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# Construction of Partially Balanced Semi-Latin Rectangles with Block Size 4

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## SUMMARY

Semi-Latin rectangles represent row-column designs where each row-column intersection contains the same number of experimental units, denoted as $k > 1$. Additionally, each treatment appears an equal number of times in each row ($n_r$, say) and in each column ($n_c$, say) ($n_r \geq 1$ and $n_c \geq 1$ may or may not be same). Partially Balanced Semi-Latin rectangles (PBSLR) constitute a subset of Semi-Latin rectangles (SLR), serving as generalizations of Latin squares and Semi-Latin squares (SLS). These designs find utility in various agricultural and industrial experiments, particularly situations where one effect is considered a column effect and the other a row effect, with the intersection (block/cell) accommodating precisely four units. This article introduces two methods for constructing PBSLR designs with a block size of 4. Also, R package has been developed for generating the designs.

*Keywords:* Semi Latin Rectangle; Partially Balanced Semi-Latin rectangles; Canonical efficiency factor; Average efficiency factor.

## 1. INTRODUCTION

In various agricultural experiments focused on plant disease research, experimental units often involve the use of plant leaves as plots. For comprehensive details about these experiments, Price (1946) provides valuable information. The inherent variability among plants and the influence of leaf height introduces sources of heterogeneity into the study. To address this, commonly recommended are row-column designs like Latin squares. In these designs, columns represent individual plants, and rows correspond to different leaf heights. However, in practical scenarios, the number of available plants often exceeds the number of usable leaves per plant. Addressing this challenge, Youden (1937) invented row-column designs with fewer rows than columns, now known as Youden squares, in a groundbreaking experiment on tobacco mosaic virus.

Andersen and Hilton (1980a, 1980b) introduced the concept of a Latin rectangle $(p,q,x)$, which is a rectangular matrix with $x$ symbols such that each symbol occurs at most $p$ times in each row and at most $q$ times in each column. A further generalization of Latin squares and Latin rectangles is the semi-Latin square (SLS) and semi-Latin rectangles (SLR). In SLS, each treatment appears exactly once in each row and each column, while each row-column intersection (cell/block) contains more than one treatment. For detailed insights, references like Bailey (1988 and 1992), Bailey & Chigbu (1997), Bailey & Royle (1997), Bedford & Whitaker (2001), Chigbu (2003), Parsad (2006), Soicher (2012 and 2013), and Bailey & Soicher (2021) provide extensive discussions. SLR is a type of row–column design where $v$ treatments are arranged into $h$ rows and $p$ columns. In each row–column intersection (block/cell), $k$ treatments occur, with each treatment

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appearing at most once in each block $(k>1)$. Additionally, each treatment appears a constant number $n_r$ (say) of times in each row and a constant number $n_c$ (say) of times in each column ($n_r \geq 1$ and $n_c \geq 1$, which may or may not be equal). These designs are denoted as $(h \times p) / k$. For theoretical discussions and applications of these designs, readers can refer to Bailey and Monod (2001), Uto and Bailey (2022), Yadav *et al.* (2024) and related citations. Bailey and Monod (2001) initially introduced Semi-Latin rectangles (SLR) and provided construction methods primarily for block size two. Uto and Bailey (2020) extended this concept by introducing balanced SLR (BSLR) and offering two algorithms, one for even $v$ and another for odd $v$ to construct BSLR designs, but these were limited to block size two. Additionally, Uto and Bailey (2022) introduced regular-graph Semi-Latin rectangles, characterized by treatment concurrences between any two distinct treatments differing by at most one.

The utility of SLR with block size $k > 2$ in various experimental situations has not been explored in the existing literature. Additionally, while BSLR is an option, it demands more experimental units, implying higher resource requirements. On the other hand, by employing Partially Balanced Semi-Latin Rectangles (PBSLRs), there is a potential to reduce the number of experimental units or resources needed for a study.

Underlying block designs of a SLR is the block design obtained by ignoring its rows and columns. For a SLR, we impose the condition that the underlying block designs are binary. PBSLRs form a subclass of Semi-Latin rectangles with the property that their underlying block designs form partially balanced incomplete block designs (PBIBDs). Therefore, PBSLR are the subclass of Semi-Latin rectangles, which exhibits the added property of partial balance, ensuring that each pair of distinct treatments are not appearing constant number of times in the design. This differentiates a partially BSLR from the BSLRs and general SLRs. In this design $n_i$ number of pairs of a treatment appear $\lambda_i$ times, $i = 1, 2, \ldots, m$. Also, each treatment appears in $hn_r = pn_c$ blocks, overall, and in each of these blocks it concurs with $(k-1)$ distinct treatments. Hence, the sum of concurrences with that treatment is $\sum_i^m n_i \lambda_i$, so, the conditions for the existence and construction of a PBSLR are specified by parameters $v, k, h, p, n_r, n_c,$ and $\lambda_i$, satisfying the following equations:

