Statistics and Applications {ISSN 2454-7395 (online)}
Volume 21, No. 2, 2023 (New Series), pp 141-154
http://www.ssca.org.in/journal

Resolvable and 2-Replicate PBIB Designs based on Higher Association Schemes Using Polyhedra

Vinayaka[1] and Rajender Parsad[2]
[1] *The Graduate School, ICAR-Indian Agricultural Research Institute, New Delhi-110012*
[2] *ICAR-Indian Agricultural Statistics Research Institute, New Delhi-110012*

Received: 15 May 2022; Revised: 29 November 2022; Accepted: 01 December 2022

Abstract

In this article, three new association schemes and construction of partially balanced incomplete block (PBIB) designs based on these association schemes in three and four associate classes using polyhedra have been proposed. Construction methods use polyhedra such as icosahedron, octahedron and pentagonal prism. PBIB designs based on icosahedral and octahedral association schemes are resolvable block designs whereas designs based on pentagonal prism association scheme are 2-replicate PBIB designs. A simple analysis of these designs is outlined including generalized forms of canonical efficiency factors (CEFs) and average variances ($\bar{V}$). A catalogue of PBIB designs for k (size of each block) ≤ 20 is given along with computed efficiencies.

Key words: Icosahedral association scheme; Octahedral association scheme; Pentagonal prism association scheme; Resolvable partially balanced incomplete block design.

1. Introduction

PBIB designs based on 2-associate classes have been extensively studied in the literature and for a comprehensive catalogue of these designs; one may refer to Clatworthy (1973); Dey (1977); Sinha (1991); Ghosh and Divecha (1995); and Saurabh and Sinha (2022). A lot of literature is available on PBIB designs based on 3- or higher class association schemes. PBIB designs based on rectangular (3-class) association scheme (known as rectangular designs) are an important class of block designs with factorial structure for experiments with two factors [see *e.g.*, Vartak (1955), Sharma and Das (1985), Suen (1989), Srivastava *et al.* (2000), Parsad *et al.* (2007a, 2007b) and references cited therein]. The nested group divisible designs, a class of PBIB(3) designs, useful for 3-factor experiments was introduced by Roy (1953) were subsequently studied by Raghavarao (1960); Bhagwandas *et al.* (1992); Duan and Kageyama (1993); Miao *et al.* (1996); and Mitra *et al.* (2002). More generalized association scheme called extended group divisible association scheme and designs based on this scheme are known as extended group divisible (EGD) designs was introduced by Hinkelmann (1964). Many useful applications of these designs and their catalogue are given in Parsad *et*

Corresponding Author: Vinayaka
Email: vinayaka.b3vs@gmail.com

al. (2007a, 2007b). Rao (1956) developed circular lattices which were essentially PBIB(3) designs for $v = 2n^2$ treatments, where $n \geq 2$ and these were further generalized by Varghese and Sharma (2004) to accommodate $2sn^2$ treatments; $n, s \geq 2$. Also, Varghese *et al.* (2004) gave some PBIB(3) designs and their applications to partial diallel crosses. Sharma *et al.* (2010) introduced 3-associate-class tetrahedral and cubical association schemes and methods of constructions of PBIB(3) designs based on these schemes using polyhedra such as tetrahedron and cube (hexahedron). On the similar lines, Vinayaka and Vinaykumar (2021) extended the work on graph based 2- and 3-associate class schemes of Garg and Farooq (2014) to 3- and 4-class graphical association schemes and constructions of related PBIB designs.

Some work on investigations of 4-associate class PBIB designs was carried out by several authors such as Nair (1951), Tharthare (1963, 1965), Garg *et al.* (2011), and others. Further, investigations on 2-replicate PBIB designs are limited to only Varghese and Sharma (2004); Sharma *et al.* (2010); and Kipkemoi *et al.* (2013, 2015).

For some parameters neither a BIB design nor a PBIB design with 2-associate classes is available. The best alternative design for such situations is higher associate PBIB design, if such design exists. Hence, in this investigation, we extend the work on 3- and 4-associate class PBIB designs further by proposing three new association schemes called icosahedral association scheme with 4-associate classes; octahedral association scheme with 3-associate classes and pentagonal prism association scheme with 3-associate classes and methods of constructing related PBIB designs based on these schemes. First two schemes produces 3- and 4-class PBIB designs belongs to the resolvable block designs which are also used in information theory *i.e.*, constructing A^2-codes and low density parity-check (LDPC) codes [see *e.g.*, Pei (2006); and Xu *et al.* (2015, 2020)] and in sequential experimentation over space and time [see *e.g.*, Patterson and Silvery (1980); John and Williams (1995); and Morgan and Reck (2007)]. The third scheme give rise to the two-replicate PBIB design which is beneficial in the situation of limited resources and also for developing mating plans in the area of plant breeding experiments like Narain (1993), Kaushik (1999), and others. We can also find applications of PBIB design in cryptology; see for example, Adhikari *et al.* (2007).

However, several authors such as Harshbarger (1949); Bose and Nair (1962); David (1967); Patterson and Williams (1976); Williams *et al.* (1976, 1977); Jarrett and Hall (1978); Varghese and Sharma (2004); Sharma *et al.* (2010); *etc.*, fostered detailed information on problems of construction and analysis of resolvable incomplete block designs.

Flowchart of the article as follows: In Section 2, three new association schemes *viz.*, icosahedral association scheme, octahedral association scheme and pentagonal prism association scheme are defined along with numerical illustrations. Section 3 deals with the constructions of PBIB designs using icosahedron, octahedron, and pentagonal prism along with examples. An outline of analysis of these designs is established in Section 4. Section 5 reveals a brief discussion. A catalogue of efficient PBIB designs has been obtained for $k \leq 20$ and is presented in the Appendix.

