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Construction of α– Resolvable and nearly α–Resolvable BIB designs

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Abstract
In experimental design, α-resolvable designs are preferred for their ability to partition blocks into subsets, each containing every treatment α times. These designs offer advantages such as orthogonality to management effects, ensuring that such effects do not interfere with treatment assessments. Moreover, they provide protection against the loss of entire blocks, where all treatments are equally affected. Resolvable designs facilitate intra-block analysis of variance, breaking down the block sum of squares into replication and block-within-replication components, aiding in data interpretation. However, α-resolvable designs are not always feasible for all experimental parameters. In such cases, nearly α-resolvable designs, where blocks can be grouped into sets containing $(v-1)$ treatments α times, offer a flexible alternative. These design concepts contribute significantly to experimental robustness, orthogonality, and variance analysis in various research contexts. In this article, we have developed a construction method of α-resolvable BIB designs for even number of treatments and a general construction method of nearly α-resolvable BIB designs.

Keywords: Balanced incomplete block design (BIBD), resolvable design, α-Resolvable balanced block design, nearly α-Resolvable balanced block design

1. Introduction

In many experiments where an incomplete-block design is used, a resolvable design, that is, a design in which the blocks can be partitioned into sets containing each treatment once, is desirable. Sometimes the blocks are naturally grouped into larger blocks containing each treatment exactly once. An example from microbiology is an experiment in which jars of treated bacteria are placed in incubators so that the bacteria may multiply. Here, blocks are the shelves within the incubators and large blocks are the incubators. Use of a resolvable design ensures that the large blocks are orthogonal to treatments (treated bacteria). In other cases, particularly in agricultural trials, neighbouring blocks are grouped into large blocks for management purposes. Then a resolvable design not only keeps management effects orthogonal to treatments, it also gives some protection against the loss of a whole large block, for all treatments are affected equally by such a loss. The large blocks are often called replicates, although their existence is independent of the treatment allocation. For further details see Bailey *et al.* (1995) [2]. Other advantage of resolvable design is that, in case of intra-block analysis of variance, the block sum of squares can be split into two components as, replication sum squares and block within replication sum of squares. So, here, we get extra information by blocks within replication. The combinatorial study of resolvability in block designs goes back at least as far as the well-known Kirkman's school girl problem formulated in 1850. The notion entered the statistical lexicon with Yates' work on square lattice designs (1936, 1940), although the term "resolvable design" was firstly introduced by Bose in 1942. He defined this for a balanced incomplete block design (BIBD) which is an arrangement of v symbols (treatment) into b sets (blocks) such that (i) each block contains $k\ (< v)$ distinct treatments; (ii) each treatment appears in $r\ (> \lambda)$ different blocks and (iii) every pair of distinct treatments appears together in exactly λ blocks. Here, the parameters of balanced incomplete block design (v, b, r, k, λ) are related by the parametric relations $vr = bk, r\ (k-1) = \lambda\ (v-1)$ and $b \geq v$ (Fisher's inequality). A block design is said to be α −resolvable if the b blocks each of size k can be grouped into r resolution sets of b/r blocks each such

that in each resolution set every treatment is replicated exactly α −times. Bose (1942) proved that necessary condition for the resolvability of a balanced incomplete block design is $b \geq v + r - 1$. There has been a very rapid development in this area of experimental designs. Some of the prominent work has been seen in Bailey *et al.* (1995) [2], Banerjee *et al.* (1990) [5], Caliński *et al.* (2008) [7], Kageyama (1972, 1973, 1977) [11-13], Kageyama *et al.* (1983, 2001) [14-15], Saka *et al.* (2021) [18]. Banerjee *et al.* (2018) [3] proposed some construction methods based on symmetric BIBD and Group Divisible (GD) designs.
Further, it is observed that, α −resolvable designs may not be available for all parametric structure, so, the concept of nearly α −resolvable designs, that is, a design in which the blocks can be partitioned into sets containing $(v-1)$ treatments α − times, has been discussed. For further details see Dinitz and Colbourn (1996) [8] and Abel and Funiro (2007) [1]. There are many authors such as Haanpaa and kaski (2005) [10], Greig *et al.* (2006) [9], Morales *et al.* (2007) [16] have been developed several construction methods of nearly α −resolvable and resolvable designs. Recently, Banerjee *et al.* (2018) [3] gave four different construction methods to obtain nearly α −resolvable designs based on symmetric BIBD and Group Divisible (GD) designs.
In the existing literature, most α-resolvable and nearly α-resolvable Balanced Incomplete Block Design (BIBD) constructions rely on symmetric BIBDs and Group Divisible (GD) designs, which often involve complex construction methods and may have limitations regarding their existence. In this study, we have introduced two straightforward construction methods. One method is designed for the construction of α-resolvable BIB designs, while the other method is tailored for the creation of nearly α-resolvable designs. These methods offer a simplified approach to achieve these designs, addressing some of the complexities and limitations associated with existing methodologies in the field.