$$vhn_r = vpn_c = khp$$

$$\sum_i^m n_i \lambda_i = hn_r(k-1) = pn_c(k-1)$$

$$\sum_i^m n_i = v - 1 \qquad \ldots (1)$$

Despite the existing literature on Semi-Latin rectangles, it has predominantly focused on block size two, with a noticeable absence of work for block size $k > 2$. Acknowledging the importance of SLRs, we present two methods to constructing PBSLRs with a block size of four.

## 2. EXPERIMENTAL SET UP

PBSLRs designs are useful in various types of experiments, including agricultural studies such as plant disease experiments, food sensory experiments, glasshouse experiments, and consumer testing experiments. For example, in food sensory experiments, consider a food sensory experiment where 12 food items are to be compared. The experiment will be conducted in 3 sessions. There are 6 panellists and each of them will taste 4 food items at each session. In this case a Semi-Latin rectangle with 3 rows (sessions), 6 columns (panelist) with each row-column intersection (block/cell) having size 4 can be used. The (3×6)/4 SLR as:

**Table 1.** Semi-Latin rectangle for 12 treatments in food sensory experiment

| Row/Column | Panelist1 | Panelist2 | Panelist3 | Panelist4 | Panelist5 | Panelist6 |
|---|---|---|---|---|---|---|
| session1 | 1, 2, 4, 9 | 2, 3, 5, 10 | 3, 4, 6, 11 | 5, 6, 7, 12 | 1, 7, 8, 9 | 8, 10, 11, 12 |
| session2 | 3, 5, 6, 11 | 1, 7, 8, 11 | 2, 8, 9, 12 | 1, 3, 4, 10 | 2, 4, 10, 12 | 5, 6, 7, 9 |
| session3 | 7, 8, 10, 12 | 4, 6, 9, 12 | 1, 5, 7, 10 | 2, 8, 9, 11 | 3, 5, 6, 11 | 1, 2, 3, 4 |

## 3. PRELIMINARIES

In the context of assuming the additive effect, the model for the present study is expressed as:

$$\boldsymbol{y} = \mu 1 + \Delta'\boldsymbol{\tau} + \phi'\boldsymbol{\rho} + \boldsymbol{\psi}'\boldsymbol{\gamma} + \boldsymbol{D}'\boldsymbol{\beta} + \boldsymbol{\varepsilon} \qquad \ldots (2)$$

where,

$\boldsymbol{y} = hpk \times 1$ vector of observations,

$1 = hpk \times 1$ vector of 1,

$\Delta' = hpk \times v$ incidence matrix of observations versus treatments,

$\boldsymbol{\tau} = v \times 1$ vector of treatment effects,

$\phi' = hpk \times h$ incidence matrix of observations versus rows,

$\boldsymbol{\rho} = h \times 1$ vector of row effects,

$\boldsymbol{\psi}' = hpk \times p$ incidence matrix of observations versus columns,

$\boldsymbol{\gamma} = p \times 1$ vector of column effects,

$\boldsymbol{D}' = hpk \times hp$ incidence matrix of observations versus blocks/cells,

$\boldsymbol{\beta} = hp \times 1$ vector of block effects,

$\boldsymbol{\varepsilon} = hpk \times 1$ vector of random errors with Expectation$(\boldsymbol{\varepsilon}) = 0$ and Dispersion$(\boldsymbol{\varepsilon}) = \sigma^2 \mathbf{I}$,

$(\boldsymbol{.})' =$ Transpose of matrix

The incidence matrices $\Delta'$, $\phi'$, $\boldsymbol{\psi}'$ and $\boldsymbol{D}'$ are defined in the usual manner. For instance, if $\Delta' = (\xi_{ui})$, then

$$\xi_{ui} = \begin{cases} 1, \text{ if the } \mathrm{u^{th}} \text{ observation corresponds} \\ \qquad \text{to the } \mathrm{i^{th}} \text{ treatment} \\ 0, \text{ otherwise.} \end{cases}$$

The matrices $\phi'$, $\boldsymbol{\psi}'$ and $\boldsymbol{D}'$ are defined similarly.