2. Definition of association schemes and numerical illustrations

It is well known that any polyhedron is a three-dimensional shape with V number of vertices, E number of edges and F number of faces. Polyhedra satisfy the Euler characteristic

χ which relates the V, F and E as $\chi = \mathrm{V} + \mathrm{F} - \mathrm{E}$, for details, one may refer to Richeson (2019). Further, convex polyhedra where every face is the same kind of regular polygon with n number of edges may be found among three families *viz.*, formerly triangles: these polyhedra are called deltahedra. There are only eight strictly-convex deltahedra out of which three are regular polyhedra (such as tetrahedron, octahedron and icosahedron are indeed platonic solids), and five are Johnson solids. Secondly, squares: the hexahedron is the only convex example and thirdly, pentagons: the regular dodecahedron is the only convex example. These are useful for constructions of PBIB designs; a reference can be made to Sharma *et al.* (2010).

Now we define three association schemes using icosahedron, octahedron and pentagonal prism in the sequel.

2.1. Icosahedral association scheme

Let the number of symbols (treatments) be $v = 12m$ ($m \geq 2$). Arrange these symbols on the twelve vertices of an icosahedron such that each vertex contains exactly m distinct symbols and intersected by five distinct edges. We define the four associates of a particular treatment ϕ as follows:

(i) Treatments except ϕ appearing in the same vertex with ϕ are the first associates;

(ii) Treatments appearing in different vertices that directly meet a vertex of ϕ through single edge are the second associates;

(iii) Treatments appearing in the end vertex that is exactly opposite to the vertex of ϕ are the third associates;

(iv) The remaining treatments are the fourth associates.

The parameters of first kind and second kind (association matrices) of the association scheme are delineated in continuation. *i.e.*, $v(= 12m)$, $n_1 = m - 1$, $n_2 = 5m$, $n_3 = m$, $n_4 = 5m$, and

$$\boldsymbol{P_1} = \begin{bmatrix} m-2 & 0 & 0 & 0 \\ 0 & 5m & 0 & 0 \\ 0 & 0 & m & 0 \\ 0 & 0 & 0 & 5m \end{bmatrix}, \boldsymbol{P_2} = \begin{bmatrix} 0 & m-1 & 0 & 0 \\ m-1 & 2m & 0 & 2m \\ 0 & 0 & 0 & m \\ 0 & 2m & m & 2m \end{bmatrix}$$

$$\boldsymbol{P_3} = \begin{bmatrix} 0 & 0 & m-1 & 0 \\ 0 & 0 & 0 & 5m \\ m-1 & 0 & 0 & 0 \\ 0 & 5m & 0 & 0 \end{bmatrix}, \boldsymbol{P_4} = \begin{bmatrix} 0 & 0 & 0 & m-1 \\ 0 & 2m & m & 2m \\ 0 & m & 0 & 0 \\ m-1 & 2m & 0 & 2m \end{bmatrix}.$$

Here, n_i is the number of i^{th} (i =1, 2, 3, 4) associates of a given treatment. Given any two treatments that are mutually i^{th} associates, the number of treatments common to the j^{th} associates of the first and k^{th} associates of the second is p^i_{jk} (i, j, k =1, 2, 3, 4) reflected in P_i matrices.

Moreover, this association scheme may also be defined alternatively as follows: Arrange $v = 12m$ ($m \geq 2$) treatments in 12 columns and m rows then the treatment δ, say, is the first associate of specific treatment ϕ, say, if δ belongs to the same column of ϕ; the second associate, if δ occur either in second or third or fourth or fifth or sixth column; the third associate, if δ occur in seventh column; and the fourth associate, otherwise.

Illustration 1: Let $v = 24(= 12 \times 2)$ treatments arranged on the twelve vertices of an icosahedron such that each vertex comprises exactly two distinct treatments are shown in Figure 1 or arrange these $v = 24(= 12 \times 2)$ treatments in 12 columns and 2 rows as given below.

1	3	5	7	9	11	13	15	17	19	21	23
2	4	6	8	10	12	14	16	18	20	22	24

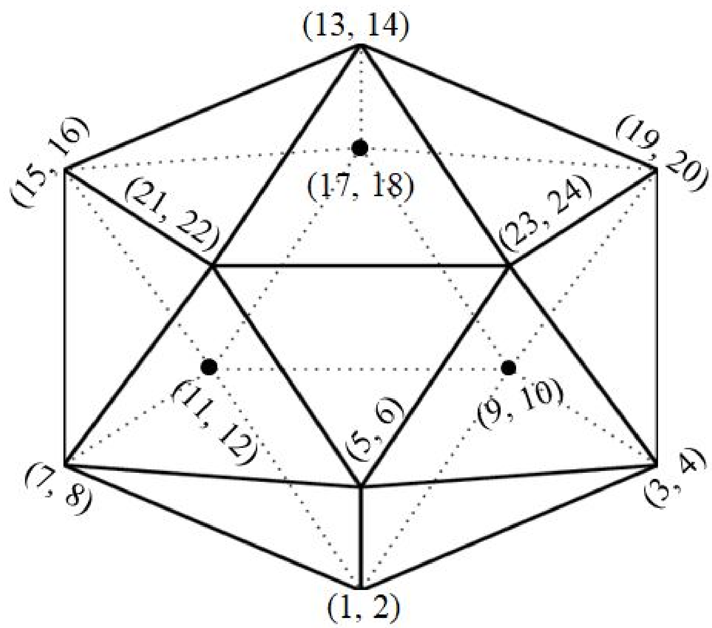

Figure 1: Arrangement of 24 treatments on the vertices of an icosahedron

The parameters of this association scheme are $v = 24$, $n_1 = 1$, $n_2 = 10$, $n_3 = 2$, $n_4 = 10$, and association matrices as:

