_2E_2 \equiv B_1C_2F_2$.	3	3	3	3	3	3
$A_2C_2E_2 \equiv B_2D_2F_2 \equiv A_2C_2F_2 \equiv A_2D_2E_2 \equiv A_2D_2F_2 \equiv B_2D_2E_2 \equiv B_2C_2E_2 \equiv B_2C_2F_2$.	1	1	1	1	1	*
	63	46	37	52	52	56

* denotes the effects confounded alongwith aliases, if any.

SUMMARY:

In the thesis, the role of transformations in getting asymmetrical rotatable designs from the already existing symmetrical ones has been brought forth. The usefulness of transformations has been established beyond doubt since, apart from getting asymmetrical rotatable designs, it has been possible to obtain, through transformations, symmetrical rotatable designs which have the same number of levels as, or larger or smaller number of levels than in the original design. The method of getting such designs is very simple and consists in the multiplication of the design matrix with some suitable transformation matrix where the coefficients have to be chosen in accordance with two restrictions laid out. The resulting matrix gives the required transformed design and is symmetrical or asymmetrical - a fact which will depend upon the choice of transformation matrix. The number of levels that the factors may have in the transformed design can also be controlled to some extent, through proper choice of transformation.

Application of such transformations has been extended to simple factorial experiments to get some interesting fractions of 3^n design. Inter-relationship between the central composite designs and the rotatable designs obtained through the incomplete block design has been shown in some of the cases and it is expected that for the cases of larger number of factors also, similar relationship exists.

REFERENCES:

1. Box, G.E.P. & Hunter, J.S. (1957). Multifactorial experimental designs for exploring response surfaces. Ann. Math. Stat. Vol. 28; 195 - 241.

2. Box, G.E.P. & Behnken, D.W (1958). A class of three level second order designs for surface fitting. Stat. Techniques Research Group Tech. Report No. 26. Princeton University.

3. Bose, R.C. & Draper, N.R (1959). Second order rotatable designs in three dimensions. Ann. Math. Stat. Vol. 30; 1097 - 1112.

4. Box, G.E.P. & Behnken, D.W (1960). Simplex-sum-designs: A class of second order rotatable designs derivable from those of first order. Ann. Math. Stat. Vol. 31, 838 - 864.

5. Cochran, W.G. & Cox, G.M. (1951). Experimental Designs. Wiley, New York.

6. Das, M.N. (1961). Construction of rotatable designs from factorial designs. J. Indian Soc. Agric. Stat. Vol. 13. 169 - 194.

7. Das, M.N. & Narasimham, V.L. (1962). Construction of rotatable designs through balanced incomplete block designs. Ann. Math. Stat. Vol. 33, 1421 - 1439.

8. Das, M.N. (1963). On construction of second order rotatable designs through balanced incomplete block designs with blocks of unequal sizes; Cal. Stat. Assoc. Bult. Vol. 12, 31 - 46.

9. Draper, N.R. (1960). Second order rotatable designs in four or more dimensions. Ann. Math. Stat. Vol. 31; 23-33.

10. Ramachander, P.R. (1963). Asymmetrical response surface designs. Thesis submitted to I.A.R.S., New Delhi.