Under model (2), the information matrix for estimating the linear functions of treatment effects is given by

$$\mathbf{C}_d = \mathbf{R}_\tau - (\mathbf{N}_1\mathbf{A}_{11}\mathbf{N}_1' + \mathbf{N}_2\mathbf{A}_{21}\mathbf{N}_1' + \mathbf{N}_3\mathbf{A}_{31}\mathbf{N}_1' + \mathbf{N}_1\mathbf{A}_{12}\mathbf{N}_2' + \mathbf{N}_2\mathbf{A}_{22}\mathbf{N}_2' + \mathbf{N}_3\mathbf{A}_{32}\mathbf{N}_2' + \mathbf{N}_1\mathbf{A}_{13}\mathbf{N}_3' + \mathbf{N}_2\mathbf{A}_{23}\mathbf{N}_3' + \mathbf{N}_3\mathbf{A}_{33}\mathbf{N}_3')$$

Where,

$$\mathbf{A}_{11} = \frac{\mathbf{I}_{h\times h}}{pk} + \frac{1}{p^2}\Big[\mathbf{J}_{h\times p}\mathbf{A}_{22}\mathbf{J}_{p\times h} + \mathbf{J}_{h\times hp}\mathbf{A}_{32}\mathbf{J}_{p\times h} + \mathbf{J}_{h\times p}\mathbf{A}_{23}\mathbf{J}_{hp\times h} + \mathbf{J}_{h\times hp}\mathbf{A}_{33}\mathbf{J}_{hp\times h}\Big],$$

$$\mathbf{A}_{12} = -\frac{1}{p}\Big[\mathbf{J}_{h\times p}\mathbf{A}_{22} + \mathbf{J}_{h\times hp}\mathbf{A}_{32}\Big], \ \mathrm{A}_{21} = \mathrm{A}_{12}'$$

$$\mathbf{A}_{13} = \left(\frac{1}{p} - \frac{1}{h}\right)\mathbf{J}_{h\times hp}\mathbf{A}_{33}, \ \mathrm{A}_{31} = \mathrm{A}_{13}'$$

$$\mathbf{A}_{22} = \frac{1}{kh}\Big[\mathbf{I}_{p\times p} + \left(pk^2 - p^3k^2\right)\mathbf{J}_{p\times p}\Big],$$

$$\mathbf{A}_{23} = -\frac{1}{h}\mathbf{J}_{p\times hp}\mathbf{A}_{33}, \ \mathbf{A}_{33} = \left(k\mathbf{I}_{hp\times hp} - \frac{pk}{h}\mathbf{J}_{hp\times hp}\right)^{-},$$

$\mathbf{R}_\tau = hn_r\mathbf{I}_{v\times v}$, J = unity matrix

$\mathbf{N}_1 =$ incidence matrix of treatments versus rows,

$\mathbf{N}_2 =$ incidence matrix of treatments versus columns,

$\mathbf{N}_3 =$ incidence matrix of treatments versus blocks /cells.

$(\boldsymbol{.})^- =$ Generalized inverse of matrix

As we know that connected designs (i.e., Rank $(\boldsymbol{C_d}) = v - 1$) is necessary to estimate all treatment contrast. For a connected designs with $v$ treatments a matrix $\boldsymbol{M_d}$ can be defined as

$$\boldsymbol{M}_d = \boldsymbol{R}_{\hat{\delta}}^{-1/2}\boldsymbol{C}_d\boldsymbol{R}_{\hat{\delta}}^{-1/2}$$

Where, $\boldsymbol{R}_\tau = \mathrm{diag}(r_1, \ldots, r_v)$ and $r_i$ $(1 \le i \le v)$ is the number replication of $i^{th}$ treatment. Then the positive eigenvalues of the matrix $\boldsymbol{M_d}$ say $\lambda_i$, $1 \le i \le v-1$ are known as the canonical efficiency factor of the designs. If we consider an equireplicate designs i.e. $r_i = r$, $1 \le i \le v$ then the canonical efficiency factors are $1/r$ times the positive eigenvalues of $\boldsymbol{C_d}$. Average efficiency factor of a design is determined by taking the harmonic mean of the canonical efficiency factors.

## 4. METHODS FOR CONSTRUCTING THE DESIGNS

In this section, we present two construction methods for generating PBSLR. The first method is employed when the number of treatments $(v)$ is even, while the second method is utilized for constructing designs when $v$ is odd.

**Method 1: Construction of PBSLR for even number of treatments in block size 4**

This method consists of two series for generating PBSLR designs. The first series is used when $v=4(n+1)$, while the second series is employed for constructing the designs when $v=4n+2$, where $n$ is any positive integer.

**Series1 (when $v=4(n+1)$)**

This series approach for constructing PBSLR designs is based on the concept of generating the design from an initial matrix. Once the initial matrix has been developed, the entire design can be generated from the initial matrix using a specified procedure. Step 1 outlines the method to obtain the initial matrix, and Step 2 details the procedure for generating the entire design from the initial matrix, as described in the following sequel.