$$\boldsymbol{P_1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 10 \end{bmatrix}, \boldsymbol{P_2} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 4 & 0 & 4 \\ 0 & 0 & 0 & 2 \\ 0 & 4 & 2 & 4 \end{bmatrix}, \boldsymbol{P_3} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 10 \\ 1 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 \end{bmatrix}, \boldsymbol{P_4} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 4 & 2 & 4 \\ 0 & 2 & 0 & 0 \\ 1 & 4 & 0 & 4 \end{bmatrix}.$$

2.2. Octahedral association scheme

Let the number of treatments be $v = 6m$ $(m \geq 2)$. Arrange these $v = 6m$ treatments on the six vertices of an octahedron such that each vertex filled with m number of distinct treatments. Now we define the three associates of a specific treatment θ as follows:

(i) Treatments other than θ present in the same vertex of θ are the first associates;

(ii) Treatments present in different vertices that intersect the vertex of θ through their respective edges are the second associates;

(iii) The remaining treatments are the third associates.

The parameters of first kind of this association scheme are $v(=6m)$, $n_1 = m-1$, $n_2 = 4m$, $n_3 = m$. Further, association matrices (parameters of second kind) are as follows:

$$\boldsymbol{P_1} = \begin{bmatrix} m-2 & 0 & 0 \\ 0 & 4m & 0 \\ 0 & 0 & m \end{bmatrix}, \boldsymbol{P_2} = \begin{bmatrix} 0 & m-1 & 0 \\ m-1 & 2m & m \\ 0 & m & 0 \end{bmatrix}, \boldsymbol{P_3} = \begin{bmatrix} 0 & 0 & m-1 \\ 0 & 4m & 0 \\ m-1 & 0 & 0 \end{bmatrix}.$$

The alternative definition for the above association scheme is as follows: Arrange $v = 6m$ ($m \geq 2$) treatments in six columns and m rows then the treatment δ, say, is the first associate of specific treatment θ, say, if δ belongs to the same column of θ; the second associate, if δ appears in any column except fourth column; and the third associate, otherwise.

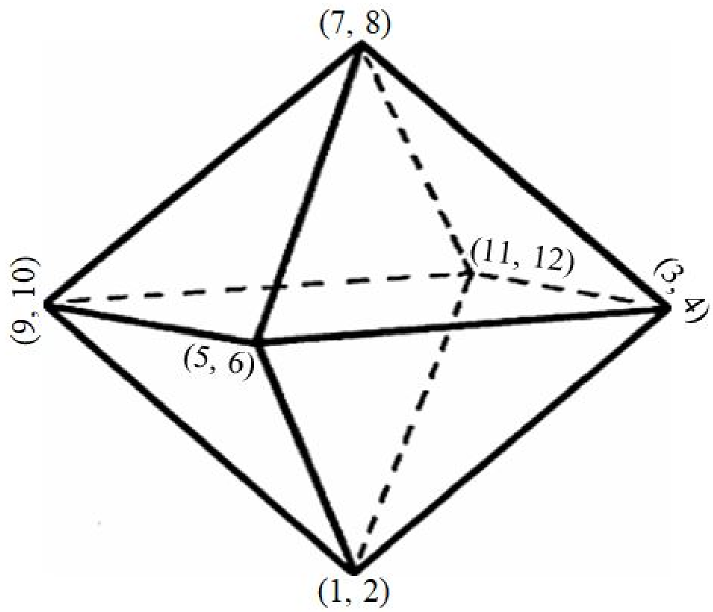

Figure 2: Arrangement of 12 treatments on the vertices of an octahedron

Illustration 2: Let $v = 12(= 6 \times 2)$ treatments arranged on the six vertices of an octahedron such that each vertex contains 2 distinct treatments are shown in Figure 2 or arrange these $v = 12(= 6 \times 2)$ treatments in six columns and two rows as given below.

1	3	5	7	9	11
2	4	6	8	10	12

The parameters of this association scheme are $v = 12$, $n_1 = 1$, $n_2 = 8$, $n_3 = 2$, and

$$\boldsymbol{P_1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \boldsymbol{P_2} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 4 & 2 \\ 0 & 2 & 0 \end{bmatrix}, \boldsymbol{P_3} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 8 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

2.3. Pentagonal prism association scheme

A pentagonal prism is also polyhedron and a type of three-dimensional solid objects which comprises the two identical five sided pentagonal bases (ends) and remaining five faces are rectangles or parallelograms. Interestingly, two identical five sided pentagons contact each other with five edges respectively. Let $v = 10m$ ($m \geq 1$) be the number of treatments. Arrange these treatments on the ten vertices of a pentagonal prism such that each vertex contains exactly m distinct treatments. Now we define the three associates of a specific treatment ψ as follows:

(i) Treatments other than ψ present in the two vertices of the same edge E_y $\forall$ $y = 1, 2, 3, 4, 5$ are the first associates;

(ii) Treatments present in the different vertices of any two rectangles that contain common edge E_y except treatments lie on both terminals of E_y are the second associates;

(iii) The remaining treatments are the third associates.