2. Construction method of α −Resolvable BIB designs

We can construct α-Resolvable Balanced Incomplete Block (BIB) designs for v treatments where v is a positive even number, by following the steps provided below:

Step-1: Consider U_i, $i = 1, 2, \dots, (v-1)$ be a sequence of treatments which is obtained as

$$U_i = \left\{(i, v) \cup \left(i+1, i+v-2, i+2, i+v-3, \dots, i+\frac{v-2}{2}, i+\frac{v}{2}\right) \text{with mod } (v-1)\right\}$$

Step-2: Find a design D_i, $i = 1, 2, \dots, (v-1)$ from the sequence U_i, $i = 1, 2, \dots, (v-1)$ by pairwise cyclic rotation of elements as

i,	*v,*	*i+1,*	*i+v-2,*	*i+2,*	*i+v-3,*	…	*i+(v-2)/2,*	*i+v/2*
i+1,	*i+v-2,*	*i+2,*	*i+v-3,*	…	*i+(v-2)/2,*	*i + v/2*	*i,*	*v*
i+2,	*i+v-3,*	…	*i+(v-2)/2,*	*i+v/2,*	*i,*	*v,*	*i+1,*	*i+v-2*
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
i+(v-2)/2,	*i+v/2,*	*i,*	*v,*	*i+1,*	*i+v-2,*	*i+2,*	*i+v-3,*	…

Step-3: After consider vertical set of treatments of each design D_i, $i = 1, 2, \dots, (v-1)$ as block, we get a α −Resolvable BIB design D^* with parameters

$$v^* = v,\ b^* = v(v-1),\ r^* = v(v-1)/2, k^* = v/2, \lambda^* = v/2 \text{ and } \alpha = v/2$$

Example: Using steps of above method, we construct a α −Resolvable BIB design for $v = 6$
Step-1: for $v = 6$, the sets U_i, $i = 1, 2, 3, 4, 5$ will be $U_1 = \{1, 6, 2, 5, 3, 4\}$, $U_2 = \{2, 6, 3, 1, 4, 5\}$, $U_3 = \{3, 6, 4, 2, 5, 1\}$, $U_4 = \{4, 6, 5, 3, 1, 2\}$, $V_5 = \{5, 6, 1, 4, 2, 3\}$.

Step-2: Now, finding D_i, $i = 1, 2, \dots, 5$ from the sequence U_i, $i = 1, 2, \dots, 5$ by pairwise cyclic rotation of elements as

$$D_1 = \begin{matrix} 1, & 6, & 2, & 5, & 3, & 4 \\ 2, & 5, & 3, & 4, & 1, & 6 \\ 3, & 4, & 1, & 6, & 2, & 5 \end{matrix} \qquad D_2 = \begin{matrix} 2, & 6, & 3, & 1, & 4, & 5 \\ 3, & 1, & 4, & 5, & 2, & 6 \\ 4, & 5, & 2, & 6, & 3, & 1 \end{matrix}$$

$$D_3 = \begin{matrix} 3, & 6, & 4, & 2, & 5, & 1 \\ 4, & 2, & 5, & 1, & 3, & 6 \\ 5, & 1, & 3, & 6, & 4, & 2 \end{matrix} \qquad D_4 = \begin{matrix} 4, & 6, & 5, & 3, & 1, & 2 \\ 5, & 3, & 1, & 2, & 4, & 6 \\ 1, & 2, & 4, & 6, & 5, & 3 \end{matrix}$$

$$D_5 = \begin{matrix} 5, & 6, & 1, & 4, & 2, & 3 \\ 1, & 4, & 2, & 3, & 5, & 6 \\ 2, & 3, & 5, & 6, & 1, & 4 \end{matrix}$$

Step-3: After consider vertical set of treatments of each design D_i, $i = 1, 2, 3, 4, 5$, as block, we get a α −Resolvable BIB design D^* as