**Step-1:** First construct a matrix of order $\frac{v}{2}\times 4$ where, $v=4(n+1)$ for any positive integer $n$, such that the sum of the elements of rows of the matrix are constant which is $(2v+2)$. First column of such matrix is obtained by arranging the numbers from top to bottom as $1,2,\ldots,\frac{v}{2}$; second column obtained by arranging the numbers as $\left(\frac{v}{2}+1\right),\left(\frac{v}{2}+2\right),\ldots,v$ starting from bottom to top; third column obtained by arranging the numbers from bottom to top as $1,2,\ldots,\frac{v}{2}$ and fourth column obtained by arranging the numbers as $\left(\frac{v}{2}+1\right),\left(\frac{v}{2}+2\right),\ldots,v$ starting from top to bottom.

For example, for $n=1, v=8,$ matrix of order 4×4 will be $\begin{pmatrix} 1 & 8 & 4 & 5 \\ 2 & 7 & 3 & 6 \\ 3 & 6 & 2 & 7 \\ 4 & 5 & 1 & 8 \end{pmatrix}$.

**Step-2:** Considering the matrix of order $\frac{v}{2}\times 4$ which is obtained in step-1 as first (initial) column of size four and adding one to each element with modulo$v$ we got total $v$ column of size four. Then the resulting design is $\left(\frac{v}{2}\times v\right)/4$ PBSLR of block size four.

General parameters for the obtained designs are $v=4(n+1)$, $h=2(n+1)$, $p=4(n+1)$, $k=4$, $n_r=4$, $n_c=2$, $\lambda_1=8(n+1)$, $\lambda_2=8$, $\lambda_3=0$, $n_1=1$, $n_2=2n+2$, $n_3=2n.$

For this construction method association scheme is defined as:

Arrange the half treatments ($v$) in one row and other in second row in ascending order, then two treatments are

First associates if they lie in the same column,

Second associates if they lie in the nearest column(s) and after that alternative column,

Third associates otherwise.

The parameters of the association scheme are:

$v=4(n+1), n_1=1, n_2=2n+2, n_3=2n$

$$P_1=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 2n+2 & 0 \\ 0 & 0 & 2n \end{pmatrix},\quad P_2=\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 2n \\ 0 & 2n & 0 \end{pmatrix},$$

$$P_3=\begin{pmatrix} 0 & 0 & 1 \\ 0 & 2n+2 & 0 \\ 2n-1 & 0 & 0 \end{pmatrix}$$

**For example,** for

$v=8$, $h=4$, $p=8$, $k=4$, $n_r=4$, $n_c=2$, $\lambda_1=16$, $\lambda_2=8$, $\lambda_3=0$, $n_1=1$, $n_2=4$, $n_3=2$, a $(4x8)/4$ PBSLR is obtained following above steps and is given below

| row/col | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| **1** | 1, 8, 4, 5 | 2, 1, 5, 6 | 3, 2, 6, 7 | 4, 3, 7, 8 | 5, 4, 8, 1 | 6, 5, 1, 2 | 7, 6, 2, 3 | 8, 7, 3, 4 |
| **2** | 2, 7, 3, 6 | 3, 8, 4, 7 | 4, 1, 5, 8 | 5, 2, 6, 1 | 6, 3, 7, 2 | 7, 4, 8, 3 | 8, 5, 1, 4 | 1, 6, 2, 5 |
| **3** | 3, 6, 2, 7 | 4, 7, 3, 8 | 5, 8, 4, 1 | 6, 1, 5, 2 | 7, 2, 6, 3 | 8, 3, 7, 4 | 1, 4, 8, 5 | 2, 5, 1, 6 |
| **4** | 4, 5, 1, 8 | 5, 6, 2, 1 | 6, 7, 3, 2 | 7, 8, 4, 3 | 8, 1, 5, 4 | 1, 2, 6, 5 | 2, 3, 7, 6 | 3, 4, 8, 7 |

Now, for association scheme arrange $v$ as

1, 2, 3, 4

5, 6, 7, 8

Then, association scheme for above designs will be

| treatment | 1st association | 2nd association | 3rd association |
|---|---|---|---|
| 1 | 5 | 2, 6, 4, 8 | 3, 7 |
| 2 | 6 | 1, 5, 3, 7 | 4, 8 |
| 3 | 7 | 2, 6, 4, 8 | 1, 5 |
| 4 | 8 | 3, 7, 1, 5 | 2, 6 |
| 5 | 1 | 2, 6, 4, 8 | 3, 7 |
| 6 | 2 | 1, 5, 3, 7 | 4, 8 |
| 7 | 3 | 2, 6, 4, 8 | 1, 5 |
| 8 | 4 | 3, 7, 1, 5 | 2, 6 |

with parameters of the association scheme as

$v=8, n_1=1, n_2=4, n_3=2$

$$P_1=\begin{pmatrix}0&0&0\\0&4&0\\0&0&2\end{pmatrix},\ P_2=\begin{pmatrix}0&1&0\\1&0&2\\0&2&0\end{pmatrix},\ P_3=\begin{pmatrix}0&0&1\\0&4&0\\1&0&0\end{pmatrix}$$