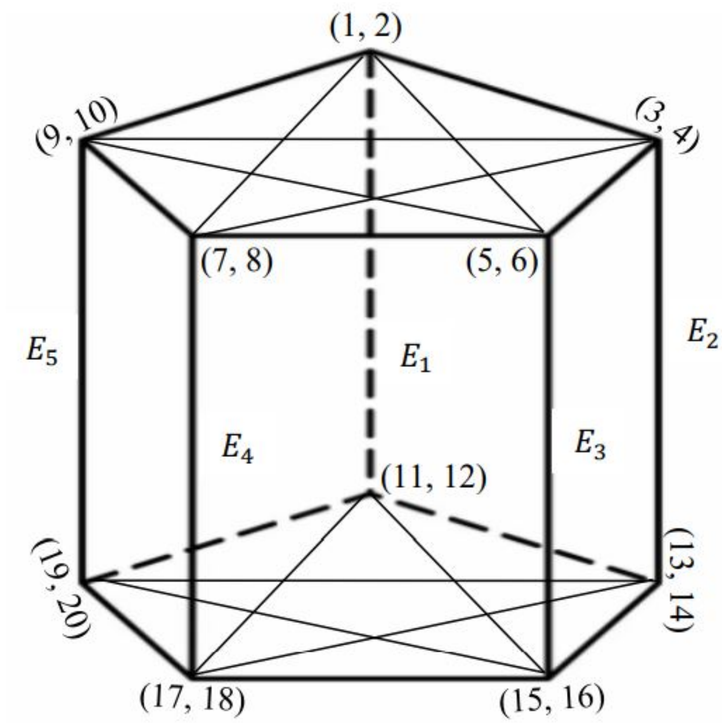

Figure 3: Arrangement of 20 treatments on the vertices of pentagonal prism

Here, the edges within identical five sided pentagons (both upper and lower) are of not interested, hence these are not named in Figure 3. The parameters of the association scheme are: $v(= 10m)$, $n_1 = 2m - 1$, $n_2 = 4m$, $n_3 = 4m$, and

$$\boldsymbol{P_1} = \begin{bmatrix} 2(m-1) & 0 & 0 \\ 0 & 4m & 0 \\ 0 & 0 & 4m \end{bmatrix}, \boldsymbol{P_2} = \begin{bmatrix} 0 & 2m-1 & 0 \\ 2m-1 & 0 & 2m \\ 0 & 2m & 2m \end{bmatrix}, \boldsymbol{P_3} = \begin{bmatrix} 0 & 0 & 2m-1 \\ 0 & 2m & 2m \\ 2m-1 & 2m & 0 \end{bmatrix}$$

Alternatively, the above association scheme may be defined as follows: arrange $v = 10m$ ($m \geq 1$) treatments in 10 columns and m rows then the treatment δ, say, is the first associate of particular treatment ψ, say, if δ belongs to either same column of ψ or sixth column; the second associate, if δ appears either in second or fifth or seventh column; and the third associate, otherwise.

Illustration 3: Let $v = 20(= 10\times 2)$ treatments arranged on the ten vertices of a pentagonal prism such that each vertex contains $m = 2$ distinct treatments as shown in Figure 3 or arrange these treatments in 10 columns and 2 rows as given below.

1	3	5	7	9	11	13	15	17	19
2	4	6	8	10	12	14	16	18	20

The parameters of this association scheme are: $v = 20$, $n_1 = 3$, $n_2 = 8$, $n_3 = 8$, and

$$\boldsymbol{P_1} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}, \boldsymbol{P_2} = \begin{bmatrix} 0 & 3 & 0 \\ 3 & 0 & 4 \\ 0 & 4 & 4 \end{bmatrix}, \boldsymbol{P_3} = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 4 & 4 \\ 3 & 4 & 0 \end{bmatrix}.$$

3. Construction methods of PBIB designs

In this section, we give two construction methods of resolvable PBIB designs and a construction method of 2-replicate PBIB design based on aforesaid association schemes *i.e.*, icosahedral association scheme, octahedral association scheme, and pentagonal prism association scheme, respectively.

3.1. Method of constructing icosahedral PBIB(4) design

An arrangement $v = 12m$ ($m \geq 2$) treatments on the vertices of an icosahedron such that each vertex contains m number of distinct treatments is given in Figure 1. Evidently, each vertex is intersected by five edges. Let $v = 12m$ treatments are defined on the icosahedral association scheme. In order to form a block, combine the treatments of a chosen vertex and five distinct vertices which intersect this chosen vertex. Applying this process to all twelve vertices of an icosahedron yields a PBIB(4) design based on icosahedral association scheme with parameters $v = 12m$, $b = 12$, $r = 6$, $k = 6m$, $\lambda_1 = 6$, $\lambda_2 = 4$, $\lambda_3 = 0$, $\lambda_4 = 2$.

Example 1: Let $v = 24(= 12 \times 2)$ treatments are defined on the icosahedral association scheme. One can get an idea about arrangement of treatments on vertices of icosahedron with the help of Figure 1. Now, by following the procedure of Method 3.1, one gets a PBIB(4) design based on the icosahedral association scheme with parameters are as $v = 24$, $b = 12$, $r = 6$, $k = 12$, $\lambda_1 = 6$, $\lambda_2 = 4$, $\lambda_3 = 0$, $\lambda_4 = 2$. This design is a resolvable class of incomplete block designs wherein twelve blocks can be grouped into six sets of two blocks each, that is, {(B1, B2); (B3, B4); (B5, B6); (B7, B8); (B9, B10); (B11, B12)} such that every treatment appears in each set exactly once. The block structure of the design is given below.