[(1, 2, 3), (6, 5, 4), (2, 3, 1), (5, 4, 6), (3, 1, 2), (4, 6, 5)]

[(2, 3, 4), (6, 1, 5), (3, 4, 2), (1, 5, 6), (4, 2, 3), (5, 6, 1)]
[(3, 4, 5), (6, 2, 1), (4, 5, 3), (2, 1, 6), (5, 3, 4), (1, 6, 2)]
[(4, 5, 1), (6, 3, 2), (5, 1, 4), (3, 2, 6), (1, 4, 5), (2, 6, 3)]
[(5, 1, 2), (6, 4, 3), (1, 2, 5), (4, 3, 6), (2, 5, 1), (3, 6, 4)]

with parameters

$v^* = 6,\ b^* = 30,\ r^* = 15, k^* = 3, \lambda^* = 3$ and $\alpha = 3$

3. Construction method of Nearly α −Resolvable BIB designs

We can construct nearly α-Resolvable Balanced Incomplete Block (BIB) designs for v treatments where v is a positive number, by following the steps provided below:

Step-1: Consider U_i, $i = 1, 2, \ldots, v$ be a set of $(v-1)$ treatments out of v treatments in which i^{th} treatment is missing. Arrange each $(v-1)$ treatments in each set U_i, $i = 1, 2, \ldots, v$ in increasing order. Let $(v-1)$ treatments in increasing order are t_1, t_2, t_3, …, t_{v-1}.

Step-2: Find a design D_i, $i = 1, 2, \ldots, v$ from the set U_i, $i = 1, 2, \ldots, v$ by arranging the treatments t_1, t_2, t_3, …, t_{v-1} as

$$\begin{array}{ccccc}
t_1, & t_2, & t_3, & \cdots\cdots, & t_{v-1} \\
t_2, & t_3, & t_4, & \cdots\cdots, & t_1 \\
t_3, & t_4, & t_5, & \cdots\cdots, & t_2 \\
\vdots & \vdots & \vdots & \cdots\cdots, & \vdots \\
\vdots & \vdots & \vdots & \cdots\cdots, & \vdots \\
t_{v-2}, & t_{v-1}, & t_1, & \cdots\cdots, & t_{v-3}
\end{array}$$

Step-3: After consider vertical set of treatments of each design D_i, $i = 1, 2, \ldots, (v-1)$ as block, we get a nearly α −Resolvable BIB design D^* with parameters

$v^* = v,\ b^* = v(v-1),\ r^* = (v-1)(v-2), k^* = v-2, \lambda^* = (v-2)(v-3)$ and $\mu = v-2$

Example: Using steps of above method, we construct a nearly α −Resolvable BIB design for $v = 7$

Step-1: for $v = 7$, the sets U_i, $i = 1, 2, \ldots, 7$ will be $U_1 = \{2, 3, 4, 5, 6, 7\}$, $U_2 = \{1, 3, 4, 5, 6, 7\}$, $U_3 = \{1, 2, 4, 5, 6, 7\}$, $U_4 = \{1, 2, 3, 5, 6, 7\}$, $U_5 = \{1, 2, 3, 4, 6, 7\}$, $U_6 = \{1, 2, 3, 4, 5, 7\}$, $U_7 = \{1, 2, 3, 4, 5, 6\}$

Step-2: for $U_1 = \{2, 3, 4, 5, 6, 7\}$, $t_1 = 2$, $t_2 = 3$, $t_3 = 4$, $t_4 = 5$, $t_5 = 6$, $t_6 = 7$ so, D_1 will be

$$D_1 = \begin{array}{cccccc}
2, & 3, & 4, & 5, & 6, & 7 \\
3, & 4, & 5, & 6, & 7, & 2 \\
4, & 5, & 6, & 7, & 2, & 3 \\
5, & 6, & 7, & 2, & 3, & 4 \\
6, & 7, & 2, & 3, & 4, & 5
\end{array}$$