**Series2 (when $v=4n+2$)**

This series approach for constructing PBSLR designs is also based on the concept of generating the design from an initial matrix. Step 1 outlines the method to obtain the initial matrix, and Step 2 details the procedure for generating the entire design from the initial matrix, as described in the following sequel.

**Step-1** First construct a matrix of order $\frac{v}{2}\times 4$ where, $v=4n+2$ for any positive integer $n$, such thatsum of the elements of rows of the matrix except last row are constant which is $(2v+4)$ and sum of the elements of last row is $(v+4)$. First column of such matrix is obtained by arranging the numbers from top to bottom as $1,2,\ldots,\frac{v}{2}$; second column obtained by arranging the numbers as $\left(\frac{v}{2}+1\right),\left(\frac{v}{2}+2\right),\ldots,v$ starting from bottom to top; third column obtained by arranging the numbers from bottom to top as $2,\ldots,\frac{v}{2}+1$ and fourth column obtained by arranging the numbers as $\left(\frac{v}{2}+2\right),\left(\frac{v}{2}+3\right),\ldots,v,1$ starting from top to bottom.

For example, for $n=2, v=10,$ matrix of order 5×4 will be $\begin{pmatrix}1&10&6&7\\2&9&5&8\\3&8&4&9\\4&7&3&10\\5&6&2&1\end{pmatrix}$.

**Step-2:** Considering the matrix of order $\frac{v}{2}\times 4$ which is obtained in step-1 as first (initial) column of size four and adding one to each element with modulo$v$ we get total $v$ column of size four. Then the resulting design is $\left(\frac{v}{2}\times v\right)/4$ PBSLR of block size four.

General parameters for the obtained designs are:

$v=4n+2,\ h=2n+1,\ p=4n+2,\ k=4,\ n_r=4,$ .

$n_c=2,\ \lambda_1=8,\ \lambda_2=0,\ \lambda_3=4n+2,\ n_1=2n+1,$

$n_2=2(n-1),\ n_3=2$

For this construction method, association scheme is defined as:

Arrange the treatments ($v$) on the circle serially, then two treatments are

First associates if they lie on the odd places from that,

Second associates if they lie on the even places from both side of that except last two places,

Third associates otherwise.

The parameters of the association scheme are:

$v=4n+2, n_1=2n+1, n_2=2(n-1), n_3=2$

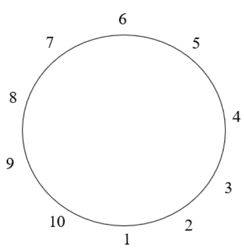
$$P_1=\begin{pmatrix}0&2(n-1)&2\\2(n-1)&0&0\\2&0&0\end{pmatrix},$$

$$P_2=\begin{pmatrix}2n+1&0&0\\0&2(n-2)&1\\0&1&1\end{pmatrix},\ P_3=\begin{pmatrix}2n+1&0&0\\0&2n-3&1\\0&1&0\end{pmatrix}$$

**For example,** for $v=10,\ h=5,\ p=10,\ k=4,$ $n_r=4,\ n_c=2,\ \lambda_1=8,\ \lambda_2=0,\ \lambda_3=10 n_1=5,\ n_2=2,$ , $n_3=2$ a (5×10)/4 PBSLR is obtained following above steps and is given below.

| row/col | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| **1** | 1, 10, 6, 7 | 2, 1, 7, 8 | 3, 2, 8, 9 | 4, 3, 9, 10 | 5, 4, 10, 1 | 6, 5, 1, 2 | 7, 6, 2, 3 | 8, 7, 3, 4 | 9, 8, 4, 5 | 10, 9, 5, 6 |
| **2** | 2, 9, 5, 8 | 3, 10, 6, 9 | 4, 1, 7, 10 | 5, 2, 8, 1 | 6, 3, 9, 2 | 7, 4, 10, 3 | 8, 5, 1, 4 | 9, 6, 2, 5 | 10, 7, 3, 6 | 1, 8, 4, 7 |
| **3** | 3, 8, 4, 9 | 4, 9, 5, 10 | 5, 10, 6, 1 | 6, 1, 7, 2 | 7, 2, 8, 3 | 8, 3, 9, 4 | 9, 4, 10, 5 | 10, 5, 1, 6 | 1, 6, 2, 7 | 2, 7, 3, 8 |
| **4** | 4, 7, 3, 10 | 5, 8, 4, 1 | 6, 9, 5, 2 | 7, 10, 6, 3 | 8, 1, 7, 4 | 9, 2, 8, 5 | 10, 3, 9, 6 | 1, 4, 10, 7 | 2, 5, 1, 8 | 3, 6, 2, 9 |
| **5** | 5, 6, 2, 1 | 6, 7, 3, 2 | 7, 8, 4, 3 | 8, 9, 5, 4 | 9, 10, 6, 5 | 10, 1, 7, 6 | 1, 2, 8, 7 | 2, 3, 9, 8 | 3, 4, 10, 9 | 4, 5, 1, 10 |