Remark 1: For $m = 1$, this scheme also reduced to 3-associate class rectangular association scheme. The PBIB(3) design so obtained is symmetric rectangular design with parameters $v = 12 = b$, $r = 6 = k$, $\lambda_1 = 4$, $\lambda_2 = 0$, $\lambda_3 = 2$. This design seems to be new and not reported in the Varghese *et al.* (2004) and Parsad *et al.* (2007b).

Replication No.	Block No.	Block Contents
I	B1	(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12)
	B2	(13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24)
II	B3	(1, 2, 3, 4, 5, 6, 9, 10, 19, 20, 23, 24)
	B4	(7, 8, 11, 12, 13, 14, 15, 16, 17, 18, 21, 22)
III	B5	(1, 2, 3, 4, 5, 6, 7, 8, 21, 22, 23, 24)
	B6	(9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20)
IV	B7	(1, 2, 5, 6, 7, 8, 11, 12, 15, 16, 21, 22)
	B8	(3, 4, 9, 10, 13, 14, 17, 18, 19, 20, 23, 24)
V	B9	(1, 2, 7, 8, 9, 10, 11, 12, 15, 16, 17, 18)
	B10	(3, 4, 5, 6, 13, 14, 19, 20, 21, 22, 23, 24)
VI	B11	(1, 2, 3, 4, 9, 10, 11, 12, 17, 18, 19, 20)
	B12	(5, 6, 7, 8, 13, 14, 15, 16, 21, 22, 23, 24)

3.2. Method of constructing octahedral PBIB(3) design

An octahedron has eight triangular faces and twelve edges, each face enclosed by the three vertices. Let $v = 6m$ ($m \geq 2$). Arrangement of these v treatments on the six vertices of an octahedron such that each vertex contains m number of distinct treatments as indicated in the association scheme. Now form the contents of a block by taking treatments that lie on three vertices of specific triangular face. Likewise, obtain the other seven blocks using remaining triangular faces of octahedron. The eight blocks thus obtained, each corresponding to one triangular face. This process results in a PBIB(3) design based on the octahedral association scheme with parameters $v = 6m$, $b = 8$, $r = 4$, $k = 3m$, $\lambda_1 = 4$, $\lambda_2 = 2$, $\lambda_3 = 0$.

Example 2: Let $v = 12(= 6 \times 2)$ treatments are defined on the octahedral association scheme. Figure 2 gives an idea about arrangement of treatments on vertices of octahedron. Now applying the procedure of Method 3.2, we can get a PBIB(3) design based on the octahedral association scheme with parameters as $v = 12$, $b = 8$, $r = 4$, $k = 6$, $\lambda_1 = 4$, $\lambda_2 = 2$, $\lambda_3 = 0$. This design is resolvable as its eight blocks can be grouped into four sets of two blocks each, that is, {(B1, B2); (B3, B4); (B5, B6); (B7, B8)} such that every treatment appears in each set exactly once. The block layout of the design is displayed below.

Replication No.	Block No.	Block Contents
I	B1	(1, 2, 3, 4, 5, 6)
	B2	(7, 8, 9, 10, 11, 12)
II	B3	(1, 2, 3, 4, 11, 12)
	B4	(5, 6, 7, 8, 9, 10)
III	B5	(1, 2, 5, 6, 9, 10)
	B6	(3, 4, 7, 8, 11, 12)
IV	B7	(1, 2, 9, 10, 11, 12)
	B8	(3, 4, 5, 6, 7, 8)

Remark 2: For $m = 1$, this scheme also reduced to two-class group divisible (GD) association scheme. The design so obtained is a semi-regular group divisible (SRGD) design with parameters as $v = 6$, $b = 8$, $r = 4$, $k = 3$, $\lambda_1 = 0$, $\lambda_2 = 2$, $n_1 = 1$, $n_2 = 4$ which is SR19 in the Clatworthy (1973).

3.3. Method of constructing pentagonal prism PBIB(3) design

Arrange $v = 10m$ ($m \geq 1$) treatments on the vertices of pentagonal prism such that each vertex contains m number of distinct treatments. Let $v = 10m$ treatments are defined on the pentagonal prism association scheme. Evidently, one can form five distinct rectangular shapes through diagonals using upper and lower pentagons given in Figure 3, so these are named as diagonal rectangles. Form five blocks of the design each one corresponding to a diagonal rectangular shape by combining the treatments situated on four vertices of that diagonal rectangle as the block contents. This process yields a PBIB(3) design based on pentagonal prism association scheme with parameters as $v = 10m$, $b = 5$, $r = 2$, $k = 4m$, $\lambda_1 = 2$, $\lambda_2 = 0$, $\lambda_3 = 1$.

Example 3: Let $v = 20(= 10 \times 2)$ treatments are defined on the pentagonal prism association scheme. For the arrangement of the treatments given in Figure 3, Now, by following the procedure of Method 3.3, one can get a PBIB(3) design based on pentagonal prism association scheme with block contents are given below:

Block No.	Block Contents
B1	$(1, 2, 5, 6, 11, 12, 15, 16)$
B2	$(1, 2, 7, 8, 11, 12, 17, 18)$
B3	$(3, 4, 7, 8, 13, 14, 17, 18)$
B4	$(3, 4, 9, 10, 13, 14, 19, 20)$
B5	$(5, 6, 9, 10, 15, 16, 19, 20)$

The design so obtained is a pentagonal prism design with parameters as $v = 20$, $b = 5$, $r = 2$, $k = 8$, $\lambda_1 = 2$, $\lambda_2 = 0$, $\lambda_3 = 1$.