Similarly, we can obtain designs D_2, D_3, D_4, D_5, D_6 and D_7 from the sets U_2, U_3, U_4, U_5, U_6 and U_7 respectively as

$$D_2 = \begin{array}{cccccc}
1, & 3, & 4, & 5, & 6, & 7 \\
3, & 4, & 5, & 6, & 7, & 1 \\
4, & 5, & 6, & 7, & 1, & 3 \\
5, & 6, & 7, & 1, & 3, & 4 \\
6, & 7, & 1, & 3, & 4, & 5
\end{array}
\quad
D_3 = \begin{array}{cccccc}
1, & 2, & 4, & 5, & 6, & 7 \\
2, & 4, & 5, & 6, & 7, & 1 \\
4, & 5, & 6, & 7, & 1, & 2 \\
5, & 6, & 7, & 1, & 2, & 4 \\
6, & 7, & 1, & 2, & 4, & 5
\end{array}
\quad
D_4 = \begin{array}{cccccc}
1, & 2, & 3, & 5, & 6, & 7 \\
2, & 3, & 5, & 6, & 7, & 1 \\
3, & 5, & 6, & 7, & 1, & 2 \\
5, & 6, & 7, & 1, & 2, & 3 \\
6, & 7, & 1, & 2, & 3, & 5
\end{array}$$

$$D_5 = \begin{array}{cccccc}
1, & 2, & 3, & 4, & 6, & 7 \\
2, & 3, & 4, & 6, & 7, & 1 \\
3, & 4, & 6, & 7, & 1, & 2 \\
4, & 6, & 7, & 1, & 2, & 3 \\
6, & 7, & 1, & 2, & 3, & 4
\end{array}
\quad
D_6 = \begin{array}{cccccc}
1, & 2, & 3, & 4, & 5, & 7 \\
2, & 3, & 4, & 5, & 7, & 1 \\
3, & 4, & 5, & 7, & 1, & 2 \\
4, & 5, & 7, & 1, & 2, & 3 \\
5, & 7, & 1, & 2, & 3, & 4
\end{array}
\quad
D_7 = \begin{array}{cccccc}
1, & 2, & 3, & 4, & 5, & 6 \\
2, & 3, & 4, & 5, & 6, & 1 \\
3, & 4, & 5, & 6, & 1, & 2 \\
4, & 5, & 6, & 1, & 2, & 3 \\
5, & 6, & 1, & 2, & 3, & 4
\end{array}$$

Step-3: After consider vertical set of treatments of each design D_i, $i = 1, 2, 3, 4, 5, 6, 7$, as block, we get a nearly α −Resolvable BIB design D^* as

[(2 ,3, 4, 5, 6), (3, 4, 5, 6, 7), (4, 5, 6, 7, 2), (5, 6, 7, 2, 3), (6, 7, 2, 3, 4), (7, 2, 3, 4, 5)]
[(1, 3, 4, 5, 6), (3, 4, 5, 6, 7), (4, 5, 6, 7, 1), (5, 6, 7, 1, 3), (6, 7, 1, 3, 4), (7, 1, 3, 4, 5)]
[(1, 2, 4, 5, 6), (2, 4, 5, 6, 7), (4, 5, 6, 7, 1), (5, 6, 7, 1, 2), (6, 7, 1, 2, 4), (7, 1, 2, 4, 5)]
[(1, 2, 3, 5, 6), (2, 3, 5, 6, 7), (3, 5, 6, 7, 1), (5, 6, 7, 1, 2), (6, 7, 1, 2, 3), (7, 1, 2, 3, 5)]
[(1, 2, 3, 4, 6), (2, 3, 4, 6, 7), (3, 4, 6, 7, 1), (4, 6, 7, 1, 2), (6, 7, 1, 2, 3), (7, 1, 2, 3, 4)]
[(1, 2, 3, 4, 5), (2, 3, 4, 5, 7), (3, 4, 5, 7, 1), (4, 5, 7, 1, 2), (5, 7, 1, 2 ,3), (7, 1, 2, 3, 4)]
[(1, 2, 3, 4, 5), (2, 3, 4, 5, 6), (3, 4, 5, 6, 1), (4, 5, 6, 1, 2), (5, 6, 1, 2, 3), (6, 1, 2, 3, 4)]

with parameters $v^* = 7$, $b^* = 42$, $r^* = 30, k^* = 5, \lambda^* = 20$ and $\alpha = 5$ as

4. Summary and Concluding remarks

The construction methods of α -Resolvable and nearly α -Resolvable BIB designs given are simple and easy to apply for practical situations. Since the designs are resolvable, these can be used in an information theory *i.e*., constructing A^2 codes and low-density parity-checks (LDPC) codes [Xu *et al*. (2015)] and also in sequential experimentation over space and time [Morgan and Reck (2007)]. We can also find application of these designs in cryptography and cryptology.

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