Now, for association scheme arrange $v$ as




Then, association scheme for above designs will be

| treatment | 1st association | 2nd association | 3rd association |
|---|---|---|---|
| 1 | 2, 4, 6, 8, 10 | 3, 9 | 5, 7 |
| 2 | 3, 5, 7, 9, 1 | 4, 10 | 6, 8 |
| 3 | 4, 6, 8, 10, 2 | 5, 1 | 7, 9 |
| 4 | 5, 7, 9, 1, 3 | 6, 2 | 8, 10 |
| 5 | 6, 8, 10, 2, 4 | 7, 3 | 9, 1 |
| 6 | 7, 9, 1, 3, 5 | 8, 4 | 10, 2 |
| 7 | 8, 10, 2, 4, 6 | 9, 5 | 1, 3 |
| 8 | 9, 1, 3, 5, 7 | 10, 6 | 2, 4 |
| 9 | 10, 2, 4, 6, 8 | 1, 7 | 3, 5 |
| 10 | 1, 3, 5, 7, 9 | 2, 8 | 4, 6 |

with parameters of the association scheme as

$$v=10, n_1 = 5, n_2 = 2, n_3 = 2$$

$$P_1 = \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \quad P_3 = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

**Method 2: Construction of PBSLR for odd number of treatments in block size 4**

This construction method for PBSLR designs with an odd number of treatments follows the idea of generating designs from an initial column. Once the initial column is developed, the entire designs can be generated using a defined procedure. Step 1 outlines the process of obtaining the initial column, and Step 2 provides the details of the procedure for generating the entire designs from this initial column, as explained in the following sequels.

**Step-1:** First construct a matrix of order $v \times 4$ where, $v = 2n + 3$ for any integer $n$, such that the first column of the matrix is obtained by arranging the numbers from top to bottom as $1, 2, \ldots, v$; second column obtained by arranging the numbers as $2, 3, \ldots, v, 1$ starting from top to bottom; third column obtained by arranging the numbers from top to bottom as $3, 4, \ldots, v, 1, 2$ and fourth column obtained by arranging the numbers as $4, 5, \ldots, v, 1, 2, 3$ starting from top to bottom.

For example, for $n = 3, v = 9,$ matrix of order 9×4 will be

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \\ 5 & 6 & 7 & 8 \\ 6 & 7 & 8 & 9 \\ 7 & 8 & 9 & 1 \\ 8 & 9 & 1 & 2 \\ 9 & 1 & 2 & 3 \end{pmatrix}.$$

**Step-2:** Considering the matrix of order $v \times 4$ which is obtained in step-1 as first column of size four and adding one to each element with mod $v$ we got total $v$ column of size four. Then the resulting design is $(v \times v)/4$ partially balanced Semi-Latin Rectangle of block size four.

General parameters for the obtained designs are $v = 2n + 3,\ h = 2n + 3,\ p = 2n + 3,\ k = 4,\ n_r = 4,$ $n_c = 4,\ \lambda_1 = 3v,\ \lambda_2 = 2v,\ \lambda_3 = v,\ \lambda_4 = 0,\ n_1 = 2,$ $n_2 = 2,\ n_3 = 2,\ n_4 = 2(n-2)$

For this construction method, association scheme is defined as:

Arrange the treatments ($v$) on the circle serially, then two treatments are

First associates if they lie on the first nearest places from that,

Second associates if they lie on the second nearest places from that,

Third associates if they lie on the third nearest places from that,

Fourth associates otherwise.