4. Analysis

The above designs *viz.*, icosahedral, octahedral, and pentagonal prism designs can be analyzed as general PBIB designs. For completeness, simple steps for method of analysis are as follows: we know that the liner additive fixed effect model *i.e.*,

$$\mathbf{y} = \mu\mathbf{1} + \mathrm{Z}_1'\boldsymbol{\alpha} + \mathrm{Z}_2'\boldsymbol{\beta} + \boldsymbol{\varepsilon}$$

where, $\mathbf{y}$ = vector of n observations, μ = general mean, $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_v)$ = vector of treatment effects, $\boldsymbol{\beta} = (\beta_1, \beta_2, \ldots, \beta_b)$ = vector of block effects, $\mathbf{1}$ = vector of unities with order $(n \times 1)$, $\mathbf{Z_1'}$ = treatments vs observations incidence matrix with order $(v \times n)$, $\mathbf{Z}_2'$ = blocks vs observations incidence matrix with order $(b \times n)$ and $\boldsymbol{\varepsilon} \sim \mathrm{N}\left(\mathbf{0}, \sigma^2\mathbf{I}_n\right)$ = vector of errors with order $(n \times 1)$.

The general expressions of the information (C) matrices, Eigen values (η_l, $\forall\ l = 1$, 2, 3, 4) and corresponding multiplicities (ω_l, $\forall\ l = 1$, 2, 3, 4) of these information matrices for aforementioned designs (*i.e.*, icosahedral, octahedral, and pentagonal prism designs) are displayed in the Table 1. Here, $\boldsymbol{C}_{id}$, $\boldsymbol{C}_{od}$, and $\boldsymbol{C}_{pd}$ are the information matrices of icosahedral, octahedral, and pentagonal prism designs, respectively and also their corresponding incidence matrices denoted as $\mathbf{N_1}$, $\mathbf{N_2}$, and $\mathbf{N_3}$. Further, concurrence matrices and associates using these incidence matrices are also mentioned in the Table 2.

Table 1: Eigen values and corresponding multiplicities of C-matrices of designs

Particulars	C-matrix	Eigen values		Multiplicities	
Icosahedral design	$\boldsymbol{C}_{id} = 6\mathbf{I}_{12m} - (6m)^{-1}\mathbf{N_1N_1'}$	η_1	6	ω_1	$12m-7$
		η_2	5.745	ω_2	3
		η_3	4.255	ω_3	3
		η_4	0	ω_4	1
Octahedral design	$\boldsymbol{C}_{od} = 4\mathbf{I}_{6m} - (3m)^{-1}\mathbf{N_2N_2'}$	η_1	4	ω_1	$2(3m-2)$
		η_2	2.667	ω_2	3
		η_3	0	ω_3	1
Pentagonal prism design	$\boldsymbol{C}_{pd} = 2\mathbf{I}_{10m} - (4m)^{-1}\mathbf{N_3N_3'}$	η_1	2	ω_1	$5(2m-1)$
		η_2	1.809	ω_2	2
		η_3	0.691	ω_3	2
		η_4	0	ω_4	1

It is well known that the canonical efficiency factors (CEFs) is $1/r$ times of harmonic mean of non-zero and positive Eigen values of the information matrix for a given block design. *i.e.*,

$$CEFs = \frac{1}{r}\left[\frac{(\omega_1+\omega_2+\ldots+\omega_l)}{\left(\frac{\omega_1}{\eta_1}+\frac{\omega_2}{\eta_2}+\ldots+\frac{\omega_l}{\eta_l}\right)}\right]$$

Table 2: Concurrence matrices and associates using incidence matrices of designs

Particulars	$\mathbf{N_1N_1'} = ((n_{ii'}))$	$\mathbf{N_2N_2'} = ((n_{ii'}))$	$\mathbf{N_3N_3'} = ((n_{ii'}))$
if $i = i'(= 1, 2, \ldots, v)$	$= r(= 6)$	$= r(= 4)$	$= r(= 2)$
if i and i' are the 1st associates	$= \lambda_1(= 6)$	$= \lambda_1(= 4)$	$= \lambda_1(= 2)$
if i and i' are the 2nd associates	$= \lambda_2(= 4)$	$= \lambda_2(= 2)$	$= \lambda_2(= 0)$
if i and i' are the 3rd associates	$= \lambda_3(= 0)$	$= \lambda_3(= 0)$	$= \lambda_3(= 1)$
if i and i' are the 4th associates	$= \lambda_4(= 2)$	–	–

Suppose for icosahedral design, there are four Eigen values (η_l) and their corresponding multiplicities (ω_l) as in Table 1, then its canonical efficiency factors are derived as follows:

$$CEFs = \frac{1}{6}\left[\frac{(12m-7+3+3)}{\left(\frac{12m-7}{6}+\frac{3}{5.745}+\frac{3}{4.255}\right)}\right] = \left[\frac{(12m-1)}{(12m+0.364)}\right] = \frac{11(12m-1)}{4(33m+1)}$$

Similarly, expressions of canonical efficiency factors (CEFs) and average variances ($\bar{V}$) of these designs are generalized in Table 3.

Table 3: Canonical efficiency factors (CEFs) and average variances ($\bar{V}$)

Particulars	CEFs	$\bar{V}$
Icosahedral design	$11(12m-1)/4(33m+1)$	$4(33m+1)/33(12m-1)$
Octahedral design	$2(6m-1)/(12m+1)$	$(12m+1)/4(6m-1)$
Pentagonal prism design	$(10m-1)/(10m+3)$	$(10m+3)/(10m-1)$

For more details and a comprehensive bibliography on canonical efficiency factors (CEFs), one may refer to Dey (2008). At last, a list of these designs using aforementioned three methods of construction is given along with computed efficiencies as Table 4, Table 5, and Table 6 respectively in the Appendix.