The parameters of the association scheme are:

$$v = 2n+3, n_1 = 2, n_2 = 2, n_3 = 2, n_4 = 2(n-2)$$

$$P_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2n-5 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 2n-6 \end{pmatrix},$$

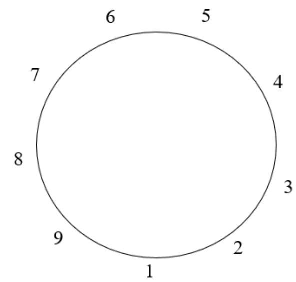
$$P_3 = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 2n-6 \end{pmatrix}, \quad P_3 = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 2n-6 \end{pmatrix}$$

**For example**, for

$v = 9,\ h = 9,\ p = 9,\ k = 4,\ n_r = 4,\ n_c = 4,\ \lambda_1 = 27,$ $\lambda_2 = 18,\ \lambda_3 = 9,\ \lambda_4 = 0,\ n_1 = 2,\ n_2 = 2,\ n_3 = 2,\ n_4 = 2$, a (9 x 9)/4 PBSLR is obtained following above steps and is given below.

| row/ col | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| **1** | 1, 2, 3, 4 | 2, 3, 4, 5 | 3, 4, 5, 6 | 4, 5, 6, 7 | 5, 6, 7, 8 | 6, 7, 8, 9 | 7, 8, 9, 1 | 8, 9, 1, 2 | 9, 1, 2, 3 |
| **2** | 2, 3, 4, 5 | 3, 4, 5, 6 | 4, 5, 6, 7 | 5, 6, 7, 8 | 6, 7, 8, 9 | 7, 8, 9, 1 | 8, 9, 1, 2 | 9, 1, 2, 3 | 1, 2, 3, 4 |
| **3** | 3, 4, 5, 6 | 4, 5, 6, 7 | 5, 6, 7, 8 | 6, 7, 8, 9 | 7, 8, 9, 1 | 8, 9, 1, 2 | 9, 1, 2, 3 | 1, 2, 3, 4 | 2, 3, 4, 5 |
| **4** | 4, 5, 6, 7 | 5, 6, 7, 8 | 6, 7, 8, 9 | 7, 8, 9, 1 | 8, 9, 1, 2 | 9, 1, 2, 3 | 1, 2, 3, 4 | 2, 3, 4, 5 | 3, 4, 5, 6 |
| **5** | 5, 6, 7, 8 | 6, 7, 8, 9 | 7, 8, 9, 1 | 8, 9, 1, 2 | 9, 1, 2, 3 | 1, 2, 3, 4 | 2, 3, 4, 5 | 3, 4, 5, 6 | 4, 5, 6, 7 |
| **6** | 6, 7, 8, 9 | 7, 8, 9, 1 | 8, 9, 1, 2 | 9, 1, 2, 3 | 1, 2, 3, 4 | 2, 3, 4, 5 | 3, 4, 5, 6 | 4, 5, 6, 7 | 5, 6, 7, 8 |
| **7** | 7, 8, 9, 1 | 8, 9, 1, 2 | 9, 1, 2, 3 | 1, 2, 3, 4 | 2, 3, 4, 5 | 3, 4, 5, 6 | 4, 5, 6, 7 | 5, 6, 7, 8 | 6, 7, 8, 9 |
| **8** | 8, 9, 1, 2 | 9, 1, 2, 3 | 1, 2, 3, 4 | 2, 3, 4, 5 | 3, 4, 5, 6 | 4, 5, 6, 7 | 5, 6, 7, 8 | 6, 7, 8, 9 | 7, 8, 9, 1 |
| **9** | 9, 1, 2, 3 | 1, 2, 3, 4 | 2, 3, 4, 5 | 3, 4, 5, 6 | 4, 5, 6, 7 | 5, 6, 7, 8 | 6, 7, 8, 9 | 7, 8, 9, 1 | 8, 9, 1, 2 |

Now, for association schemearrange $v$ as




Then, association scheme for above designs will be

| treatment | 1st association | 2nd association | 3rd association | 4th association |
|---|---|---|---|---|
| **1** | 2, 9 | 3, 8 | 4, 7 | 5, 6 |
| **2** | 3, 1 | 4, 9 | 5, 8 | 6, 7 |
| **3** | 4, 2 | 5, 1 | 6, 9 | 7, 8 |
| **4** | 5, 3 | 6, 2 | 7, 1 | 8, 9 |
| **5** | 6, 4 | 7, 3 | 8, 2 | 9, 1 |
| **6** | 7, 5 | 8, 4 | 9, 3 | 1, 2 |
| **7** | 8, 6 | 9, 5 | 1, 4 | 2, 3 |
| **8** | 9, 7 | 1, 6 | 2, 5 | 3, 4 |
| **9** | 1, 8 | 2, 7 | 3, 6 | 4, 5 |

with parameters of the association scheme as

$$v = 9, n_1 = 2, n_2 = 2, n_3 = 2, n_4 = 2$$

$$P_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix},$$

$$P_3 = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}, \quad P_3 = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

**Remark1: (i)** As per the definition of PBSLR, every treatment occurs an equal number of times in each row, an equal number of times in each column, and at most once in a block. Consequently, PBSLR exists only when $v > k$, if $v < k$, treatments would appear in a block at least once. Similarly, if $v = k$, each block, row, and column would have identical content, lacking meaningful interpretation in the context of PBSLR. Additionally, the proposed method for constructing PBSLR is limited to $5 < v \leq 20$, because for $v = 5$ it gives BSLR.