5. Discussion

The designs obtained from the icosahedral and octahedral association schemes fall into the resolvable class of incomplete block designs with minimal replications (*i.e.*, $r \leq 6$). The benefit of resolvable design is that its replications can be applied over different locations or over distinct time periods. Further, pentagonal prism association scheme provide 2-replicate PBIB designs which are beneficial when the experimenters facing the situation of constraint of resources. Additionally, efficiencies of these designs are quite high. Hence, these designs can be used to test a large number of cultivars in agricultural trials. The association schemes of these designs also find application in obtaining efficient partial diallel cross plans in plant/animal breeding experiments.

Acknowledgements

The authors thank the Editor-in-Chief and anonymous reviewers for their valuable suggestions and helpful comments which led to considerable improvement in the article.

References

Adhikari, A., Bose, M., Kumar, D., and Roy, B. (2007). Applications of partially balanced incomplete block designs in developing (2, n) visual cryptographic schemes. *IEICE Transactions on Fundamentals of Electronics, Communications and Computer Sciences*, **90**, 949-951.

Bhagwandas, Sinha, K., and Kageyama, S. (1992). Constructions of PBIB designs based on nested group divisible association scheme. *Utilitas Mathematica*, **41**, 169-174.

Bose, R. C. and Nair, K. R. (1962). Resolvable incomplete block designs with two replications. *Sankhya: The Indian Journal of Statistics, Series A*, 9-24.

Clatworthy, W. H. (1973). *Tables of Two-associate-class Partially Balanced Designs*. National Bureau of Standards, Applied Mathematics, Series No. 63, Washington D.C.

David, H. A. (1967). Resolvable cyclic designs. *Sankhya: The Indian Journal of Statistics, Series A*, 191-198.

Dey, A. (1977). Construction of regular group divisible designs. *Biometrika*, **64**, 647-649.

Dey, A. (2008). Canonical efficiency factors and related issues revisited. *Journal of Indian Society of Agricultural Statistics*, **62**, 169-173.

Duan, X. and Kageyama, S. (1993). Constructions of nested group divisible designs. *Statistics and Probability Letters*, **18**, 41-48.

Garg, D. K. and Farooq, S. A. (2014). Construction of PBIB designs through chosen lines and triangles of graphs. *International Journal of Mathematics Trends and Technology*, **8**, 25-32.

Garg, D. K., Jhaji, H. S., and Mishra, G. (2011). Construction of some new triangular and four associate class PBIB designs with two replicates. *International Journal of Mathematical Sciences and Applications*, **1**, 808-821.

Ghosh, D. K. and Divecha, J. (1995). Some new semi-regular group divisible designs. *Sankhya: The Indian Journal of Statistics, Series B*, **57**, 453-455.

Harshbarger, B. (1949). Triple rectangular lattices. *Biometrics*, **5**, 1-13.

Hinkelmann, K. (1964). Extended group divisible partially balanced incomplete block designs. *The Annals of Mathematical Statistics*, **35**, 681-695.

Jarrett, R. G. and Hall, W. B. (1978). Generalized cyclic incomplete block designs. *Biometrika*, **65**, 397-401.

John, J. A. and Williams, E. R. (1995). *Cyclic and Computer Generated Designs.* 2^{nd} Edition. Chapman and Hall London.

Kaushik, L. S. (1999). Partial diallel crosses based on three associate class association schemes. *Journal of Applied Statistics*, **26**, 195-201.

Kipkemoi, E. C., Koske, J. K., and Mutiso, J. M. (2013). Construction of three-associate class partially balanced incomplete block designs in two replicates. *American Journal of Mathematical Science and Applications*, **1**, 61-65.

Kipkemoi, E. C., Mutiso, J. M., and Bekolle, D. (2015). Construction of some new three associate class partially balanced incomplete block designs in two replicates. *International Journal of Academic Research in Progressive Education and Development*, **1**, 188-192.

Miao, Y., Kageyama, S., and Duan, X. (1996). Further constructions of nested group divisible designs. *Journal of the Japan Statistical Society*, **26**, 231-239.

Mitra, R. K., Sinha, K., Kageyama, S., and Singh, M. K. (2002). Constructions of group divisible and nested group divisible designs. *Utilitas Mathematica*, **61**, 167-174.

Morgan, J. P. and Reck, B. H. (2007). Resolvable designs with large blocks. *The Annals of Statistics*, **35**, 747-771.

Nair, K. R. (1951). Rectangular lattices and partially balanced incomplete block designs. *Biometrics*, **7**, 145-154.

Narain, P. (1993). *Statistical Genetics.* Wiley Eastern Ltd., New Delhi.

Parsad, R., Gupta, V. K., and Srivastava, R. (2007a). Designs for cropping systems research. *Journal of Statistical Planning and Inference*, **137**, 1687-1703.

Parsad, R., Kageyama, S., and Gupta, V. K. (2007b). Use of complementary property of block designs in PBIB designs. *Ars Combinatoria*, **85**, 173-182.

Patterson, H. D. and Silvey, V. (1980). Statutory and recommended list trials of crop varities in the united kingdom. *Journal of the Royal Statistical Society: Series A*, **143**, 219-252.