**(ii)** Although for $v = 4n + 2$ proposed method gives three associate designs but for $v = 6$ it gives two associate designs. Similarly, for $v = 2n + 3$ proposed method gives four associate designs but for $v = 7$ it gives three associate designs.

**Remark2:** An R package *slr* has been developed to implement the proposed methods of constructions of partially balanced semi-Latin rectangles. The package is available on CRAN in the webpage https://cran.r-project.org/web/packages/slr/index.html. Yadav *et al.*, (2023). It contains a function *pbslr*$(v,k)$ which can be used to construct PBSLR, putting $k = 4$ and for any given number of treatments $v$.

## 5. EFFICIENCY OF SEMI-LATIN RECTANGLES

In this Section, we provide the average efficiency factor of PBSLR designs generated using the developed methods. The average efficiency factor (John, 1992) of a design is determined by taking the harmonic mean of the canonical efficiency factors, which are obtained by multiplying the nonzero eigenvalues of its information matrix by $1/r$, here $r = hn_r = pn_c$. A list of PBSLR designs for $5 < v \leq 20$are given in Table 2. Average efficiency factor ($\bar{E}$) was computed for each parametric combination of designs obtained based on PBSLR.

**Table 2.** Average efficiency factor of partially balanced semi-Latin rectangles with $5 < v \leq 20$

| $v$ | $h$ | $p$ | $n_r$ | $n_c$ | $\lambda_1$ | $\lambda_2$ | $\lambda_3$ | $\lambda_4$ | $\bar{E}$ | Methods |
|---|---|---|---|---|---|---|---|---|---|---|
| 5 | 5 | 5 | 4 | 4 | 15 | - | - | - | 0.9375 | Method2 |
| 6 | 3 | 6 | 4 | 2 | 8 | 6 | - | - | 0.8974 | Method1(Series2) |
| 7 | 7 | 7 | 4 | 4 | 21 | 14 | 7 | - | 0.8517 | Method2 |
| 8 | 4 | 8 | 4 | 2 | 16 | 8 | 0 | - | 0.7778 | Method1(Series1) |
| 9 | 9 | 9 | 4 | 4 | 27 | 18 | 9 | 0 | 0.7711 | Method2 |
| 10 | 5 | 10 | 4 | 2 | 8 | 0 | 10 | - | 0.8074 | Method1(Series2) |
| 11 | 11 | 11 | 4 | 4 | 33 | 22 | 11 | 0 | 0.7026 | Method2 |
| 12 | 6 | 12 | 4 | 2 | 24 | 8 | 0 | - | 0.7333 | Method1(Series1) |
| 13 | 13 | 13 | 4 | 4 | 39 | 26 | 13 | 0 | 0.6443 | Method2 |
| 14 | 7 | 14 | 4 | 2 | 8 | 0 | 14 | - | 0.7739 | Method1(Series2) |
| 15 | 15 | 15 | 4 | 4 | 45 | 30 | 15 | 0 | 0.5945 | Method2 |
| 16 | 8 | 16 | 4 | 2 | 32 | 8 | 0 | - | 0.7143 | Method1(Series1) |
| 17 | 17 | 17 | 4 | 4 | 51 | 34 | 17 | 0 | 0.5517 | Method2 |
| 18 | 9 | 18 | 4 | 2 | 8 | 0 | 18 | - | 0.7570 | Method1(Series2) |
| 19 | 19 | 19 | 4 | 4 | 57 | 38 | 19 | 0 | 0.5143 | Method2 |
| 20 | 10 | 20 | 4 | 2 | 40 | 8 | 0 | - | 0.7037 | Method1(Series1) |

## 6. CONCLUDING REMARKS

Semi-Latin rectangles (SLR) are more useful in various agricultural as well as industrial experiments in which one of the effects can be consider as column effect and another as row effect, where the intersection of effects can be accommodate with 4 units. In this article we proposed methods for obtaining PBSLR designs for any possible parametric combinations. These methods enable for construction of PBSLR for all permissible number of treatments. We also have provided the average efficiency factor of designs generated using the proposed methods of constructions for the number of treatments up to 20. Also, an R package has been developed to facilitate the application of the proposed methods.

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