Patterson, H. D. and Williams, E. (1976). A new class of resolvable incomplete block designs. *Biometrika*, **63**, 83-92.

Pei, D. (2006). A survey of construction for A2-codes. *In Authentication Codes and Combinatorial Designs. Discrete Mathematics and its Applications*, ed. K. H. Rosen, 215-30. Chapman and Hall/CRC, New York.

Raghavarao, D. (1960). A generalization of group divisible designs. *The Annals of Mathematical Statistics*, 756-771.

Rao, C. R. (1956). A general class of quasifactorial and related designs. *Sankhya: The Indian Journal of Statistics*, **17**, 165-174.

Richeson, D. S. (2019). *Euler's Gem: the Polyhedron Formula and the Birth of Topology.* Princeton University Press, Princeton, New Jersey.

Roy, P. M. (1953). Hierarchical group divisible incomplete block designs with m-associate classes. *Science and Culture*, **19**, 210-211.

Saurabh, S. and Sinha, K. (2022). A list of new partially balanced designs. *Communications in Statistics-Theory and Methods*, 1-4. doi: 10.1080/03610926.2022.2059685.

Sharma, V. K. and Das, M. N. (1985). On resolvable incomplete block designs 1. *Australian Journal of Statistics*, **27**, 298-302.

Sharma, V. K., Varghese, C., and Jaggi, S. (2010). Tetrahedral and cubical association schemes with related PBIB(3) designs. *Model Assisted Statistics and Applications*, **5**, 93-99.

Sinha, K. (1991). A list of new group divisible designs. *Journal of Research of the National Institute of Standards and Technology*, **96**, 613-615.

Srivastava, R., Parsad, R., and Gupta, V. K. (2000). Structure resistant factorial designs. *Sankhya: The Indian Journal of Statistics, Series B*, **62**, 257-265.

Suen, C. Y. (1989). Some rectangular designs constructed by the method of differences. *Journal of Statistical Planning and Inference*, **21**, 273-276.

Tharthare, S. K. (1963). Right angular designs. *The Annals of Mathematical Statistics*, 1057-1067.

Tharthare, S. K. (1965). Generalized right angular designs. *The Annals of Mathematical Statistics*, 1535-1553.

Varghese, C. and Sharma, V. K. (2004). A series of resolvable PBIB(3) designs with two replicates. *Metrika*, **60**, 251-254.

Varghese, C., Sharma, V. K., Jaggi, S., and Sharma, A. (2004). *Three-Associate Class Partially Balanced Incomplete Block Designs and their Application to Partial Diallel Crosses*. Project report, IASRI publication, New Delhi.

Vartak, M. N. (1955). On an application of Kronecker product of matrices to statistical designs. *The Annals of Mathematical Statistics*, **26**, 420-438.

Vinayaka and Vinaykumar, L. N. (2021). Higher associate class partially balanced incomplete block designs using graphs for agricultural experiments. *The Pharma Innovation Journal*, **10**, 967-972. doi: 10.22271/tpi.2021.v10.i12Sn.11252.

Williams, E. R., Patterson, H. D., and John, J. A. (1976). Resolvable designs with two replications. *Journal of the Royal Statistical Society. Series B (Methodological)*, 296-301.

Williams, E. R., Patterson, H. D., and John, J. A. (1977). Efficient two-replicate resolvable designs. *Biometrics*, 713-717.

Xu, H., Feng, D., Sun, C., and Bai, B. (2015). Construction of LDPC codes based on resolvable group divisible designs. *In 2015 International Workshop on High Mobility Wireless Communications (HMWC)*, 111-115.

Xu, H., Yu, Z., Feng, D., and Zhu, H. (2020). New construction of partial geometries based on group divisible designs and their associated LDPC codes. *Physical Communication*, **39**, 1-38. doi: https://doi.org/10.1016/j.phycom.2019.100970.

Appendix

Table 4: PBIB(4) designs based on Icosahedral association scheme with $k \leq 20$ using Method 3.1

SI. No.	m	v	b	r	k	λ_1	λ_2	λ_3	λ_4	E_1	E_2	E_3	E_4	E
1	2	24	12	6	12	6	4	2	0	1	0.9649	0.8979	0.9282	0.9440
2	3	36	12	6	18	6	4	2	0	1	0.9763	0.9295	0.9510	0.9625

Table 5: PBIB(3) designs based on Octahedral association scheme with $k \leq 20$ using Method 3.2

SI. No.	m	v	b	r	k	λ_1	λ_2	λ_3	E_1	E_2	E_3	E
1	2	12	8	4	6	4	2	0	1	0.8889	0.8000	0.8800
2	3	18	8	4	9	4	2	0	1	0.9230	0.8571	0.9189
3	4	24	8	4	12	4	2	0	1	0.9411	0.8889	0.9388
4	5	30	8	4	15	4	2	0	1	0.9524	0.9090	0.9508
5	6	36	8	4	18	4	2	0	1	0.9600	0.9231	0.9589

Table 6: PBIB(3) designs based on Pentagonal prism association scheme with $k \leq 20$ using Method 3.3

SI. No.	m	v	b	r	k	λ_1	λ_2	λ_3	E_1	E_2	E_3	E
1	1	10	5	2	4	2	0	1	1	0.5882	0.7692	0.6923
2	2	20	5	2	8	2	0	1	1	0.7407	0.8695	0.8261
3	3	30	5	2	12	2	0	1	1	0.8108	0.9090	0.8788
4	4	40	5	2	16	2	0	1	1	0.8511	0.9302	0.9070
5	5	50	5	2	20	2	0	1	1	0.8772	0.9433	0.9245